NONSMOOTH AND VARIATIONAL ANALYSIS (NAVAL) CONFERENCE

UNIVERSITY OF BURGUNDY,
JUNE 26-28, 2023
DIJON, FRANCE

The international conference Nonsmooth And Variational AnaLysis (NAVAL) is organised in honour of Professor **Lionel THIBAULT** of the University of Montpellier 2 from June 26th to 28th 2023. Its objective is to gather eminent experts from different countries (Australia, Austria, Chile, France, Germany, Israel, Italy, Saudi Arabia, Senegal, Spain, USA and many others, i.e. about 80 participants) to exchange on the latest contributions and to present the state of the art in the fields of nonsmooth and variational analysis, vector optimization, optimal control, operations research and nonlinear dynamics. The development of nonsmooth analysis tools has led to important advances in optimisation, optimal control and nonlinear dynamics. These theories have burgeoned tremendously due to rich applications in economics, engineering, mechanics, and many others. Because of these important applications, these topics are vivrant research areas and expanding branches of applied mathematics. The main objective of this conference is to discuss new results, develop new ideas and encourage collaborations between participants that will benefit for practical problems. Special issue in the journal "Optimization" will be devoted to the conference.

This meeting is dedicated to recent developments in the theory, algorithms and applications of optimization

The site of the conference is:
http://jourani.perso.math.cnrs.fr/Colloque_Thibault/index.html

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Conference program

Venue : Amphi Bernard, UFR Sciences et Techniques

Monday 26/06/2023

8h-8h30	**Welcome**
8h30-9h	**Opening:** **Mr. Vincent Thomas,** *President of the University of Burgundy* **Mr. François Blais,** *Director of the UFR Sciences et Techniques* **Dariusz Zagrodny,** *Scientific Committee* **Abderrahim Jourani,** *Organizing Committee*
	Chairman : S. Adly
9h-9h30	**T. Rockafellar,** *Variational Convexity and Local Duality in Nonconvex Optimization*
9h30-10h	**M. Bounkhel,** *V-Proximal Analysis in Smooth Banach Spaces*
10h-10h30	**M. Lopez,** *How to avoid the normal cone in the subdifferential formulas?*
10h30-11h	**Coffee break**
	Chairman : D. Zagrodny
11h-11h30	**M. Fabian,** *Clarke Jacobians, Bouligand Jacobians, and compact connected sets of matrices*
11h30-12h	**M. A. Goberna,** *Relaxed Lagrangian duality in convex infinite optimization*
12h-12h30	**P. Combettes,** *Proximal compositions and cocompositions*
12h30-14h	**Lunch**
	Chairman : A. Hantoute
14h-14h30	**I. Ekeland,** *TBA*
14h30-15h	**T. Bayen,** *The hybrid maximum principle for optimal control problems governed by spatially heterogeneous control systems*
15h-15h30	**M.Studniarski,** *Evolutionary Multitasking in Multiobjective Optimization*
15h30-16h	**Coffee break**
	Chairman : R. Henrion
16h-16h30	**C. Zalinescu,** *On Gâteaux differentiability of convex locally Lipschitz functions defined on locally convex spaces*
16h30-17h	**A. Zaffaroni,** *Minimal sublinear functions, recession hull, and appications to Cut Generating Functions*
17h-17h30	**D. Salas,** *Determination of functions in complete metric spaces*
17h30-18h	**J. Ugon,** *Variational properties of the abstract subdifferential operator*

18h15-20h : Function at Salle du Conseil of UFR Sciences et Techniques

Conference program

Venue : Amphi Bernard, UFR Sciences et Techniques

Tuesday 27/06/2023

	Chairman : R. Henrion
8h30-9h	**A. D. Ioffe,** *On the strong minimum in the classical problem of calculus of variations*
9h-9h30	**E.M. Bednarczuk,** *Inexact Forward-Backward algorithm for problems involving weakly convex functions*
9h30-10h	**C. Gutiérrez,** *On quasi efficient solutions of multiobjective optimization problems*
10h-10h30	**J. Outrata,** *On the isolated calmness property of inverse and implicit multifunctions*
10h30-11h	**Coffee break**
	Chairman : A. Hantoute
11h-11h30	**A. Kruger,** *Fuzzy multiplier rules in non-Lipschitzian settings : Decoupling approach revisited*
11h30-12h	**R. Henrion,** *Optimality conditions for a PDE-constrained control problem with probabilistic and almost-sure state constraints*
12h-12h30	**A. Calogero,** *On the engulfing property for convex functions*
12h30-14h	**Lunch**
	Chairman : D. Zagrodny
14h-14h30	**G. Colombo,** *On the optimal control of Moreau's sweeping process*
14h30-15h	**M. Quincampoix,** *On Metric Regularity of some Variational Inequalities Applications to Optimal Control*
15h-15h30	**J. E. Martinez-Legaz,** *Feasibility problems via paramonotone operators in a convex setting*
15h30-16h30	**Posters & Coffee break**
	Chairman : S. Adly
16h30-17h	**H. Ramirez,** *Extensions of Constant Rank Qualification Constrains condition to Nonlinear Conic Programming*
17h-17h30	**M. Sene,** *Geometric characterizations of regularity of sets by tangent cones*
17h30-18h	**F. Nacry,** *Selected works in Variational Analysis*

8:00 p.m. : Gala dinner at Restaurant La Closerie, 18 rue Sainte Anne 21000 DIJON

Conference program

Venue : Amphi Bernard, UFR Sciences et Techniques

Wednesday 28/06/2023

	Chairman : S. Adly
8h30-9h	**B. Mordukhovich,** *Local Monotonicity and Variational Convexity in Variational Analysis*
9h-9h30	**N. Zlateva,** *Epigraphical characterization of uniformly lower regular functions in Hilbert spaces*
9h30-10h	**A. Hantoute,** *How much is the variational value function far from being convex?*
10h-10h30	**M. Thera,** *Nonlinear weak sharp minimum and the stability of a local minimum on metric spaces*
10h30-11h	**Coffe break**
	Chairman : R. Henrion
11h-11h30	**J. P. Penot,** *Semiconvex regularization*
11h30-12h	**A. Daniilidis,** *Determination of functions by Metric Slopes*
12h-12h30	**D. Zagrodny,** *Determining functions by slopes*
12h30-14h	**Lunch**
	Chairman : A. Hantoute
14h-14h30	**F. Flores-Bazan,** *Characterizing simultaneous diagonalization for two matrices*
14h30-15h	**M. De Lara,** *Hidden Convexity in the ℓ_0 Pseudonorm*
15h15h30	**N. Sukhorukova,** *Deep learning and its mathematical nature*
15h30-16h	**P. Perez-Aros,** *Some connexions between subdifferential and weak compactness of sublevels*
16h-16h30	**Coffe break**
	Chairman : A. Jourani
16h30-17	**B. Hernandez-Jimenez,** *gH-differentiability for interval-valued functions: applications*
17h-17h30	**C. Pintea,** *Various types of closed convex sets*
17h30-18h	**R. Diaz-Milan,** *Minimal faces of convex sets in general vector spaces*
18h	**Closing conference**

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ACCELERATION OF FIRST-ORDER OPTIMIZATION ALGORITHMS VIA DAMPED INERTIAL DYNAMICS

HEDY ATTOUCH

We report on recent progress in accelerating first-order algorithms for convex continuous optimization in a general Hilbert space framework. We rely on damped inertial dynamics driven by the gradient of the function to be minimized, and on algorithms obtained by temporal discretization. The fast optimization properties come from the design of the damping term.

We focus on first-order methods, which play a central role in solving large-scale optimization problems in machine learning, data science, and image processing. They have regained popularity as datasets and problems continue to increase in size. We review classical results, from Polyak's heavy ball with friction method to Nesterov's accelerated gradient method.

We analyze the subtle tuning of the vanishing viscous damping coefficient, α/t, in the Su-Boyd-Candès dynamic approach of the Nesterov method. We complete this portrait of first-order accelerated gradient methods by considering the Ravine method of Gelfand and Tsetlin, which shares convergence properties very similar to Nesterov's method and has often been confused with the latter.

Then, we show that the high-resolution ODE of the Nesterov and Ravine methods reveals the geometric damping driven by the Hessian of the function to be minimized. Indeed, the introduction of geometric damping controlled by the Hessian brings significant progress in the performance of the algorithms. This gives rise to new first-order methods that significantly dampen the oscillations that occur naturally with inertial systems.

We then present a new acceleration method based on time scaling and averaging of the continuous steepest descent method. Finally, we present some recent first results concerning the extension of the above results to the stochastic framework and the stochastic differential equation approach to solving convex optimization problems with a noisy gradient input.

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THE HYBRID MAXIMUM PRINCIPLE FOR OPTIMAL CONTROL PROBLEMS GOVERNED BY SPATIALLY HETEROGENEOUS CONTROL SYSTEMS

TÉRENCE BAYEN, ANAS BOUALI AND LOIC BOURDIN

We consider a general control system that is described by a differential equation involving a spatially heterogeneous dynamics. We suppose that the sequence of dynamics followed by the trajectory and the corresponding switching times (*i.e.*, the instants at which the control system moves from one dynamics to another) are fully constrained by the state position. The objective is to provide first order necessary optimality conditions often referred to as *Hybrid Maximum Principle* (HMP in short). In [3], a HMP was derived thanks to an excellent strategy that relies on augmentation techniques. We emphasize that in [3], the switching times are determined only by the time variable (and not by the state position) and that our setting differs from the one considered in [3] or in [2]. We prove in [1] with an explicit counterexample that the augmentation techniques cannot be fully applied to our setting, but we show that it can be adapted by introducing a new notion of local solution to classical optimal control problems and by establishing a corresponding Pontryagin maximum principle. Thanks to this method we derive a HMP adapted to our setting, with a simple proof that does not require any technical tool (such as implicit function arguments) to handle the dynamical discontinuities w.r.t. the state. This presentation is based on the submitted paper [1] in collaboration with A. Bouali and L. Bourdin.

References

[1] T. Bayen, A. Bouali, L. Bourdin, *The hybrid maximum principle for optimal control problems with spatially heterogeneous dynamics is a consequence of a Pontryagin maximum principle for* $\mathrm{L}^1_\square$ *local solutions*, preprint `hal-03985420v1`, submitted, 2023.

[2] F.H. Clarke, *Functional Analysis, Calculus of Variation, Optimal control*, Graduate Texts in Mathematics, 264, Springer, London, 2013.

[3] A. Dmitruk and A. Kaganovich, *The hybrid maximum principle is a consequence of the Pontryagin maximum principle*, Systems Control Lett., 57, pp. 964–970, 2008.

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INEXACT FORWARD-BACKWARD ALGORITHM FOR PROBLEMS INVOLVING WEAKLY CONVEX FUNCTIONS

EWA M.BEDNARCZUK

Given a Hilbert space $\mathcal{X}$, we consider a problem of the form

$$\text{minimize}_{x\in\mathcal{X}}\, f(x) + g(x) \tag{1}$$

where the function $f\colon \mathcal{X} \to (-\infty, +\infty]$ is weakly convex with modulus $\rho \geq 0$ (ρ-weakly convex) and function $g\colon \mathcal{X} \to (-\infty, +\infty]$ is convex and Fréchet differentiable with a L_g-Lipschitz continuous (Fréchet) gradient ∇g.

The main scope of this talk is to analyse the convergence properties of the inexact Forward-Backward (FB) splitting when applied to problem (1), with stepsize $\alpha_t >$ and accuracy level $\varepsilon_t \geq 0$

$$x_{t+1} \in \varepsilon_t - \text{prox}_{\alpha_t f}(x_t - \alpha_t \nabla g(x_t)). \tag{2}$$

In finite dimensions, there exist algorithms which are devoted to solving problems that involve weakly convex functions. These algorithms are based on an explicit (sub)gradient update step for the weakly convex function. In particular, in the authors discuss the local convergence rate of standard (projected) subgradient methods for the minimisation of a weakly-convex function h over a constraint set C which satisfy the following sharpness condition: there exists $\mu > 0$ such that

$$h(y) - \min_{x\in C} h(x) \geq \mu\, \text{dist}(z, S) \qquad (\forall y \in C) \tag{3}$$

where S denotes the set of minimisers of h.

In our approach, we exploit a sharpness requirement that is analogous to (3) which we adapt to our framework, to investigate the strong convergence of the scheme (2). We take into account a possibly inexact proximal step for the weakly convex function. The exact proximal step is also considered to infer a complexity bound for the exact forward backward algorithm. We use the concept of ε-proximal operator. Its properties are expressed in terms of global proximal ε-subdifferentials. The concept of inexact proximal operator and its computational tractability lie at the core of many proximal methods referring to fully convex case, where both f and g are convex.

The talk is based on joint works with Giovanni Bruccola, Gabriele Scrivanti and The Hung Tran.

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V-PROXIMAL ANALYSIS IN SMOOTH BANACH SPACES

MESSAOUD BOUNKHEL

In this talk, I will present various results obtained recently on V-proximal subdifferential, V-proximal normal cone, and some important results extended from Hilbert spaces to smooth Banach spaces and from convex cases to nonconvex cases. The concept of V-prox-regularity will be also defined and many existing results will be extended from Hilbert spaces to smooth Banach spaces. The applications to nonconvex variational inequalities and to sweeping processes in Banach spaces will be stated.

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ON THE ENGULFING PROPERTY FOR CONVEX FUNCTIONS

ANDREA CALOGERO

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $\partial\varphi$ denote its normal mapping (or subdifferential). The Monge–Ampère measure μ_φ associated to φ is defined on any Borel set E by $\mu_\varphi(E) = |\partial\varphi(E)|$, where $|\cdot|$ stands for Lebesgue measure. The study of the properties of μ_φ is strictly related to a property of φ called *engulfing*. In this talk we introduce the sections and the related engulfing notion for convex functions on Euclidean spaces, together with its main properties. In particular, making use only of results peculiar of convex analysis, we show that differentiability and strict convexity are conditions intrinsic to the engulfing property. The notions of sections and engulfing can be extended in a suitable way to convex functions on the Heisenberg group $\mathbb{H}$. These sections, that arise as suitable unions of horizontal sections, will lead to the definition of a quasi-distance in $\mathbb{H}$. These results are in some publications in collaboration with R. Pini.

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ON THE OPTIMAL CONTROL OF MOREAU'S SWEEPING PROCESS

GIOVANNI COLOMBO

Moreau's sweeping process with a drift f is the evolution inclusion

$$\dot{x}(t) \in -N_{C(t)}(x(t)) + f(x(t), u(t)) \quad \text{a.e.,} \tag{EI}$$

where $C(t)$ is a moving closed set, with external normal cone $N_{C(t)}(\cdot)$, f is Lipschitz and u is a control. The above problem together with the initial condition $x(0) = x_0 \in C(0)$ admits a unique forward in time solution, provided the sets $C(t)$ satisfy a regularity condition called uniform prox-regularity. This condition generalizes both convexity and $\mathcal{C}^{1,1}$-regularity of the boundary of $C(t)$. The dynamics (EI) can be seen as a constrained evolution, where the constraint is active in it.

The optimal control of (EI) provides significant challenges, as its right hand side severely lacks Lipschitz regularity with respect to the state variable x. I will present the state of the art on necessary optimality conditions, that are of the type of Pontryagin Maximum Principle, together with some toy examples. The main available approaches rely either on discretization or on different types of penalizations, and are due to a number of different authors, including Mordukhovich and collaborators, Hermosilla and Palladino, De Pinho, Ferreira and Smirnov, Zeidan and collaborators, and myself with collaborators.

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PROXIMAL COMPOSITIONS AND COCOMPOSITIONS

PATRICK L. COMBETTES

We introduce the resolvent composition, a monotonicity-preserving operation between a linear operator and a set-valued operator, as well as the proximal composition, a convexity-preserving operation between a linear operator and a function. Dual cocompositions notions are also presented. A large core of properties of these compositions is established. Applications to the relaxation of monotone inclusion and convex optimization problems are presented.

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DETERMINATION OF FUNCTIONS BY METRIC SLOPES

ARIS DANIILIDIS

Two smooth, convex and bounded from below functions in a Hilbert space are equal up to a constant if and only if their derivatives have the same norm everywhere. We shall give an analogous determination property for the class of continuous, coercive functions in compact metric spaces using the notion of metric slope and discuss extensions in a more general case.

Talk based on several works in collaboration with:
T. M. Le (TU Wien, Austria), L. Miclo (TSE, France) and D. Salas (UOH, Chile).

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HIDDEN CONVEXITY IN THE ℓ_0 PSEUDONORM

MICHEL DE LARA

The so-called ℓ_0 pseudonorm counts the number of nonzero components of a vector. In this talk, we review a series of recent results on a class of Capra (Constant Along Primal Rays) conjugacies (and polarities) that reveal hidden convexity in the ℓ_0 pseudonorm.

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MINIMAL FACES OF CONVEX SETS IN GENERAL VECTOR SPACES

REINIER DIAZ MILLAN

We study relations between the minimal faces of convex sets and the well know relative interiors in general vector spaces. We propose a new notion of relative interior that can be seen as an alternative to the quasi-relative interior. We call this new notion face-relative interior, as the definition stems from the facial structure of convex sets.

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CLARKE JACOBIANS, BOULIGAND JACOBIANS, AND COMPACT CONNECTED SETS OF MATRICES

MARIAN FABIAN

We show that every compact convex set of m x n matrices can be understood as Clarke Jacobian of a suitable Lipschitz mapping. This can be extended to compact CONNECTED sets and to Bouligand Jacobians. We will use this for discussing the possibility of finding Lipschitz right inverses of Lipschitz mappings. The lecture is based on a recent joint paper by D. Bartl, me, and J. Kolar.

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CHARACTERIZING SIMULTANEOUS DIAGONALIZATION FOR TWO MATRICES

FABIÁN FLORES-BAZÁN

Given two real symmetric matrices, we analize when they are diagonalizable simultaneously (SD via congruence). Firstly, we present some sufficient conditions for SD, all of them equivalent each other. These conditions are derived under a different perspective from those appearing in the literature. In addition, following our reasoning started in [1], we establish sufficient and necessary conditions for SD, whose nature differs from those provided in [3]. It is expected to propose an scheme for determining SD. The two-dimensional case deserves an special attention, where the characterizations are much more precise than those in higher dimension.

This is a joint work with Felipe Opazo (Chile).

References

[1] Flores-Bazán, F.; Opazo, F., Characterizing the convexity of joint-range for a pair of inhomogeneous quadratic functions and strong duality. *Minimax Theory Appl*, **1** (2016), 257–290.

[2] Flores-Bazán, F.; Opazo, F., Characterizing convexity of images for quadratic-linear mappings with applications in nonconvex quadratic optimization, *SIAM J. Optim.* **31** (2021), 1774–1796.

[3] Jiang, Q.; Li, D., Simultaneous diagonalization of matrices and its applications in quadratically constrained quadratic programming, *SIAM J. Optim*, **26** (2016), 1649–1668.

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ON SCALARIZATIONS FOR PROPER AND APPROXIMATE PROPER EFFICIENT POINTS

FERNANDO GARCÍA CASTAÑO, MIGUEL ANGEL MELGUIZO-PADIAL AND G. PARZANESE

In this talk, we will show that under a separation property, a $\mathcal{Q}$-minimal point in a normed space is the minimum of a given sublinear function. This fact provides sufficient conditions, via scalarization, for nine types of proper efficient points; establishing a characterization in the particular case of Benson proper efficient points. We will see also some necessary and sufficient conditions in terms of scalarization for approximate Benson and Henig proper efficient points. The separation property we handle is a variation of another property introduced in [1] and our scalarization results do not require convexity or boundedness assumptions.

References

[1] Kasimbeyli, R. A Nonlinear Cone Separation Theorem and Scalarization in Nonconvex Vector Optimization. SIAM Journal on Optimization, 20(3), 1591-1619. https://doi.org/10.1137/070694089

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RELAXED LAGRANGIAN DUALITY IN CONVEX INFINITE OPTIMIZATION

MIGUEL A. GOBERNA

We associate with each convex optimization problem P, posed on some locally convex space, and a given non-empty family H of finite subsets of its index set, a suitable Lagrangian-Haar dual problem. We present results on the equivalence of P to some subproblem obtained by replacing its whole index set by some element of H, zero duality theorems, strong and reverse strong theorems, and optimality conditions for this new type of duality.

The talk is based on the following two papers:

N. Dinh, M.A. Goberna, M.A. López, M. Volle, Relaxed Lagrangian duality in convex infinite optimization: reverse strong duality and optimality, Journal of Applied Numerical Optimization 4, 3-18, 2022.

N. Dinh, M.A. Goberna, M.A. López, M. Volle, Relaxed Lagrangian duality and convex infinite optimization duality: reducibility and strong duality, Optimization 72, 189-214, 2023.

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ON QUASI EFFICIENT SOLUTIONS OF MULTIOBJECTIVE OPTIMIZATION PROBLEMS

CÉSAR GUTIÉRREZ

To the best of our knowledge, the concept of quasi solution of an optimization problem was introduced by Pierre Loridan [1] in 1982. These solutions, whose existence can be guaranteed by the Ekeland variational principle, are important as they can be identified as approximate stationary points. In other words, they are in the core of many numerical algorithms to solve optimization problems.

The aim of this talk is to introduce several concepts of quasi nondominated (efficient) solutions of multiobjective optimization problems and to show their basic properties as approximations of the well-known set of efficient solutions. In addition, some characterizations in convex problems by solutions of scalar optimization problems are stated. The obtained results are derived by basic variational mathematical tools and clarify and generalize many others of the literature.

References

[1] P. Loridan: *Necessary conditions for ε-optimality*, Math. Programming Stud., 19, 140–152, 1982.

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HOW MUCH IS THE VARIATIONAL VALUE FUNCTION FAR FROM BEING CONVEX?

ABDERRAHIM HANTOUTE

We consider the variational value function

$$\psi(a) := \inf \left\{ \int_T f_0(t, x(t)) d\mu : x(t) \in \mathcal{D}, \int_T g_0(t, x(t)) d\mu \in a - C \right\},$$

where $f_0 : T \times X \to \overline{\mathbb{R}}$ and $g_0 : T \times X \to Y$, $\mathcal{D} \subset X$ is decomposable, $C \subset Y$ is a cone, X, Y are Banach spaces, and μ is a complete, non-negative and non-atomic measure. The function ψ is always convex if Y is finite-dimensional, a consequence of the convexity Lyapunov theorem. This is not always the case in general because non-convex variational value functions already exist in $l^2(\mathbb{N})$. However, in the infinite-dimensional framework, we can find a convex function with the following properties (i) it is also a variational value function, which is intrinsically related to the original value function ψ (ii) it has the same closure as ψ (with appropriate topologies) (iii) it coincides with ψ in the finite-dimensional setting (iv) it coincides with ψ on the effective domain of the last one. Such a function, called representative, makes it possible to solve by indirect methods the associated variational problem $\psi(0)$ and to study its stability and/or duality. The results presented in this talk come from a joint research project with F. Flores.

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OPTIMALITY CONDITIONS FOR A PDE-CONSTRAINED CONTROL PROBLEM WITH PROBABILISTIC AND ALMOST-SURE STATE CONSTRAINTS

CAROLINE GEIERSBACH AND RENÉ HENRION

Probabilistic constraints, originally introduced in the framework of operations research, have recently attracted a quickly growing interest in risk averse PDE-constrained optimization under uncertainty. The talk presents optimality conditions for a control problem related with a simple linear elliptical PDE with random source term and involving a uniform state constraint over the given domain. The latter are supposed to hold true with a given minimum probability. The obtained necessary and sufficient optimality conditions are formulated with the help of a spherical integral which results from the spherical-radial decomposition of Gaussian (more generally: elliptically distributed) random vectors. In the extreme case of choosing a probability level equal to one, the probabilistic constraint turns into an almost-sure constraint. The resulting description, however, is degenerate and the derivation of optimality conditions requires a different approach. The relation with robust constraints (with respect to the support of the random vector) is discussed in the context of the given PDE.

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ABOUT DISCRETE SET-VALUED DYNAMICAL SYSTEMS

ELVIRA HERNÁNDEZ AND JUAN PERÁN

The aim of our poster is to illustrate discrete dynamical systems defined by set-valued maps. We present a general framework and consider some asymptotic properties which are motivated by the classical theory in the one-dimensional case. For these purposes we consider basic results on topological and dynamical properties and present a set-valued dynamical system which generalizes other existing in the literature. Finally, invariant sets, ω-limit sets and a stability notion in the Lyapunov sense are studied.

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GH-DIFFERENTIABILITY FOR INTERVAL-VALUED FUNCTIONS: APPLICATIONS

BEATRIZ HERNÁNDEZ-JIMÉNEZ, RAFAELA OSUNA-GÓMEZ, YURILEV CHALCO-CANO AND TIAGO M. DA COSTA

Differential Calculus is a branch of Mathematics that allows us to solve problems where the change of variables can be modeled in a numerical continuum to determine, from it, the variation of these elements in specific moment or interval. The Optimization Theory is a basic part of Applied Mathematics, and the development of differential calculus has enabled powerful mathematical tools for this area.

The Differential Calculus has provided essential mathematical tools to areas as physics, biology, engineering, economics, among others. In particular, since Fermat and Lagrange's work, Differential Calculus has played a leading role in the Optimization Theory. In order to optimize a differentiable function or to solve an optimization problem with constraints, derivative is crucial in both situations and numerical algorithms for computing approximately optimal solutions because the main iterative optimization methods are based on the evaluation of hessian matrices or gradients.

Under the hypothesis that observations and estimates in the real world are incomplete to accurately represent the actual data, the Interval Analysis was introduced by Moore with the aim of managing the imprecision or lack of accurate information that appears on many mathematical models or computational of some real-world deterministic phenomena.

Moreover, interval differentiability and its application in fuzzy environment is an active research area as you can see in literature. But it has not been developed without problems, in order to define correctly the operations between intervals and to establish the appropriate differentiability concept due to the no linearity of the space of intervals. Therefore, it is of interest to establish the definitions and equivalences correctly, such that they allow a successful development of the theory and applications based on them.

So, in this talk we present necessary and sufficient conditions for generalized Hukuhara differentiability of interval-valued functions and counterexamples of some equivalences previously presented in the literature, for which important results are based on. Moreover, we present some applications and futures line of research.

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ON THE STRONG MINIMUM IN THE CLASSICAL PROBLEM OF CALCULUS OF VARIATIONS

ALEXANDER IOFFE

The main result to be presented at the talk contains a second order necessary condition for a strong minimum in the standard problem of calculus of variations. Novelty of this result is emphasized by the fact that no idea of a possibility of such a condition has ever appeared in the classical theory. Simple example show that the theorem can work when other known necessary conditions fail.

Originally the theorem was obtained as a consequence of a corresponding result for a sufficiently general class of optimal control problems (Calculus of Variations and PDEs (2020), **59**:83). But an independent proof of the theorem is noticeably simpler and will be described in some details.

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FUZZY MULTIPLIER RULES IN NON-LIPSCHITZIAN SETTINGS DECOUPLING APPROACH REVISITED

ALEXANDER Y. KRUGER

We are going to revisit the *decoupling approach* to optimality conditions and subdifferential calculus developed and discussed in [1, 2, 3, 4, 5] as well as some recent developments. Given extended-real-value functions f_1 and f_2 on a metric space and a subset U, properties of the type

$$\inf_U (f_1 + f_2) \le \liminf_{\substack{d(x_1,x_2)\to 0\\ x_1,x_2\in U}} (f_1(x_1) + f_2(x_2)) \tag{1}$$

are of major importance in many areas of analysis and appear (often implicitly) in many publications. The quantity in the right-hand side of the above inequality is known as *uniform infimum*, while the property itself is often referred to as *uniform lower semicontinuity*. The talk is about some extensions of such properties and their consequences.

In particular, we are going to show that in many cases it suffices to consider the weaker than (1) *quasiuniform lower semicontinuity* property:

$$\inf_U (f_1 + f_2) \le \sup_{V\in EI(U)} \liminf_{\substack{d(x_1,x_2)\to 0\\ x_1,x_2\in V}} (f_1(x_1) + f_2(x_2))$$

where $EI(U)$ denotes the collection of *essentially interior* subsets of U: $V \in EI(U)$ if and only if $B_\rho(V) \subset U$ for some $\rho > 0$.

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Joint work with Marián Fabian and Patrick Mehlitz.

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PROXIMAL POINT TYPE ALGORITHMS FOR NONCONVEX PSEUDOMONOTONE EQUILIBRIUM PROBLEMS

FELIPE LARA

We present a brief resume of 4 papers regarding proximal point type algorithms for pseudomonotone equilibrium problems when the second argument of the bifunction is strongly quasiconvex in the sense of Poljak [5]. The proximal point type algorithms that we present are the classical version of the proximal point algorithm, its relaxed-inertial version and both extragradient versions (the 2-step and proximal projection versions), all of them for pseudomonotone equilibrium problems. Furthermore, a new class of strongly quasiconvex functions which are not convex is provided and sufficient conditions for quadratic fractional functions to be strongly quasiconvex are given. Finally, we provide applications in mixed variational inequalities.

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HOW TO AVOID THE NORMAL CONE IN THE SUBDIFFERENTIAL FORMULAS?

MARCO A. LÓPEZ

We start by providing alternative characterizations of the normal cone to the effective domain of the supremum of an arbitrary family of convex functions. These results are then applied to give new formulas for the subdifferential of the supremum function, which use both the active and nonactive functions at the reference point. In contrast with previous works, the main feature of our subdifferential characterization is that the normal cone to the effective domain of the supremum (or to finite-dimensional sections of this domain) does not appear. The talk also includes a new type of optimality conditions for convex optimization. The results presented in this talk were established in a joint research project with R. Correa and A. Hantoute.

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FEASIBILITY PROBLEMS VIA PARAMONOTONE OPERATORS IN A CONVEX SETTING

JUAN ENRIQUE MARTÍNEZ-LEGAZ

This joint paper with J. Camacho, M. J. Cánovas and J. Parra is focused on some properties of paramonotone operators on Banach spaces and their application to certain feasibility problems for convex sets in a Hilbert space and convex systems in the Euclidean space. In particular, it shows that operators that are simultaneously paramonotone and bimonotone are constant on their domains, and this fact is applied to tackle two particular situations. The first one, closely related to simultaneous projections, deals with a finite amount of convex sets with an empty intersection and tackles the problem of finding the smallest perturbations (in the sense of translations) of these sets to reach a nonempty intersection. The second is focused on the distance to feasibility; specifically, given an inconsistent convex inequality system, our goal is to compute/estimate the smallest right-hand side perturbations that reach feasibility. This work derives lower and upper estimates of such a distance, which become the exact value when confined to linear systems.

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LOCAL MONOTONICITY AND VARIATIONAL CONVEXITY IN VARIATIONAL ANALYSIS

BORIS MORDUKHOVICH

This talk addresses two interrelated themes of advanced variational analysis and nonsmooth optimization. The first theme revolves around local maximal monotonicity of set-valued mappings while the second one concerns the recently introduced notions of variational convexity of extended-real-valued functions and its strong counterpart. Our consideration are given in general Banach spaces with further specifications in Hilbert and finite-dimensional settings. The obtained results include establishing close relationships between the notions under consideration and their strong versions. Various second-order characterization of these notions are established and applied to devised aspects of nonsmooth optimization. Based on join works with P. D. Khanh, V. V. H. Khoa and V. T. Phat.

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SELECTED WORKS

FLORENT NACRY

The talk will present various contributions of Professor Lionel Thibault to the field of Variational Analysis.

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ON THE ISOLATED CALMNESS PROPERTY OF INVERSE AND IMPLICIT MULTIFUNCTIONS

HELMUT GFRERER AND JIŘÍ V. OUTRATA

The contribution deals with the isolated calmness property around the given reference point which is a valuable Lipschitzian stability notion appearing in parameterized equilibria. Apart from a general characterization in terms of a new generalized derivative we obtain also some special workable criteria ensuring this property in a class of SCD (subspace containing derivative) and semismooth* mappings. Such mappings arise frequently in a number of equilibrium models including parameterized complementarity problems and variational inequalities of the first and second kind. In this development we employ a powerful theory of SCD mappings which proved its efficiency also outside the area of Lipschitzian stability.

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SOME CONNEXIONS BETWEEN SUBDIFFERENTIAL AND WEAK COMPACTNESS OF SUBLEVELS

PEDRO PÉREZ-AROS

In this talk, we present a review of some recent results that establish a link between the weak compactness of sublevel sets of a given function and the non-emptiness of the preimage of its subddiferential. Formally, we prove that if X is a complete locally convex space, and $f : X \to \mathbb{R} \cup +\infty$ is such that $(\partial f)^{-1}(x^*) \neq \emptyset$ for every $x^* \in U$, where U is an open set with respect to the Mackey topology in X^*, then the set $\{x \in X : f(x) - \langle x^*, x\rangle \leq \gamma\}$ is relatively weakly compact for every $\gamma \in \mathbb{R}$ and $x^* \in U$.

Moreover, we demonstrate that our results allow for direct derivation of a large number of recent results established in Banach spaces for the connection between the subdifferential and weak compactness of sublevel sets. Our arguments also extend these results to locally convex spaces. This talk is based on two joint works with Professor Lionel Thibault, as reported in [1, 2].

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VARIOUS TYPES OF CLOSED CONVEX SETS

CORNEL PINTEA

We consider and study various classes of closed convex sets, such as Motzkin decomposable and Minkowski sets. We either characterize such sets or provide sufficient conditions, mostly in terms of the Gauss range, for a closed convex set to be inside a certain considered class.

The content of the presentation is based on a few joint papers with Juan Enrique Martinez Legaz.

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EXTENSIONS OF CONSTANT RANK QUALIFICATION CONSTRAINS CONDITION TO NONLINEAR CONIC PROGRAMMING

HÉCTOR RAMÍREZ

We present new constraint qualification conditions for nonlinear conic programming that extend some of the constant rank-type conditions from nonlinear programming. Specifically, we propose a general and geometric approach, based on the study of the faces of the cone, for defining a new extension of this condition to the conic context. We then compare these new conditions with some of the existing ones, including the nondegeneracy condition, Robinson's constraint qualification, and the metric subregularity constraint qualification. The main advantage of the latter is that we are able to recast the strong second-order properties of the constant rank condition in a conic context. In particular, we obtain a second-order necessary optimality condition that is stronger than the classical one obtained under Robinson's constraint qualification, in the sense that it holds for every Lagrange multiplier, even though our condition is independent of Robinson's condition.

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VARIATIONAL CONVEXITY AND LOCAL DUALITY IN NONCONVEX OPTIMIZATION

TERRY ROCKAFELLAR

An extended-real-valued function is variationally convex relative to a pair in the graph of its subgradient mapping if, in a primal-dual local sense, it is impossible to distinguish the properties of that mapping and its associated function values from those coming from a convex function. This is closely tied to the subgradient mapping being maximal monotone locally.

Although variational convexity may at first seem puzzling and obscure, it leads in the format of an optimization problem with perturbation parameters to a natural sufficient condition for local optimality that provides a far-reaching extension of the workhorse second-order conditions in nonlinear programming and other standard domains. It characterizes circumstances in which the local optimality can be described by a convex-concave saddle point of an augmented Lagrangian function. In that way it opens the way to understanding how computational methods in nonconvex optimization will behave locally as if articulated in convex optimization.

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DETERMINATION OF FUNCTIONS IN COMPLETE METRIC SPACES

ARIS DANIILIDIS, TRI MINH LE AND DAVID SALAS

Recently, Thibault and Zagrodny [4] show that whenever two bounded from below continuous functions over a complete metric space have a common infimum and their global slope coincide everywhere, the functions must be equal. This is a very impressive result, since it doesn't require compactness of the space [1, 2], nor convexity of the function [3]. In this talk, we will extend the aforementioned result to a larger family of slope-like operators, called metrically compatible descent moduli. We show that, by controlling the critical points and controlling divergent sequences that fit a notion of criticality, we can determine functions in complete metric spaces. We derive Thibault-Zagrodny result as corollary, since we show that the global slope gives you the required control for free.

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GEOMETRIC CHARACTERIZATIONS OF REGULARITY OF SETS BY TANGENT CONES

MOUSTAPHA SENE

This work is concerned by geometrical characterizations of regularity of some type of sets through tangent cones. Two category of sets are considered. In the first categories we prove that for subanalytic sets, the Clarke regularity corresponds to the representation of its Clarke tangent cone by right derivatives at the origin of some Lipschitz mapping. In the second case, after proving some additional properties for epi-Lipschitzian sets, we introduce the notion of strictly Hadamard differentiable set. Then, we prove that an epi-Lipschitzian set S is strictly Hadamard differentiable at $\bar{x} \in S$ iff its Clarke tangent cone at $\bar{x}$ contains a closed hyperplane. Many consequences of the results are discussed in the talk.

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EVOLUTIONARY MULTITASKING IN MULTIOBJECTIVE OPTIMIZATION

MARCIN STUDNIARSKI, AISHA YOUNUS AND RADHWAN AL-JAWADI

In [1], for the purpose of resolving Constrained Multi-objective Optimization Problems (CMOPs), an Evolutionary Multitasking-based Constrained Multi-objective Optimization (EMCMO) framework is created. In EMCMO, a CMOP optimization is split into two tasks that are related to one another: one task is for the original CMOP, while the other task just considers the objectives while ignoring all constraints. The major goal of the second task is to help solve CMOP by providing reasonable knowledge about the first task's goals.

In this paper, we apply another idea which is similar to the one described above. Consider a multi-objective problem with two objectives, for example, one of the test problems listed in Table 1 of [2]. We apply the evolutionary multitasking method to this problem, where one task is to solve the original multiobjective problem, and the other two tasks are for minimizing two objective functions separately. In this way, we obtain three different solutions: one for the multiobjective problem, and the other two are the minimum points of the two objective functions, respectively. Of course, the same can be done for more than two objectives.

Then we compare the results of the multitasking method with the results obtained by applying some multiobjective optimization algorithms (for example, NSGA-II [2] but also some other recently introduced algorithms) to check if multitasking really helps to solve the multiobjective problem. We present preliminary numerical results which are promising.

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DEEP LEARNING AND ITS MATHEMATICAL NATURE

NADEZDA SUKHORUKOVA

Deep learning is one of the key tools in the modern area of Artificial Intelligence due to many practical applications, where deep learning demonstrated its efficiency, including data analysis and signal and image processing and many others. The origin of deep learning is mathematical in its nature. Essentially, the objective of deep learning is to solve an approximation problem: optimise the weights (parameters) of the network. These weights can be considered as decision variables of certain optimisation problems, whose objective functions represent inaccuracy. Therefore, it is natural to approach this problem can be treated using modern optimisation.

Solid mathematical background of deep learning relies on the results of the celebrated Kolmogorov–Arnold representation theorem which states that every multivariate continuous function can be represented as a composition of continuous univariate functions. This theorem, however, does not provide an algorithm for constructing these composition functions. Essentially, modern deep learning techniques approximate this composition function by a composition of affine transformations and the so-called activation functions.

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NONLINEAR WEAK SHARP MINIMUM AND THE STABILITY OF A LOCAL MINIMUM ON METRIC SPACES

MICHEL THÉRA

This presentation focuses on the study of nonlinear (weak) sharp minimizers with respect to a gauge function. This concept generalizes the well known tilt stability and plays an important role in both theoretical and numerical aspects of optimization. In the last part of the talk we will be going to talk about stability (in some appropriate sense) of local/global minimizers of an objective function perturbed by a function belonging to a suitable class of Lipschitz functions defined on metric spaces. It is a joint work with Huynh van Ngai.

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VARIATIONAL PROPERTIES OF THE ABSTRACT SUBDIFFERENTIAL OPERATOR

R. DÍAZ MILLÁN, N. SUKHORUKOVA AND J. UGON

convexity generalises classical convexity by considering the suprema of functions taken from an arbitrarily defined set of functions. These are called the abstract linear (abstract affine) functions. The purpose of this paper is to study the abstract subdifferential. We obtain a number of results on the calculus of this subdifferential: summation and composition rules, and prove that under some reasonable conditions the subdifferential is a maximal abstract monotone operator. Another contribution of this paper is a counterexample that demonstrates that the separation theorem between two abstract convex sets is generally not true. The lack of the extension of separation results to the case of abstract convexity is one of the obstacles in the development of abstract convexity based numerical methods.

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MINIMAL SUBLINEAR FUNCTIONS, RECESSION HULL, AND APPLICATIONS TO CUT GENERATING FUNCTIONS

ALBERTO ZAFFARONI

We say that a sublinear function $f : \mathbb{R}^n \to \mathbb{R}$ is recession-minimal if it is minimal among those for which the 1-lower level set $L^+ = \{x \in \mathbb{R}^n : f(x) \leq 1\}$ has a prescribed recession cone. The problem of characterizing such notion of minimality was posed in a paper [2] by Conforti, Cornuéjols, Daniilidis, Lemaréchal, and Malick. They provide partial results based on previous research by Basu, Cornuéjols and Zambelli [1] and by Zaffaroni [3], who study minimality of sublinear functions which share the same 1-lower lever set L^+

The sets L^+ and $L^- = [f \leq -1]$ can be used to identify any lower semicontinuous positively homogeneous function. We build on known results to provide a novel characterization of sublinear functions using the sets L^+ and L^-, together with $L = [f \leq 0]$, which gives also the recession cone of both L^+ and L^-, provided L^- is nonempty. And we exploit the latter result to cheracterize recession minimality of f for a given L.

A major role is played here by a regularization of L^+ and L^- that we call recession hull. Given a convex set $V \subseteq \mathbb{R}^n$ with recession cone V_∞, its recession hull $h(V)$ is given by the intersection of all translates of the recession cone which contain V. That is

$$h(V) = \bigcap_{z \in o(V)} z + V_\infty,$$

where $o(V) = \{z \in \mathbb{R}^n : V \subseteq z + V_\infty\}$ is the set of bounding elements of V.

In a number of simple cases the recession hull is itself a translate of V_∞. More generally it is a superset of V which shows a sort of conic shape, thus regularizing V.

We first study its main features and characterizations in primal and dual terms. And then prove that f is recession minimal if and only if it is the least sublinear function representing L^+, and L^+ is regular, in the sense that it coincides with its recession hull.

As in [2], we apply previous results to Cut Generating Functions, a fundamental tool for the relaxation of Mixer Integer Linear Programs. We obtain sensible improvements in the description of minimal CGFs, and answer some questions posed in [2].

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DETERMINING FUNCTIONS BY SLOPES

DARIUSZ ZAGRODNY

During the talk results concerning characterization of convex functions by its slope, jointly obtained with L. Thibault, will be presented, see [4]. In order to achieve this we define the global slope at a given point in a metric space taking infimum over all tilts at this point. Next we propose a definition of the local slope, in the Banach space setting, at a given point for directionally differentiable functions by taking infimum over the unit sphere from directional derivatives at this point. These two notions are the same whenever the involved function is convex. Moreover, the Moreau envelope of a convex function is shown to have always smaller slope than the function, in the general Banach space setting. In the first result concerning slopes of two functions on a complete metric space, it is revealed that if a certain growth of one of them is smaller than the corresponding growth of the other one, then the slopes related to them repeat the relation at least at one point. This is enough to observe in the Banach space setting that if two bounded from below lower semicontinuous convex functions are such that the origin has the same distance to their subdifferentials at each point, then they must be equal up to an additive constant. Thus a result generalizing achievements of [2, Theorem 3.1 and Corollary 3.1] and [3, Theorem B] is obtained.

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ON GATEAUX DIFFERENTIABILITY OF CONVEX LOCALLY LIPSCHITZ FUNCTIONS DEFINED ON LOCALLY CONVEX SPACES

CONSTANTIN ZĂLINESCU

The generic Fréchet and/or Gateaux differentiability of a continuous convex function on an open convex subset of a Banach space is well studied and has important theoretical implications. An example of J. Rainwater (1988) shows that similar results are not true when the openness of the domain is not ensured, but such properties can be obtained if the respective convex function is assumed to be locally Lipschitz, as shown by M.E. Verona (1988), J. Rainwater (1988), D. Noll (1990) and others.

It is our aim to extend such results for convex functions defined on (not necessarily open) convex subsets of locally convex spaces. In the same framework we extend Gale's duality theorem (1967), as well as a result of Ergin & Sarver on the unique dual representation of a convex function (2010).

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EPIGRAPHICAL CHARACTERIZATION OF UNIFORMLY LOWER REGULAR FUNCTIONS IN HILBERT SPACES

MATEY KONSTANTINOV AND NADIA ZLATEVA

We provide a characterization of uniformly lower regular functions defined on a Hilbert space. To this end we introduce and study a property we call epi prox-regularity of an epigraph set which slightly differs from the well-known prox-regularity property of a set.

Key words: prox-regular set, uniformly prox-regular set, primal lower nice function, proximal normal, distance function, metric projection mapping, Hilbert space

AMS Subject Classification: 49J52, 49J53

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