Characterizations of convex approximate subdifferential calculus in Banach spaces*

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Abstract

We establish subdifferential calculus rules for the sum of convex functions defined on normed spaces. This is achieved by means of a condition relying on the continuity behaviour of the inf-convolution of their corresponding conjugates, with respect to any given topology intermediate between the norm and the weak* topologies on the dual space. Such a condition turns out to be also necessary in Banach spaces. These results extend both the classical formulas by Hiriart Urruty-Phelps [17] and by Thibault [27].

Key words. Convex functions, approximate subdifferential, calculus rules, approximate variational principle.

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1 Introduction

Given two lower semicontinuous (lsc) convex proper functions $f, g : X \to \mathbb{R} \cup \{+\infty\}$, defined on a normed space X, we consider two families $f_k, g_k : X \to \mathbb{R} \cup \{+\infty\}$ of (lsc convex proper) functions approximating them respectively, in a sense which will be made precise later on. Under a condition relying on the continuity behaviour of the inf-convolution of their corresponding Fenchel conjugates, it will be established that the approximate subdifferential of $f + g$ can be written as the τ-limit of the sum of the approximate subdifferentials of f_k and g_k at the reference point, where τ is any given topology intermediate between the weak* and the norm topology on the dual space. This extends the classical Hiriart Urruty-Phelps formula [17] (see, also, [16]). Moreover, when X is Banach, it is shown that the subdifferential of the sum is written as the τ-limit of the sum of the subdifferentials of f_k and g_k at nearby points. The same characterization holds for the approximate subdifferential. This generalizes some results by Thibault [27], established in reflexive spaces (see, also, Jourani [20]), and others by Attouch-Baillon-Théra [2] in Hilbert spaces. The condition just mentioned is naturally verified under

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any one of the well-known qualification conditions like those of Moreau-Rockafellar, Robinson, Attouch-Brézis and so on [29], which guarantee exact calculus rules. However, our condition is automatically satisfied in reflexive spaces, and also in other frameworks. For instance, it holds for the inf-convolution of the biconjugates at all points of the primal space. This leads to the characterization of the subgradients, of the biconjugates sum, living in the dual space.

The problem dealing with the subdifferential of the sum of convex functions can also be handled from the point of view of the variational convergence theory [1], [8]. This approach guarantees that the sum $f_k + g_k$ slice (Mosco in the reflexive setting) converges whenever both f_k and g_k converge in the same sense. These results also require conditions on the continuity of the inf-convolution of the conjugates together with some constraint qualifications in the spirit of Moreau-Rockafellar condition (see, e.g., [5], [22], [13], [21], [25], [30]). Consequently, using results on quantitative stability of the subdifferentials (see, e.g., [1], [8], [28]), it can be deduced that the subdifferential of $f_k + g_k$ (but not necessarily the sum of subdifferentials) converges in the sense of Painlevé-Kuratowski to the subdifferential of $f + g$. It is important to observe that the approach in this paper leads to characterizations of the approximate subdifferential of $f + g$ by means of those of each approximating function. This goes into the spirit of subdifferential calculus where one is interested in decoupling the subdifferential of the involved approximating functions f_k, g_k.

This paper is organized as follows. Section 2 is reserved to fix the main notation and present some preliminary results which shed light on the orientation of this work. Despite their simple appearance, Theorems B and C should be considered a starting point for other subdifferential formulas studied in the following sections.

In Section 3, the main result is given in Theorem 1, which provides two different characterizations of the approximate subdifferential mapping $\partial_\varepsilon(f + g)$, valid in normed spaces under a continuity assumption on $f^*\square g^*$ (the inf-convolution of the Fenchel conjugates). In Theorem 3 this condition is shown to be necessary when X is Banach.

In Section 4, it is shown that the inf-convolution of the biconjugates always verifies, for all points in X, the required continuity assumption of Theorem 1. Consequently, Theorem 7 gives formulas for characterizing the mapping $X^* \cap \partial_\varepsilon(f^{**} + g^{**})$ using approximate subdifferentials of the approximating sequences f_k and g_k, without appealing to the continuity condition mentioned above.

Finally, in section 5 we give the limiting formulas characterizing the approximate subdifferential of the sum $f + g$ in line with [27], where the approximate subdifferentials of the involved functions are evaluated at nearby points. For the sake of simplicity, we don't consider approximating sequences f_k and g_k in this section. The main tool toward this objective is an approximate version of the Ekeland's variational principle given in Lemma 13.

2 Preliminary results and notation

To motivate this work, we begin by considering the natural and useful role of the Moreau-Yosida envelopes within the theory of subdifferential calculus of convex functions. This will allow simple and nice proofs of well-known formulas in line with [17] (see, also, [15]).

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function defined on a real normed space $(X, \|\cdot\|)$, where proper means that $\operatorname{dom} f := \{x \in X \mid f(x) < \infty\}$ is nonempty. For $\lambda > 0$, we denote by f_λ the function

$$f_\lambda(x) := \inf_{y \in X} \left\{ f(y) + \frac{\|y - x\|^2}{2\lambda} \right\},$$

usually known as the Moreau-Yosida envelope of f [23].

The first result expresses the approximate subdifferential of f at a given point $x \in \operatorname{dom} f := \{x \in X \mid f(x) < \infty\}$, defined for $\varepsilon \geq 0$ by

$$\partial_\varepsilon f(x) := \{x^* \in X^* \mid \langle x^*, y - x\rangle \leq f(y) - f(x) + \varepsilon \ \ \forall y \in X\},$$

where X^* denotes the topological dual of X and $\langle \cdot, \cdot\rangle$ the duality product on $X^* \times X$. When $\varepsilon = 0$, we recover the (Fenchel) subdifferential $\partial f(x)$. Recall that $\partial_\varepsilon f(x) = \{x^* \in X^* \mid f(x) + f^*(x^*) \leq \langle x^*, x\rangle + \varepsilon\}$, where $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$ is the Fenchel conjugate of f given by

$$f^*(x^*) := \sup_{y \in X}\{\langle x^*, y\rangle - f(y)\}.$$

Theorem A: *For $f : X \to \mathbb{R} \cup \{+\infty\}$ proper, convex, and lsc, we have that*

$$\partial_\varepsilon f(x) = \bigcap_{\delta > \varepsilon} \bigcup_{\eta > 0} \bigcap_{\lambda \in (0,\eta)} \partial_\delta f_\lambda(x) \ \ \forall x \in X, \ \forall \varepsilon \geq 0. \tag{1}$$

Proof. Taking $x^* \in \partial_\varepsilon f(x)$, we get

$$f_\lambda^*(x^*) = f^*(x^*) + \lambda \left\|x^*\right\|^2 \leq \langle x^*, x\rangle - f(x) + \lambda \left\|x^*\right\|^2 + \varepsilon \ \ \forall \lambda > 0.$$

Then, for all $\delta > 0$ and $\lambda \in (0, \frac{\delta}{\|x^*\|^2})$ we have $f_\lambda^*(x^*) \leq \langle x^*, x\rangle - f(x) + \delta + \varepsilon \leq \langle x^*, x\rangle - f_\lambda(x) + \delta + \varepsilon$ and so $x^* \in \partial_{\varepsilon+\delta} f_\lambda(x)$. Then, $x^* \in \bigcap_{\lambda \in (0,\eta)} \partial_{\varepsilon+\delta} f_\lambda(x)$ for all $\delta > 0$ and η sufficiently small, showing that $x^* \in \bigcup_{\eta>0} \bigcap_{\lambda \in (0,\eta)} \partial_{\varepsilon+\delta} f_\lambda(x)$ for all $\delta > 0$. Because the converse inclusion is straightforward we obtain the desired formula. ■

Formula (1) provides a pure algebraic representation of the approximate subdifferential of the function f. Similar results hold for nonconvex functions in Asplund spaces [19], in terms of upper limits involving the sequential weak* topology. An interesting question is how to adapt formula (1) in order to express the approximate subdifferential of the sum of two convex functions in terms of approximate subdifferentials of their Moreau-Yosida envelopes. For this purpose, one needs to include a topological operation as we do in the following theorem.

Recall that $\overline{h}^\tau$ and $\operatorname{cl}^\tau(A)$ denote the τ-closures of a function $h : X^* \to \mathbb{R} \cup \{+\infty\}$ and a set $A \subset X^*$, respectively. When τ is the norm topology, we simply write $\overline{h}$ and $\operatorname{cl}(A)$. Moreover, if $\operatorname{co} h$ denotes the convex envelope of h, then $\overline{\operatorname{co}}^\tau h = \overline{\operatorname{co} h}^\tau$. We shall use w^* to denote the weak* topology on X^* (also denoted by $\sigma(X^*, X)$).

Theorem B: *For $f, g : X \to \mathbb{R} \cup \{+\infty\}$ convex and lsc functions such that $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$, we have that*

$$(f + g)^* = \overline{\inf_{\lambda > 0} (f_\lambda + g_\lambda)^*}^{w^*} \tag{2}$$

and, consequently,

$$\partial_\varepsilon(f + g)(x) = \bigcap_{\delta > \varepsilon} \operatorname{cl}^{w^*}\left(\bigcup_{\eta > 0} \bigcap_{\lambda \in (0,\eta)} \bigcup_{\substack{\delta_1, \delta_2 \geq 0 \\ \delta_1 + \delta_2 = \delta}} \partial_{\delta_1} f_\lambda(x) + \partial_{\delta_2} g_\lambda(x)\right) \ \ \forall x \in X, \ \forall \varepsilon \geq 0. \tag{3}$$

Proof. Let us prove (2). We denote $h_{(\lambda)} := f_\lambda + g_\lambda$. Since $f + g = \sup_{\lambda > 0} h_{(\lambda)}$, we have $(f + g)^* \leq \inf_{\lambda > 0} h_{(\lambda)}{}^*$ and then $(f + g)^* \leq \overline{\operatorname{co}}^{w^*}(\inf_{\lambda > 0} h_{(\lambda)}{}^*) = \operatorname{cl}^{w^*}(\inf_{\lambda > 0} h_{(\lambda)}{}^*)$; the last equality follows from the fact that $(h_{(\lambda)}{}^*)_\lambda$ is nonincreasing (as $\lambda \searrow 0$) so that $\inf_{\lambda > 0} h_{(\lambda)}{}^*$

is a convex function. To obtain the converse inequality, we observe that $\mathrm{cl}^{w^*}(\inf_{\lambda>0} h_{(\lambda)}{}^*) \leq \inf_{\lambda>0} h_{(\lambda)}{}^*$. Then, considering X^* with its weak* topology, we get $[\mathrm{cl}^{w^*}(\inf_{\lambda>0} h_{(\lambda)}{}^*)]^* \geq [\inf_{\lambda>0} h_\lambda^*]^* = \sup_{\lambda>0} h_{(\lambda)}{}^{**} = \sup_{\lambda>0} h_{(\lambda)} = f+g$ which implies that $\mathrm{cl}^{w^*}(\inf_{\lambda>0} h_{(\lambda)}{}^*) = [\mathrm{cl}^{w^*}(\inf_{\lambda>0} h_{(\lambda)}{}^*)]^{**} \leq (f+g)^*$; for the last equality recall that the function $\mathrm{cl}^{w^*}(\inf_{\lambda>0} h_{(\lambda)}{}^*)$ never takes the value $-\infty$ because $(f+g)^*$ is proper and $\inf_{\lambda>0} h_{(\lambda)}{}^* \geq (f+g)^*$ as was shown above. The proof of (2) is complete.

To prove (3) we take $x^* \in \partial_\varepsilon(f+g)(x)$ so that for each $\delta > \varepsilon$ we have that

$$(f+g)^*(x^*) < -(f+g)(x) + \langle x^*, x\rangle + \delta.$$

Since from (2) there exists a net $(x_\gamma^*)_\gamma$ w*-convergent to x^* such that $(f+g)^*(x^*) = \lim_\gamma \inf_{\lambda>0}(f_\lambda + g_\lambda)^*(x_\gamma^*)$, for some γ_0 we get

$$\inf_{\lambda>0}(f_\lambda + g_\lambda)^*(x_\gamma^*) < -(f+g)(x) + \langle x^*, x\rangle + \delta \quad \forall\gamma \geq \gamma_0.$$

So, invoking the fact that $(f_\lambda + g_\lambda)^*$ is nonincreasing in λ for $\lambda \searrow 0$, there exists η_γ such that

$$(f_\lambda + g_\lambda)^*(x_\gamma^*) < -(f+g)(x) + \langle x^*, x\rangle + \delta \quad \forall\lambda \in (0, \eta_\gamma),\ \gamma \geq \gamma_0;$$

that is, $x_\gamma^* \in \bigcup_{\eta>0} \bigcap_{\lambda\in(0,\eta)} \partial_\delta(f_\lambda + g_\lambda)(x)$. Hence, $x^* \in \mathrm{cl}^{w^*}\left(\bigcup_{\eta>0} \bigcap_{\lambda\in(0,\eta)} \partial_\delta(f_\lambda + g_\lambda)(x)\right)$ for all $\delta > \varepsilon$. This finishes the proof since the continuity of the Moreau-Yoshida approximation implies that $\partial_\delta(f_\lambda + g_\lambda) = \bigcup_{\substack{\delta_1,\delta_2\geq 0\\ \delta_1+\delta_2=\delta}} \partial_{\delta_1} f_\lambda(x) + \partial_{\delta_2} f_\lambda(x)$, and, on the other hand, the inclusion "$\supset$" in (2) is straightforward. ■

The following theorem is a direct consequence of Theorem B.

Theorem C: *Let Y be another normed space, h a lsc convex function defined on Y, and $A : X \to Y$ a linear continuous mapping, whose adjoint is denoted A^*, satisfying* $\operatorname{dom} h \cap AX \neq \emptyset$. *Then, we have that*

$$(h \circ A)^* = \mathrm{cl}^{w^*}(\inf_{\lambda>0}(h_\lambda \circ A)^*), \tag{4}$$

and, consequently,

$$\partial_\varepsilon(h \circ A)(x) = \bigcap_{\delta>\varepsilon} \mathrm{cl}^{w^*}\left(\bigcup_{\eta>0} \bigcap_{\lambda\in(0,\eta)} A^*\partial_\delta h_\lambda(Ax)\right) \quad \forall x \in X,\ \forall\varepsilon \geq 0. \tag{5}$$

Proof. It suffices to apply Theorem B with $f, g : X \times Y \to \mathbb{R} \cup \{+\infty\}$ being defined by

$f(x,y) := h(y)$ and $g(x,y) := \mathrm{I}_{\mathrm{Gr}\, A}$ (the indicator function of the graph of A). ■

Theorems B and C encompass many well-known results of subdifferential calculus.

At this step, our objective in the following sections is to extend the formulas in Theorems B and C above, by considering general approximating sequences f_k, g_k, and working with any topology τ in X^*, which is intermediate between the weak* and the norm topology.

3 Subdifferential calculus rules

In this section, we first give calculus rules for the subdifferential of the sum of convex functions defined on a normed space X, not necessarily reflexive, which extend the Hiriart Urruty-Phelps formula. We recall that τ is a topology defined on X^* which is intermediate between the weak*and norm topologies.

Given a sequence $(A_k)_{k\in\mathbb{N}}$ of subsets in X^*, we define the τ-closed sets

$$\tau\text{-}\liminf_{k\to\infty} A_k := \bigcap_{V\in\mathcal{N}_\tau(\theta)} \bigcup_{n\geq 0} \bigcap_{k\geq n} (A_k+V)\,,\ \tau\text{-}\limsup_{k\to\infty} A_k := \bigcap_{V\in\mathcal{N}_\tau(\theta)} \bigcap_{n\geq 0} \bigcup_{k\geq n} (A_k+V)\,,$$

where $\mathcal{N}_\tau(\theta)$ is the family of neighborhoods of the origin θ. Observe that the following inclusions always hold,

$$\{\tau\text{-}\lim_k x_k \mid x_k\in A_k\} \subset \tau\text{-}\liminf_{k\to\infty} A_k, \quad \mathrm{cl}^\tau\left(\bigcup_{n\geq 0}\bigcap_{k\geq n} A_k\right) \subset \tau\text{-}\liminf_{k\to\infty} A_k$$

with equality in the first inclusion if the topology τ is metrizable (note that both inclusions may be strict). Following the notation in [4], we write

$$\sigma\text{-}\liminf_{k\to\infty} A_k := \{w^*\text{-}\lim_k x_k \mid x_k\in A_k\}.$$

When τ is the norm topology, we omit the reference to it and simply write $\liminf_{k\to\infty} A_k$, $\limsup_{k\to\infty} A_k$, and so on.

We also use the inf-convolution of two functions $f,g: X\to\mathbb{R}\cup\{+\infty\}$ defined by

$$f\square g(x) := \inf_{y\in X}\{f(y)+g(x-y)\}.$$

Theorem 1 *Let $f,g: X\to\mathbb{R}\cup\{+\infty\}$ be two lsc convex functions, with $\operatorname{dom} f\cap\operatorname{dom} g\neq\emptyset$, and let (f_k), (g_k) be two sequences of lsc convex functions pointwise converging to f and g, respectively. We assume that for all $x^*\in X^*$ there exist two sequences (u_k^*), (v_k^*), τ-convergent to x^* such that*

$$\limsup_k f_k^*(u_k^*) \leq f^*(x^*),\ \ \limsup_k g_k^*(v_k^*) \leq g^*(x^*). \tag{6}$$

If f and g verify the condition

$$\overline{f^*\square g^*}^{w^*} = \overline{f^*\square g^*}^{\tau}, \tag{7}$$

then we have

$$\partial_\varepsilon(f+g)(x) = \bigcap_{\delta>\varepsilon} \tau\text{-}\liminf_{k\to\infty} \bigcup_{\substack{\delta_1+\delta_2=\delta\\ \delta_1,\delta_2\geq 0}} \partial_{\delta_1} f_k(x)+\partial_{\delta_2} g_k(x) \quad \forall x\in X,\ \forall\varepsilon\geq 0. \tag{8}$$

In addition, if the sequences (f_k), (g_k), are nondecreasing, the last formula is also written as

$$\partial_\varepsilon(f+g)(x) = \bigcap_{\delta>\varepsilon} \mathrm{cl}^\tau\left(\bigcup_{n\geq 0}\bigcap_{k\geq n}\bigcup_{\substack{\delta_1+\delta_2=\delta\\ \delta_1,\delta_2\geq 0}} \partial_{\delta_1} f_k(x)+\partial_{\delta_2} g_k(x)\right). \tag{9}$$

Proof. We begin by showing that (8) holds. Pick $x^* \in \partial_\varepsilon(f+g)(x)$ for given $x \in X$ and $\varepsilon \geq 0$. If $\delta > \varepsilon$ is fixed we write

$$(f+g)(x) + (f+g)^*(x^*) \leq \langle x^*, x\rangle + \varepsilon < \langle x^*, x\rangle + \frac{\delta+\varepsilon}{2}.$$

Since $f+g$ is proper, it holds $(f+g)^* = \overline{f^* \Box g^*}^{w^*}$ [23] and so, invoking (7),

$$(f+g)(x) + \overline{f^* \Box g^*}^{\tau}(x^*) \leq \langle x^*, x\rangle + \varepsilon < \langle x^*, x\rangle + \frac{\delta+\varepsilon}{2}.$$

Let $(x_i^*) \subset X^*$ be a τ-convergent net to x^* such that for all i,

$$f^* \Box g^*(x_i^*) \leq \overline{f^* \Box g^*}^{\tau}(x^*) + \frac{\delta-\varepsilon}{4}, \;\; |\langle x_i^* - x^*, x\rangle| \leq \frac{\delta-\varepsilon}{4}.$$

Hence, we obtain $(f+g)(x) + f^* \Box g^*(x_i^*) < \langle x_i^*, x\rangle + \delta$, and so there are $u_i^*, v_i^* \in X^*$ such that $u_i^* + v_i^* = x_i^*$ and

$$(f+g)(x) + f^*(u_i^*) + g^*(v_i^*) < \langle u_i^* + v_i^*, x\rangle + \delta. \tag{10}$$

Now, using the convergence assumption on the conjugates, for each i we find sequences $u_{i,k}^* \xrightarrow{\tau}_k u_i^*$ and $v_{i,k}^* \xrightarrow{\tau}_k v_i^*$ such that, for all k sufficiently large,

$$(f_k + g_k)(x) + f_k^*(u_{i,k}^*) + g_k^*(v_{i,k}^*) < \left\langle u_{i,k}^* + v_{i,k}^*, x\right\rangle + \delta. \tag{11}$$

Whence, taking $\delta_{1,k} := f_k(x) + f_k^*(u_{i,k}^*) - \left\langle u_{i,k}^*, x\right\rangle$ and $\delta_{2,k} := \delta - \delta_{1,k}$, we check that $\delta_{1,k}, \delta_{2,k} \geq 0$ together with $u_{i,k}^* \in \partial_{\delta_{1,k}} f_k(x)$ and $v_{i,k}^* \in \partial_{\delta_{2,k}} g_k(x)$. Consequently, for k sufficiently large we obtain

$$u_{i,k}^* + v_{i,k}^* \in \partial_{\delta_{1,k}} f_k(x) + \partial_{\delta_{2,k}} g_k(x) \subset \bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2 \geq 0}} \partial_{\delta_1} f_k(x) + \partial_{\delta_2} g_k(x),$$

which implies that $x_i^* \in \tau\text{-}\liminf_{k\to\infty} \bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2 \geq 0}} \partial_{\delta_1} f_k(x) + \partial_{\delta_2} g_k(x)$. Thus, to get the direct inclusion it suffices to take the limit on i and next make the intersection on $\delta > \varepsilon$. This completes the proof of (8) in view of the straightforwardness of the opposite inclusion.

To establish the last statement (9), we suppose that $f_k \nearrow f$ and $g_k \nearrow g$. If $x^* \in \partial_\varepsilon(f+g)(x)$ is given for fixed $\varepsilon \geq 0$ and $x \in X$, we take $\delta > \varepsilon$ and argue as above to conclude that (10) holds for all i. Then, by the current assumptions (the convergence property of the conjugates and monotonicity of the functions) we find sequences $u_{i,k}^* \xrightarrow{\tau}_k u_i^*$ and $v_{i,k}^* \xrightarrow{\tau}_k v_i^*$ such that for sufficiently large k we have that

$$(f_{k'} + g_{k'})(x) + f_{k'}^*(u_{i,k}^*) + g_{k'}^*(v_{i,k}^*) < \left\langle u_{i,k}^* + v_{i,k}^*, x\right\rangle + \delta \quad \forall k' \geq k.$$

In other words, $u_{i,k}^* + v_{i,k}^* \in \bigcup_{k\geq 0} \bigcap_{k'\geq k} \bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2 \geq 0}} \partial_{\delta_1} f_{k'}(x) + \partial_{\delta_2} g_{k'}(x)$ and, so, the desired inclusion is obtained by taking the limits on k and i, consecutively. This finishes the proof since the opposite inclusion always holds. ■

Formula (9) is of algebraic type; in fact, the limit that appears in (8) is replaced in (9) by intersections and unions of the involved sequence of sets. In addition, we observe that the equality in (9) is not a consequence of (8), only one inclusion ($\supset$) is direct from (8). On the other hand, as the following inclusion is straightforward, for every $x \in X$ and $\varepsilon \geq 0$,

$$\bigcap_{\delta>\varepsilon} \tau\text{-}\limsup_{k\to\infty} \bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\geq 0}} \partial_{\delta_1} f_k(x) + \partial_{\delta_2} g_k(x) \subset \partial_\varepsilon (f+g)(x),$$

relation (8) in Theorem 1 also ensures that

$$\partial_\varepsilon (f+g)(x) = \bigcap_{\delta>\varepsilon} \tau\text{-}\limsup_{k\to\infty} \bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\geq 0}} \partial_{\delta_1} f_k(x) + \partial_{\delta_2} g_k(x).$$

Moreover, by reading carefully the proof of Theorem 1, the last formula is still valid if, instead of (6), we suppose in Theorem 1 above that for each $x^* \in X^*$ the sequences (u_k^*), (v_k^*), satisfy the weaker condition

$$\liminf_k f_k^*(u_k^*) \leq f^*(x^*), \quad \liminf_k g_k^*(v_k^*) \leq g^*(x^*). \tag{12}$$

Let us also observe, as it comes out of the same proof of Theorem 1, that (6) (together with (7)) gives rise to the following more explicit characterization of the subdifferential of the sum,

$$\partial_\varepsilon (f+g)(x) = \bigcap_{\delta>\varepsilon} \mathrm{cl}^\tau \left\{ \tau\text{-}\lim_{k\to\infty} x_k^*, \; x_k^* \in \bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\geq 0}} \partial_{\delta_1} f_k(x) + \partial_{\delta_2} g_k(x) \right\}. \tag{13}$$

For instance, if τ is the weak* topology on X^*, then the last formula reads

$$\partial_\varepsilon (f+g)(x) = \bigcap_{\delta>\varepsilon} \mathrm{cl}^{w^*} \left(\sigma\text{-}\liminf_{k\to\infty} \bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\geq 0}} \partial_{\delta_1} f_k(x) + \partial_{\delta_2} g_k(x) \right),$$

while for τ being the norm topology one gets

$$\partial_\varepsilon (f+g)(x) = \bigcap_{\delta>\varepsilon} \left\{ \lim_{k\to\infty} x_k^*, \; x_k^* \in \bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\geq 0}} \partial_{\delta_1} f_k(x) + \partial_{\delta_2} g_k(x) \right\}.$$

Now we are going to specify Theorem 1 for some useful topologies. Let us denote by w_b^* the topology on X^* defined by declaring that a net (x_i^*) converges to x^* for w_b^* iff (x_i^*) is bounded and w*-convergent to x^*; this topology may be strictly stronger than the usual bounded weak* topology (see, e.g., [18]). Then, as a consequence of Banach-Dieudonné Theorem [14, Theorem 3.92], for any convex functions $f, g : X \to \mathbb{R} \cup \{+\infty\}$ with $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$ we have that

$$\overline{f^* \square g^*}^{w^*}(x^*) = \overline{f^* \square g^*}^{w_b^*}(x^*) \quad \forall x^* \in X^*;$$

that is, (7) always holds for $\tau = w_b^*$. In particular, whenever the closed unit ball in X^* is w*-sequentially compact (which is the case, for instance, when X is separable, weakly compactly generated, or Asplund,) we get

$$\overline{f^*\square g^*}^{w^*}(x^*) = \overline{f^*\square g^*}^{\sigma}(x^*) \quad \forall x^* \in X^*,$$

where σ refers to the sequential weak* topology in X^*, so that

$$\overline{f^*\square g^*}^{\sigma}(x^*) = \inf_{x_n^* \rightharpoonup^{w^*} x^*} \liminf_n f^*\square g^*(x_n^*).$$

Hence, the following corollary is an immediate consequence of Theorem 1.

Corollary 2 *Let f, g, (f_k), and (g_k) be as in Theorem 1. Then,*

$$\partial_\varepsilon(f+g)(x) = \bigcap_{\delta>\varepsilon} w_b^*\text{-}\liminf_{k\to\infty} \bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\geq 0}} \partial_{\delta_1} f_k(x) + \partial_{\delta_2} g_k(x) \quad \forall x \in X, \ \forall \varepsilon \geq 0.$$

Moreover, if the closed unit ball in X^* is w^*-sequentially compact, then the above formula reduces to

$$\partial_\varepsilon(f+g)(x) = \bigcap_{\delta>\varepsilon} \sigma\text{-}\liminf_{k\to\infty} \bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\geq 0}} \partial_{\delta_1} f_k(x) + \partial_{\delta_2} g_k(x).$$

Remark 1 Theorem 1 can be stated in an equivalent form without appealing explicitly to (7). Indeed, for $D := \{x^* \in X^* \mid \overline{f^*\square g^*}^{w^*}(x^*) = \overline{f^*\square g^*}^{\tau}(x^*)\}$ we have (assuming that (f_k), (g_k) are nondecreasing)

$$D \cap \partial_\varepsilon(f+g)(x) = D \cap \bigcap_{\delta>\varepsilon} \mathrm{cl}^\tau \left(\bigcup_{n\geq 0} \bigcap_{k\geq n} \bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\geq 0}} \partial_{\delta_1} f_k(x) + \partial_{\delta_2} g_k(x) \right) \quad \forall x \in X, \ \forall \varepsilon \geq 0.$$

Remark 2 The convergence assumptions used in Theorem 1 imply that (f_k) and (g_k) slice converge to f and g, respectively; that is, for each $(x, x^*) \in X \times X^*$, there is a sequence $(x_k, x_k^*) \subset X \times X^*$ which converges to (x, x^*) such that $\lim_k f_k(x_k) = f(x)$ and $\lim_k f_k^*(x_k^*) = f^*(x^*)$, and the same for (g_k). We refer to [8] for more details on this topic. Moreover, the pointwise convergence assumption ensures also the slice convergence of the sum $(f_k + g_k)$ to $f + g$ (see, e.g., [6, Corollary 2.8]). Hence, by Attouch-Beer Theorem [3, 9] together with its normed version [28, Theorem 3.4] (see, also [12] or [30]) one concludes that $(\partial_\varepsilon(f_k + g_k))_k$ graphically converges in the sense of Painlevé-Kuratowski to $\partial_\varepsilon(f + g)$ for all $\varepsilon \geq 0$. In this line, the approximate subdifferential set $\partial_\varepsilon(f + g)(x)$, for $x \in X$ and $\varepsilon \geq 0$, is characterized by means of the mappings $\partial_\varepsilon(f_k + g_k)$ evaluated at nearby points of x. However, the formulas in Theorem 1 goes into the spirit of subdifferential calculus where one is interested in decoupling the subdifferential of the involved approximating functions f_k, g_k, and considering only the reference point x rather than nearby ones.

Condition (7) of Theorem 1 is satisfied for any topology τ (being intermediate between the weak* and the norm topologies) whenever there exists a point in $\mathrm{dom}\, f \cap \mathrm{dom}\, g$ where one of these functions is continuous (Moreau-Rockafellar's qualification condition). Indeed, in this case, the function $f^*\square g^*$ is w^*-lsc so that $\overline{f^*\square g^*}^{w^*} \leq \overline{f^*\square g^*}^{\tau} \leq f^*\square g^* = \overline{f^*\square g^*}^{w^*}$. It is worth observing that condition (7) may hold without requiring such a condition; consider the case $f = g$, or when the underlying space is reflexive. The following example shows the existence of functions defined on a nonreflexive Banach space which satisfy condition (7) and, consequently, formula (8). These functions are often used in convex analysis [27] (see the proof of Theorem C in Section 2).

Example 1 Let X be a reflexive Banach space, Y a non-reflexive normed space, $A : X \to Y$ a continuous linear operator with adjoint mapping A^*, and $h : Y \to \mathbb{R} \cup \{+\infty\}$ a lsc convex function such that $A^{-1}(\operatorname{dom} h) \neq \emptyset$. We endow the product space $X \times Y$ with the box norm and define the functions $f, g : X \times Y \to \mathbb{R} \cup \{+\infty\}$ by

$$f(x, y) := h(y), \quad g(x, y) := \mathrm{I}_{\operatorname{Gr} A}(x, y).$$

Then, although the normed space $X \times Y$ is not reflexive, the functions f, g satisfy condition (7); that is, $\overline{f^* \square g^*}^{w^*} = \overline{f^* \square g^*}^{\tau}$, for any locally convex topology τ between the weak* and the norm topology. In other words, the formula in (8) holds for any such a τ. Moreover, if the unit ball of the dual of X is w*-sequentially compact, then $\overline{f^* \square g^*}^{w^*} = \overline{f^* \square g^*}^{\sigma}$ and, so, the formula in (8) holds with σ.

Proof. It is not difficult to check that $f^* \square g^*(x^*, y^*) = (A^* h^*)(x^* + A^* y^*)$, where $A^* h^* : X^* \to \mathbb{R} \cup \{+\infty\}$ is the mapping defined as

$$A^* h^*(u^*) := \inf\{h^*(z^*) \mid A^* z^* = u^*\}. \tag{14}$$

Observe that the reflexivity assumption entails $\overline{A^* h^*}^{w^*} = \overline{A^* h^*}$. Then, for $(x^*, y^*) \in X^* \times Y^*$ there exists a net $(x_i^*, y_i^*)_i$ w*-convergent to (x^*, y^*) such that

$$\overline{f^* \square g^*}^{w^*}(x^*, y^*) = \lim_i f^* \square g^*(x_i^*, y_i^*) = \lim_i A^* h^*(x_i^* + A^* y_i^*) \geq \overline{A^* h^*}(x^* + A^* y^*).$$

On the other hand, if $\overline{A^* h^*}(x^* + A^* y^*) = \lim_j A^* h^*(u_j^*)$ for some sequence $u_j^* \to x^* + A^* y^*$, then $u_j^* - A^* y^* \to x^*$ and we get

$$\overline{A^* h^*}(x^* + A^* y^*) = \lim_j A^* h^*(u_j^* - A^* y^* + A^* y^*) = \lim_j f^* \square g^*(u_j^* - A^* y^*, y^*) \geq \overline{f^* \square g^*}(x^*, y^*).$$

Whence, $\overline{f^* \square g^*}^{w^*}(x^*, y^*) \geq \overline{f^* \square g^*}(x^*, y^*)$ and, so, the desired equality follows. ■

The next theorem gives the converse of Theorem 1.

Theorem 3 *Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be two lsc convex functions, with $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$, and let (f_k), (g_k) be two nondecreasing sequences of lsc convex functions pointwise converging to f and g, respectively. We assume that for all $x^* \in X^*$ there exist two sequences (u_k^*), (v_k^*), τ-convergent to x^* such that*

$$\limsup_k f_k^*(u_k^*) \leq f^*(x^*), \quad \limsup_k g_k^*(v_k^*) \leq g^*(x^*).$$

Then, (7) $\Longleftrightarrow$ (8) $\Longleftrightarrow$ (9) $\Longleftrightarrow$ (15), *where*

$$\partial_\varepsilon(f + g)(x) = \bigcap_{\delta > \varepsilon} \mathrm{cl}^\tau \left(\bigcup_{n \geq 0} \bigcap_{k \geq n} \bigcup_{\substack{\delta_1 + \delta_2 = \delta \\ \delta_1, \delta_2 \geq 0}} \partial_{\delta_1} f_k(x) + \partial_{\delta_2} g_k(x) \right) \quad \forall x \in X, \ \forall \varepsilon > 0. \tag{15}$$

If, in addition, X is a Banach space, then (7) $\iff$ (8) $\iff$ (9) $\iff$ (15) $\iff$ (16) $\iff$ (17), *where*

$$\partial(f+g)(x) = \bigcap_{\delta>0} \tau\text{-}\liminf_{k\to\infty} \partial_\delta f_k(x) + \partial_\delta g_k(x) \quad \forall x \in X; \tag{16}$$

$$\partial(f+g)(x) = \bigcap_{\delta>0} \mathrm{cl}^\tau \left(\bigcup_{n\geq 0} \bigcap_{k\geq n} \partial_\delta f_k(x) + \partial_\delta g_k(x) \right) \quad \forall x \in X. \tag{17}$$

Proof. According to Theorem 1, the following implications hold true,

$$(7) \implies (9) \implies (8) \text{ and } (9) \implies (15).$$

Thus, it remains to prove that (15) $\implies$ (7) because the proof of the implication (8) $\implies$ (7) is similar. Since $\overline{f^*\square g^*}^{w^*} \leq \overline{f^*\square g^*}^{\tau}$ and $(f+g)^* = \overline{f^*\square g^*}^{w^*}$, we only need to prove that $\overline{f^*\square g^*}^{\tau}(x^*) \leq (f+g)^*(x^*)$ for any given $x^* \in \mathrm{dom}(f+g)^*$. Fix $\varepsilon > 0$. Since $X \cap \partial_\varepsilon(f+g)^*(x^*)$ is always nonempty, we pick $x \in X \cap \partial_\varepsilon(f+g)^*(x^*)$ so that $x^* \in X^* \cap \partial_\varepsilon(f+g)^{**}(x) = \partial_\varepsilon(f+g)(x)$. Given $\delta > \varepsilon$, by applying the current assumption (15) we find a net $(x^*_\gamma) \subset \bigcup_{n\geq 0} \bigcap_{k\geq n} \bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\geq 0}} \partial_{\delta_1} f_k(x) + \partial_{\delta_2} g_k(x)$ which τ-converges to x^*. Fix then γ and let $n_\gamma \in \mathbb{N}$ and the vectors $u^*_{\gamma,k} \in \partial_\delta f_k(x)$ and $v^*_{\gamma,k} \in \partial_\delta g_k(x)$, $k > n_\gamma$, be such that $x^*_\gamma = u^*_{\gamma,k} + v^*_{\gamma,k}$. Hence, invoking the monotonicity assumption, for all $k > n_\gamma$ we get

$$\begin{aligned}(f_k+g_k)(x) + f^*\square g^*(x^*_\gamma) &\leq f_k(x) + g_k(x) + f^*(u^*_{\gamma,k}) + g^*(v^*_{\gamma,k}) \\ &\leq f_k(x) + g_k(x) + f^*_k(u^*_{\gamma,k}) + g^*_k(v^*_{\gamma,k}) \leq \left\langle x^*_\gamma, x \right\rangle + 2\delta.\end{aligned}$$

Therefore, taking the limits as $k \to \infty$, $\delta \to \varepsilon$ and $\varepsilon \to 0$, consecutively, we get the desired inequality.

Now we suppose that X is a Banach space. Because (9) $\implies$ (17) $\implies$ (16) we only have to prove that (16) $\implies$ (7). Indeed, it suffices as above to establish the inequality $\overline{f^*\square g^*}^{\tau}(x^*) \leq (f+g)^*(x^*)$ for any given $x^* \in \mathrm{dom}(f+g)^*$. Indeed, according to [29, Theorem 3.1.4], there exists $(x^*_n, x^{**}_n)_n \subset \partial(f+g)^*$ such that $(x^*_n)_n$ (norm-)converges to x^* and $(f+g)^*(x^*_n) \to (f+g)^*(x^*)$. Moreover, by Rockafellar's Theorem [26, Proposition 1] for each n there exist bounded nets $(x_\gamma) \subset X$ w*-converging to x^{**}_n and $(x^*_\gamma) \subset X^*$ (norm-)converging to x^*_n such that $x^*_\gamma \in \partial(f+g)(x_\gamma)$ for all γ. Let us fix γ and $\delta > 0$, and choose a τ-neighborhood U of the origin such that, for all $u \in U$,

$$\overline{f^*\square g^*}^{\tau}(x^*_\gamma + u) \geq \overline{f^*\square g^*}^{\tau}(x^*_\gamma) - \delta \text{ and } \langle u, x_\gamma \rangle \leq \delta. \tag{18}$$

Applying the current assumption (16), there exist sequences $u^*_{\gamma,k} \in \partial_\delta f_k(x_\gamma)$ and $v^*_{\gamma,k} \in \partial_\delta g_k(x_\gamma)$ such that $u^*_{\gamma,k} + v^*_{\gamma,k} \in x^*_\gamma + U$ for each sufficiently large k. Hence, with the argument above, we get

$$(f_k+g_k)(x_\gamma) + f^*\square g^*(u^*_{\gamma,k} + v^*_{\gamma,k}) \leq \left\langle u^*_{\gamma,k} + v^*_{\gamma,k}, x_\gamma \right\rangle + 2\delta.$$

Then, by the choice of U in (18) we get $(f_k+g_k)(x_\gamma) + \overline{f^*\square g^*}^{\tau}(x^*_\gamma) \leq \left\langle x^*_\gamma, x_\gamma \right\rangle + 4\delta$, which implies that $\overline{f^*\square g^*}^{\tau}(x^*_\gamma) \leq \left\langle x^*_\gamma, x_\gamma \right\rangle - (f+g)(x_\gamma)$ as k and δ go to ∞ and 0, consecutively. Finally, since (x^*_γ) (norm-)converges to x^*_n and (x_γ) is bounded, by taking the limit on γ we get

$$\overline{f^*\square g^*}^{\tau}(x^*_n) \leq \langle x^*_n, x^{**}_n \rangle - (f+g)^{**}(x^{**}_n) \leq (f+g)^{***}(x^*_n) = (f+g)^*(x^*_n).$$

Therefore, the required inequality follows when $n \to \infty$ (recall that $(f+g)^*(x^*_n) \to (f+g)^*(x^*)$).

■

Using the same arguments of the proof above, under the weaker assumption (12) instead of (6), condition (7) is also equivalent to the following characterization

$$\partial_\varepsilon (f+g)(x) = \bigcap_{\delta>\varepsilon} \tau\text{-}\limsup_{k\to\infty} \bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\geq 0}} \partial_{\delta_1} f_k(x) + \partial_{\delta_2} g_k(x).$$

In particular, when (f_k) and (g_k) are the constant sequences (f) and (g), respectively, we obtain the following generalization of Hiriart Urruty-Phelps Theorem [17].

Corollary 4 *Let $f, g : X \to \mathbb{R}\cup\{+\infty\}$ be two lsc convex functions with $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. Then, condition (7) is equivalent to*

$$\partial_\varepsilon (f+g)(x) = \bigcap_{\delta>\varepsilon} \operatorname{cl}^\tau \left(\bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\geq 0}} \partial_{\delta_1} f(x) + \partial_{\delta_2} g(x) \right) \quad \forall x\in X,\ \varepsilon \geq 0. \tag{19}$$

If, in addition, X is a Banach space, then each one of the relationships (7) and (19) is equivalent to

$$\partial (f+g)(x) = \bigcap_{\delta>0} \operatorname{cl}^\tau \left(\partial_\delta f(x) + \partial_\delta g(x) \right) \quad \forall x\in X.$$

In the following result we apply the previous theorems to the Moreau-Yoshida envelope, which is a reference example for the approximating sequences (f_k) and (g_k) used before. This result has already been established in the reflexive case in [20] (see, also, [2] for the Hilbert setting). Its proof is straightforward from the previous Theorem 3, taking into account that for $\lambda_k \searrow 0$ we have

$$\tau\text{-}\liminf_{\lambda\to 0^+} \bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\geq 0}} \partial_{\delta_1} f_\lambda(x) + \partial_{\delta_2} g_\lambda(x) = \tau\text{-}\liminf_{k\to\infty} \bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\geq 0}} \partial_{\delta_1} f_{\lambda_k}(x) + \partial_{\delta_2} g_{\lambda_k}(x) \quad \forall x\in X,\ \delta>0,$$

where $\tau\text{-}\liminf_{\lambda\to 0^+} B_\lambda := \bigcap_{V\in\mathcal{N}_\tau(\theta)} \bigcup_{\eta>0} \bigcap_{\lambda\in(0,\eta)} (B_\lambda + V)$ for any family $(B_\lambda)_{\lambda>0}$ of subsets in X^*.

Corollary 5 *Let $f, g : X \to \mathbb{R}\cup\{+\infty\}$ be two lsc convex functions with $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. Then, the following assertions are equivalent:*

(i) $\overline{f^*\square g^*}^{w^*} = \overline{f^*\square g^*}^{\tau}$;

(ii) *for every $x\in X$ and $\varepsilon\geq 0$ we have that*

$$\partial_\varepsilon (f+g)(x) = \bigcap_{\delta>\varepsilon} \operatorname{cl}^\tau \left(\bigcup_{\eta>0} \bigcap_{\lambda\in(0,\eta)} \bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\geq 0}} \partial_{\delta_1} f_\lambda(x) + \partial_{\delta_2} g_\lambda(x) \right);$$

(iii) *for every $x\in X$ and $\varepsilon\geq 0$ we have that*

$$\partial_\varepsilon (f+g)(x) = \bigcap_{\delta>\varepsilon} \tau\text{-}\liminf_{\lambda\to 0^+} \bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\geq 0}} \partial_{\delta_1} f_\lambda(x) + \partial_{\delta_2} g_\lambda(x).$$

In addition, if X is a Banach space, each one of the above statements is equivalent to

(iv) *for every $x \in X$ we have that*

$$\partial(f+g)(x) = \bigcap_{\delta>0} \tau\text{-}\liminf_{\lambda\to 0^+} \partial_\delta f_\lambda(x) + \partial_\delta g_\lambda(x).$$

Examples of useful topologies on X^* which are included within our analysis are the weak*, the sequential weak*, and the norm topology. For instance, when τ is the weak* topology, then the formulas above (7), (8), (9), (15), (16), and (17) always holds. The same conclusion follows for τ being the norm topology when X is reflexive. Moreover, when τ is the norm topology in X^*, if one of the assertions (i)–(iv) in Corollary 5 holds for every lsc convex functions $f, g : X \to \mathbb{R} \cup \{+\infty\}$ with $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$, then X is necessarily reflexive. Indeed, based on the proof of [7, Theorem 3.8], it has been proved in [30] that in every nonreflexive Banach space there exist lsc convex functions, with $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$, which violate condition (i) of Corollary 5.

4 Calculus in the bidual setting

The main stream of this section is devoted to the characterization of the approximate subdifferential of the sum of the biconjugates $f^{**} + g^{**}$, by means of strong limits involving the subdifferential of the approximating functions. The reason for considering the bidual setting is that the analysis in this case does not require supplementary conditions as in the previous section. This will also make clear the nature of the sets

$$\bigcap_{\delta>\varepsilon} \liminf_{k\to\infty} \bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\geq 0}} \partial_{\delta_1} f_k(x) + \partial_{\delta_2} g_k(x),\ \varepsilon \geq 0,\ x \in X,$$

which indeed are nothing else but the part of $\partial_\varepsilon(f^{**} + g^{**})(x)$ living in X^* (Theorem 7).

We will need the following Lemma, which may also be of independent interest.

Lemma 6 *Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be two lsc convex functions such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. *Then, we have that*

$$\overline{f^{**}\square g^{**}}^{w^*}(x) = \overline{f^{**}\square g^{**}}(x) \quad \forall x \in X.$$

Proof. We denote $A := \operatorname{epi} f$, $B := \operatorname{epi} g$ so that $\operatorname{epi} f^{**}$ and $\operatorname{epi} g^{**}$ coincide with the w*-closures of A and B in $X^{**} \times \mathbb{R}$, respectively. Then, taking into account Mazur's Theorem we write

$$\overline{A+B}^{\|\|_*} = \overline{A+B}^{w} = (X \times \mathbb{R}) \cap \overline{A+B}^{w^*} = (X \times \mathbb{R}) \cap \overline{\overline{A}^{w^*} + \overline{B}^{w^*}}^{w^*}.$$

Thus, since $\overline{A+B}^{\|\|_*} \subset (X \times \mathbb{R}) \cap \overline{\overline{A}^{w^*} + \overline{B}^{w^*}}^{\|\|_*} \subset (X \times \mathbb{R}) \cap \overline{\overline{A}^{w^*} + \overline{B}^{w^*}}^{w^*}$, we infer that $(X \times \mathbb{R}) \cap \overline{\overline{A}^{w^*} + \overline{B}^{w^*}}^{\|\|_*} = (X \times \mathbb{R}) \cap \overline{\overline{A}^{w^*} + \overline{B}^{w^*}}^{w^*}$, which in turn implies the desired equality in view of the relationships $\operatorname{epi} \overline{f^{**}\square g^{**}}^{w^*} = \overline{\operatorname{epi} f^{**}\square g^{**}}^{w^*} = \overline{\operatorname{epi} f^{**} + \operatorname{epi} g^{**}}^{w^*}$ (the same holds for the norm topology). ■

Now, we state our main results of this section.

Theorem 7 *Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be two lsc convex functions with* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$, *and let (f_k), (g_k) be two nondecreasing sequences of lsc convex functions converging to f and g, respectively. We assume that for every $x^* \in X^*$ there exist two sequences (u_k^*), (v_k^*), convergent* *to* x^* *such that*

$$\limsup_{k\to\infty} f_k^*(u_k^*) \le f^*(x^*), \;\; \limsup_{k\to\infty} g_k^*(v_k^*) \le g^*(x^*).$$

Then, for all $x^{**} \in X^{**}$ *and* $\varepsilon \ge 0$ *we have*

$$X^* \cap \partial_\varepsilon (f^{**} + g^{**})(x^{**}) = X^* \cap \bigcap_{\delta>\varepsilon} \mathrm{cl}\left(\bigcup_n \bigcap_{k\ge n} \bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\ge 0}} \partial_{\delta_1} f_k^{**}(x^{**}) + \partial_{\delta_2} g_k^{**}(x^{**}) \right).$$

Consequently, for $x \in X$ *the last formula reads*

$$X^* \cap \partial_\varepsilon (f^{**} + g^{**})(x) = \bigcap_{\delta>\varepsilon} \mathrm{cl}\left(\bigcup_n \bigcap_{k\ge n} \bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\ge 0}} \partial_{\delta_1} f_k(x) + \partial_{\delta_2} g_k(x) \right).$$

Proof. The first formula follows by combining Lemma 6 and Remark 1. To establish the second one we suppose that $x \in X$ and take x^* on the right-hand-side. Fix $y^{**} \in X^{**}$. Then, for every $\delta > \varepsilon$ there exist sequences $x_k^* \in \partial_{\delta_{k,1}} f_k(x)$, $y_k^* \in \partial_{\delta_{k,2}} g_k(x)$, with $\delta_{k,1}, \delta_{k,2} \ge 0$ and $\delta_{k,1} + \delta_{k,2} = \delta$, such that $x_k^* + y_k^* \to x^*$. Then, for all $y \in X$ we write

$$f_k(y) \ge f_k(x) + \langle x_k^*, y - x\rangle - \delta_{k,1}, \; g_k(y) \ge g_k(x) + \langle y_k^*, y - x\rangle - \delta_{k,2}. \tag{20}$$

By using Rockafellar's Theorem, for each k there exist (bounded) nets $(x_i), (y_j) \subset X$, w*-converging in X^{**} to y^{**}, such that

$$f_k(x_i) \to f_k^{**}(y^{**}), \; g_k(y_j) \to g_k^{**}(y^{**}).$$

Thus, evaluating at x_i and y_j the first and second inequality in (20), respectively, taking the limits on i, j, and next summing the resulting inequalities we obtain

$$f_k^{**}(y^{**}) + g_k^{**}(y^{**}) \ge f_k(x) + g_k(x) + \langle x_k^* + y_k^*, y^{**} - x\rangle - \delta.$$

Now, by appealing to the pointwise convergence assumption and the fact that $f_k = f_k^{**}$ and $g_k = g_k^{**}$ on X, we get

$$f^{**}(y^{**}) + g^{**}(y^{**}) \ge f^{**}(x) + g^{**}(x) + \langle x^*, y^{**} - x\rangle - \delta,$$

and so taking the limit when $\delta \to \varepsilon$ we obtain $x^* \in X^* \cap \partial_\varepsilon (f^{**} + g^{**})(x)$.

It remains to establish the direct inclusion in the last statement. For this aim, we pick $x^* \in X^* \cap \partial_\varepsilon (f^{**} + g^{**})(x)$ so that, according to the first formula, for each $\rho > 0$ we find $y^{***} \in \rho B_{X^{***}}$ and n such that $x^* + y^{***} \in \cup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\ge 0}} \partial_{\delta_1} f_k^{**}(x) + \partial_{\delta_2} g_k^{**}(x)$ for all $k \ge n$; hence, with i^* the adjoint of the canonical injection $i : X \to X^{**}$, we write

$$x^* + i^*(y^{***}) \in \bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\ge 0}} i^*(\partial_{\delta_1} f_k^{**}(x)) + i^*((\partial_{\delta_2} g_k^{**}(x)) = \bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\ge 0}} \partial_{\delta_1} f_k(x) + \partial_{\delta_2} g_k(x)$$

so that $x^* + i^*(y^{***}) \in \underset{n}{\cup} \underset{k\geq n}{\cap} \underset{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\geq 0}}{\cup} \partial_{\delta_1} f_k(x) + \partial_{\delta_2} g_k(x)$. Since $i^*(y^{***}) \in \rho B_{X^*}$ and i^* is continuous, with ρ small enough we obtain the desired conclusion. ■

By comparing both Theorems 1 and 7, it follows under condition (7) that

$$X^* \cap \partial_\varepsilon (f^{**} + g^{**})(x) = \partial_\varepsilon (f+g)(x).$$

On the other hand, Theorem 1 can be obtained from Theorem 7 in view of the relationship $X^* \cap \partial_\varepsilon (f^{**}+g^{**})(x) = X^* \cap \partial_\varepsilon (f+g)^{**}(x)$, which is a consequence of the equality $(f^{**}+g^{**}) = (f+g)^{**}$, that follows from condition (7) (see, e.g., [29]).

The following result is a direct consequence of Theorem 7.

Corollary 8 *Let $f, g : X \to \mathbb{R}\cup\{+\infty\}$ be two lsc convex functions such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. *Then, we have that*

$$X^* \cap \partial_\varepsilon (f^{**} + g^{**})(x) = \bigcap_{\delta>\varepsilon} \operatorname{cl} \left(\bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\geq 0}} \partial_{\delta_1} f(x) + \partial_{\delta_2} g(x) \right) \quad \forall x \in X,\ \forall \varepsilon \geq 0.$$

The Moreau-Yoshida envelope satisfies the assumptions in Theorem 7, and so we get

Corollary 9 *Let $f, g : X \to \mathbb{R}\cup\{+\infty\}$ be two lsc convex functions such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. *Then, we have that*

$$X^* \cap \partial_\varepsilon (f^{**} + g^{**})(x) = \bigcap_{\delta>\varepsilon} \liminf_{\lambda\to 0} \bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\geq 0}} \partial_{\delta_1} f_\lambda(x) + \partial_{\delta_2} g_\lambda(x) \quad \forall x \in X,\ \varepsilon \geq 0.$$

In the following result, we use conditions which ensures subdifferential sum rules similar to those obtained under usual qualification conditions.

Corollary 10 *Let $f, g : X \to \mathbb{R}\cup\{+\infty\}$ be two lsc convex functions such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$.
(i) *If* $\operatorname{epi} f^{***} + \operatorname{epi} g^{***}$ *is (norm-)closed, then we have that*

$$X^* \cap \partial_\varepsilon (f^{**} + g^{**})(x^{**}) = X^* \cap \bigcup_{\substack{\varepsilon_1,\varepsilon_2\geq 0 \\ \varepsilon_1+\varepsilon_2=\varepsilon}} \partial_{\varepsilon_1} f^{**}(x^{**}) + \partial_{\varepsilon_2} g^{**}(x^{**}) \quad \forall x^{**} \in X^{**},\ \varepsilon \geq 0.$$

(ii) *If* $\operatorname{epi} f^* + \operatorname{epi} g^*$ *is (norm-)closed, then we have that*

$$X^* \cap \partial_\varepsilon (f^{**} + g^{**})(x) = \bigcup_{\substack{\varepsilon_1,\varepsilon_2\geq 0 \\ \varepsilon_1+\varepsilon_2=\varepsilon}} \partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} g(x) \quad \forall x \in X,\ \varepsilon \geq 0;$$

in particular, for $\varepsilon = 0$ we write

$$X^* \cap \partial (f^{**} + g^{**})(x) = \partial f(x) + \partial g(x).$$

Proof. (i) We fix $x^{**} \in X^{**}$ and $\varepsilon \geq 0$. Pick $x^* \in X^* \cap \partial_\varepsilon (f^{**} + g^{**})(x^{**})$ and take $\delta > \varepsilon$. By Corollary 8, there exist $\alpha_n, \beta_n \geq 0$ with $\alpha_n + \beta_n = \delta$, $\zeta_n \in \partial_{\alpha_n} f^{**}(x^{**})$, and $\psi_n \in \partial_{\beta_n} g^{**}(x^{**})$

such that

$$x^* = \lim_{n\to\infty} \zeta_n + \psi_n.$$

Observe that $(\zeta_n, \langle \zeta_n, x^{**}\rangle - f^{**}(x^{**}) + \alpha_n) \in \operatorname{epi} f^{***}$ and $(\psi_n, \langle \psi_n, x^{**}\rangle - g^{**}(x^{**}) + \beta_n) \in \operatorname{epi} g^{***}$ so that $(\zeta_n + \psi_n, \langle \zeta_n + \psi_n, x^{**}\rangle - f^{**}(x^{**}) - g^{**}(y^{**}) + \delta) \in \operatorname{epi} f^{***} + \operatorname{epi} g^{***}$. Hence, taking the limit on n and making $\delta \to \varepsilon$, by the closedness assumption we infer that $(x^*, \langle x^*, x^{**}\rangle - f^{**}(x^{**}) - g^{**}(x^{**}) + \varepsilon) \in \operatorname{epi} f^{***} + \operatorname{epi} g^{***}$. Let $(\xi_1, \mu_1) \in \operatorname{epi} f^{***}$ and $(\xi_2, \mu_2) \in \operatorname{epi} g^{***}$ be such that

$$x^* = \xi_1 + \xi_2,\ \langle x^*, x^{**}\rangle - f^{**}(x^{**}) - g^{**}(x^{**}) + \varepsilon = \mu_1 + \mu_2.$$

Writing $\langle \xi_1 + \xi_2, x^{**}\rangle - f^{**}(x^{**}) - g^{**}(x^{**}) + \varepsilon = \mu_1 + \mu_2 \geq f^{***}(\xi_1) + g^{***}(\xi_2)$, we deduce that

$$0 \leq f^{***}(\xi_1) + f^{**}(x^{**}) - \langle \xi_1, x^{**}\rangle + g^{***}(\xi_2) + g^{**}(x^{**}) - \langle \xi_2, x^{**}\rangle \leq \varepsilon,$$

and so there are $\varepsilon_1, \varepsilon_2 \geq 0$ with $\varepsilon_1 + \varepsilon_2 = \varepsilon$ such that $f^{***}(\xi_1) + f^{**}(x^{**}) - \langle \xi_1, x^{**}\rangle \leq \varepsilon_1$ and $g^{***}(\xi_2) + g^{**}(x^{**}) - \langle \xi_2, x^{**}\rangle \leq \varepsilon_2$. In other words, $\xi_1 \in \partial_{\varepsilon_1} f^{**}(x^{**})$ and $\xi_2 \in \partial_{\varepsilon_2} g^{**}(x^{**})$, showing that $x^* = \xi_1 + \xi_2 \in \partial_{\varepsilon_1} f^{**}(x^{**}) + \partial_{\varepsilon_2} g^{**}(x^{**})$. This finishes the proof of (i) since the opposite inclusion always holds.

(ii) The proof of this statement follows similarly as above by taking into account the last formula in Theorem 7. ■

The following two corollaries interpret the previous limiting formulas, namely Corollary 8, in the context of the classical convex duality theory [23].

Corollary 11 *Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be two lsc convex functions such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. *Then, there exist sequences (ζ_k), $(\xi_k) \subset X^{***}$ with $\zeta_k + \xi_k \to \theta$ such that*

$$\inf_{X^{**}} f^{**} + g^{**} = \lim_k -f^{***}(\zeta_k) - g^{***}(\xi_k).$$

In particular, if $\operatorname{epi} f^{***} + \operatorname{epi} g^{***}$ *is (norm-)closed, then*

$$\inf_{X^{**}} f^{**} + g^{**} = \max_{\xi \in X^{***}} -f^{***}(\xi) - g^{***}(-\xi).$$

Proof. Let us first observe that the inequality " $\geq$ " always holds. To prove the other inequality, we may assume that $\inf_{X^{**}} f^{**} + g^{**} = -(f^{**} + g^{**})^*(\theta) \in \mathbb{R}$. Then, for every $\varepsilon > 0$ there exists $x^{**} \in X^{**}$ such that $x^{**} \in \partial_\varepsilon (f^{**} + g^{**})^*(\theta)$ or, equivalently, $\theta \in X^{**} \cap \partial_\varepsilon (f^{**} + g^{**})^{**}(x^{**}) = \partial_\varepsilon (f^{**} + g^{**})(x^{**})$. Thus, using Theorem 7, we find sequences (ζ_k), $(\xi_k) \subset X^{***}$ such that $\zeta_k \in \partial_\varepsilon f^{**}(x^{**})$, $\xi_k \in \partial_\varepsilon g^{**}(x^{**})$, and $\zeta_k + \xi_k \to \theta$; hence,

$$f^{***}(\zeta_k) + f^{**}(x^{**}) \leq \langle \zeta_k, x^{**}\rangle + \varepsilon,\ g^{***}(\xi_k) + g^{**}(x^{**}) \leq \langle \xi_k, x^{**}\rangle + \varepsilon.$$

The addition of these two inequalities gives, for sufficiently large k,

$$f^{**}(x^{**}) + g^{**}(x^{**}) \leq -f^{***}(\zeta_k) - g^{***}(\xi_k) + \langle \zeta_k + \xi_k, x^{**}\rangle + 2\varepsilon \leq -f^{***}(\zeta_k) - g^{***}(\xi_k) + 3\varepsilon,$$

so that the conclusion follows when $\varepsilon \to 0^+$.

Finally, the last statement follows similarly as above by invoking, instead of Theorem 7, Corollary 10(i) which guarantees that the sequences (ζ_k), (ξ_k) may be taken fixed. ■

Corollary 12 *Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be two lsc convex functions such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$ *and* $\inf_{X^{**}} f^{**} + g^{**} = \inf_X f + g$. *Then, there exist sequences (x_k^*), $(y_k^*) \subset X^*$ with $x_k^* + y_k^* \to \theta$ such that*

$$\inf_X f + g = \lim_k -f^*(x_k^*) - g^*(y_k^*).$$

While, in case the set $\operatorname{epi} f^* + \operatorname{epi} g^*$ *is (norm-)closed it holds that*

$$\inf_X f + g = \max_{x^* \in X^*} -f^*(x^*) - g^*(-x^*).$$

Proof. With the arguments in the proof of Corollary 11, and taking into account the current assumption, for every $\varepsilon > 0$ we find $x \in X$ such that $\theta \in \partial_\varepsilon(f^{**} + g^{**})(x)$. Thus, using Corollary 10, we find sequences $x_k^* \in \partial_\varepsilon f(x)$ and $y_k^* \in \partial_\varepsilon g(x)$ such that $x_k^* + y_k^* \to \theta$ together with

$$f(x) + g(x) \leq -f^*(x_k^*) - g^*(y_k^*) + \langle x_k^* + y_k^*, x\rangle + 2\varepsilon \leq -f^*(x_k^*) - g^*(y_k^*) + 3\varepsilon.$$

Thus, the conclusion follows when $\varepsilon \to 0^+$. ■

5 Subdifferential limiting calculus

In this section, we give the limiting formulas characterizing the approximate subdifferential of the sum, in terms of only the approximate subdifferentials at nearby points of the involved functions. The involved approximate subdifferentials are of the same order of ε; that is, the intersection over $\delta > \varepsilon$, that appears in formulas of previous sections, here will disappear. For the sake of simplicity, we don't consider approximating sequences f_k and g_k. In fact, given $\varepsilon \geq 0$ we deal with the following set

$$\tau\text{-}\limsup_{\substack{u \xrightarrow{f} x,\ v \xrightarrow{g} x \\ \langle \cdot, u-x\rangle,\ \langle \cdot, v-x\rangle \to 0}} \left(\bigcup_{\substack{\alpha+\beta=\varepsilon \\ \alpha,\beta\geq 0}} \partial_\alpha f(u) + \partial_\beta g(v) \right), \tag{21}$$

of elements of the form $\tau\text{-}\lim_i u_i^* + v_i^*$ where $u_i^* \in \partial_{\alpha_i} f(u_i)$ and $v_i^* \in \partial_{\beta_i} g(v_i)$, for some $u_i, v_i \in X$, $\alpha_i, \beta_i \geq 0$, and $\alpha_i + \beta_i = \varepsilon$ verifying

$$u_i \xrightarrow{f} x,\ v_i \xrightarrow{g} x,\ \lim_i \langle u_i^*, u_i - x\rangle = \lim_i \langle v_i^*, v_i - x\rangle = 0,$$

where $u_i \xrightarrow{f} x$ means that $u_i \to x$ and $f(u_i) \to f(x)$. For $\varepsilon = 0$, we simply write $\tau\text{-}\limsup_{\substack{u \xrightarrow{f} x,\ v \xrightarrow{g} x \\ \langle \cdot, u-x\rangle,\ \langle \cdot, v-x\rangle \to 0}} \partial f(u) + \partial g(v)$.

We begin by establishing an approximate variational principle in line with [28]-[11]-[12], where the original Ekeland's variational principle have been adapted to general normed spaces for approximate subdifferentials. The quantities which appear in the following lemma have been established in the Banach setting for $\varepsilon = 0$ by many authors, including [10, Theorem 4.3.12], [24], [27]. The following lemma, which is necessary for our purpose, is a slight refinement of these results in normed spaces.

Lemma 13 *Let* $f : X \to \mathbb{R} \cup \{\infty\}$ *be a proper lsc convex function,* $x_0 \in X$*, and* $\delta \geq 0$*. For* $\varepsilon > 0$ *(resp.,* $\varepsilon \geq 0$ *and* X *is a Banach space) and* $x_0^* \in \partial_{\varepsilon+\delta} f(x_0)$*, there exist* $(z, z^*) \in \partial_\varepsilon f$ *and* $\lambda \in [-1, 1]$ *such that*

$$\left\| z^* - (1 + \sqrt{\delta}\lambda) x_0^* \right\| \leq \sqrt{\delta},$$

$$\|z - x_0\| + |\langle x_0^*, z - x_0\rangle| \leq \sqrt{\delta},$$

$$|\langle z^*, z - x_0\rangle| \le \delta + \sqrt{\delta},$$

$$f(z) - f(x_0) \le \delta + \sqrt{\delta}, \ \textit{and}\ f(x_0) - f(z) \le \delta + \sqrt{\delta} + \varepsilon.$$

Proof. Suppose $\delta > 0$. If $\varepsilon = 0$ and X is Banach, the conclusion is known (see, e.g., [29, Theorem 3.1.1]). So, let $\varepsilon > 0$ and $x_0^* \in \partial_{\varepsilon+\delta} f(x_0)$. Consider the function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ defined by $\varphi(x) := f(x) - \langle x_0^*, x\rangle + \sqrt{\delta}(\|x - x_0\| + |\langle x_0^*, x - x_0\rangle|)$. We denote

$$A := \{x \in X \mid \varphi(x) \le \varphi(x_0) - \varepsilon\}.$$

If A is empty, we get $\varphi(x_0) \le \varphi(x) + \varepsilon$ for all $x \in X$ and so (by the approximate chain rule [29, Theorem 3.1.1])

$$x_0^* \in \partial_\varepsilon \varphi(x_0) \subset \partial_\varepsilon f(x_0) + \sqrt{\delta}\partial_\varepsilon(\|\cdot - x_0\| + |\langle x_0^*, \cdot - x_0\rangle|)(x_0) \subset \partial_\varepsilon f(x_0) + \sqrt{\delta}(B_{X^*} + [-1, 1]x_0^*).$$

Therefore, there exist $\lambda \in [-1, 1]$ and $u^* \in B_{X^*}$ such that $z^* := (1 + \sqrt{\delta}\lambda)x_0^* + \sqrt{\delta}u^* \in \partial_\varepsilon f(x_0)$. Thus, the pair (x_0, z^*) yields the desired conclusion.

From now on, we suppose that $A \neq \emptyset$. Observe that since $x_0^* \in \partial_{\varepsilon+\delta} f(x_0)$ it follows

$$\varphi(x) \ge f(x) - \langle x_0^*, x\rangle \ge \varphi(x_0) - \varepsilon - \delta \quad \text{for all } x \in X, \tag{22}$$

implying that $\inf_A \varphi \in \mathbb{R}$. Then, there exists $z \in A$ such that

$$\varphi(z) \le \varphi(x) + \varepsilon \quad \text{for all } x \in A. \tag{23}$$

Observe that for $x \notin A$ it holds (recall that $z \in A$)

$$\begin{aligned}\varphi(x) &> \varphi(x_0) - \varepsilon \\ &\ge \varphi(z) = f(z) - \langle x_0^*, z\rangle + \sqrt{\delta}(\|z - x_0\| + |\langle x_0^*, z - x_0\rangle|).\end{aligned}$$

In other words, (23) holds for every $x \in X$ and, hence,

$$\theta \in \partial_\varepsilon \varphi(z) \subset \partial_\varepsilon f(z) - x_0^* + \sqrt{\delta}(B_{X^*} + [-1, 1]x_0^*).$$

Pick $\lambda \in [-1, 1]$ and $u^* \in B_{X^*}$ such that $z^* := (1 + \sqrt{\delta}\lambda)x_0^* + \sqrt{\delta}u^* \in \partial_\varepsilon f(z)$. In particular, we have that $\left\|z^* - (1 + \sqrt{\delta}\lambda)x_0^*\right\| \le \sqrt{\delta}$ so that the first two required properties of the lemma follow.

Now, by replacing x by x_0 in (23) and using the definition of A we get (by taking $x = x_\delta$ in (22))

$$f(z) - \langle x_0^*, z\rangle + \sqrt{\delta}(\|z - x_0\| + |\langle x_0^*, z - x_0\rangle|) \le f(x_0) - \langle x_0^*, x_0\rangle - \varepsilon \le f(z) - \langle x_0^*, z\rangle + \delta, \tag{24}$$

which implies that $\|z - x_0\| + |\langle z - x_0, x_0^*\rangle| \le \sqrt{\delta}$; that is, the second inequality of the lemma holds. On the other hand, invoking the definition of z^* we obtain

$$\begin{aligned}|\langle z^* - x_0^*, z - x_0\rangle| &= \sqrt{\delta}\,|\langle u^* + \lambda x_0^*, z - x_0\rangle| \\ &\le \sqrt{\delta}(\|z - x_0\| + |\langle z - x_0, x_0^*\rangle|) \le \delta\end{aligned}$$

and, hence $|\langle z^*, z - x_0\rangle| \le \delta + \sqrt{\delta}$, yielding the third inequality of the lemma. Finally, by using

(24) we get

$$f(z) - f(x_0) \leq \langle z^*, z - x_0\rangle - \varepsilon \leq \delta + \sqrt{\delta} - \varepsilon, \text{ and } f(x_0) - f(z) \leq \delta + \sqrt{\delta} + \varepsilon,$$

yielding the last property. ■

We give now the main theorem of this section, which directly follows from Theorem 3 and Lemma 15 below. It extends the results of [27] without assuming the reflexivity of the underlying space.

Theorem 14 *Let $f, g : X \to \mathbb{R}\cup\{+\infty\}$ be two lsc convex functions such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. *If X is a Banach space, the following assertions are equivalent*:

(i) $\overline{f^*\square g^*}^{w^*} = \overline{f^*\square g^*}^{\tau}$;

(ii) $\partial_\varepsilon(f+g)(x) = \underset{\substack{u \xrightarrow{f} x,\ v \xrightarrow{g} x \\ \langle\cdot, u-x\rangle,\ \langle\cdot, v-x\rangle \to 0}}{\tau\text{-}\limsup} \left(\bigcup_{\substack{\alpha+\beta=\varepsilon \\ \alpha,\beta\geq 0}} \partial_\alpha f(u) + \partial_\beta g(v) \right) \quad \forall x \in X,\ \varepsilon \geq 0;$

(iii) $\partial_\varepsilon(f+g)(x) = \underset{\substack{u \xrightarrow{f} x,\ v \xrightarrow{g} x \\ \langle\cdot, u-x\rangle,\ \langle\cdot, v-x\rangle \to 0}}{\tau\text{-}\limsup} \left(\bigcup_{\substack{\alpha+\beta=\varepsilon \\ \alpha,\beta\geq 0}} \partial_\alpha f(u) + \partial_\beta g(v) \right) \quad \forall x \in X,\ \varepsilon > 0;$

(iii) $\partial(f+g)(x) = \underset{\substack{u \xrightarrow{f} x,\ v \xrightarrow{g} x \\ \langle\cdot, u-x\rangle,\ \langle\cdot, v-x\rangle \to 0}}{\tau\text{-}\limsup} \left(\partial f(u) + \partial g(v)\right) \ \forall x \in X.$

Lemma 15 *Let $f, g : X \to \mathbb{R}\cup\{+\infty\}$ be two lsc convex functions such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. *Then, for every $x \in X$ and $\varepsilon > 0$ (resp., $\varepsilon \geq 0$ and X is a Banach space) the following equality holds,*

$$\underset{\substack{u \xrightarrow{f} x,\ v \xrightarrow{g} x \\ \langle\cdot, u-x\rangle,\ \langle\cdot, v-x\rangle \to 0}}{\tau\text{-}\limsup} \left(\bigcup_{\substack{\alpha+\beta=\varepsilon \\ \alpha,\beta\geq 0}} \partial_\alpha f(u) + \partial_\beta g(v) \right) = \bigcap_{\delta>\varepsilon} \operatorname{cl}^\tau \left(\bigcup_{\substack{\delta_1+\delta_2=\delta \\ \delta_1,\delta_2\geq 0}} \partial_{\delta_1} f(x) + \partial_{\delta_2} g(x) \right).$$

Proof. We fix $x \in X$ and $\varepsilon > 0$ (resp., $\varepsilon \geq 0$ and X is Banach).

To prove the inclusion "$\subset$" we pick x^* in the left-hand side and take the nets $(u_i, u_i^*) \subset \partial_{\alpha_i} f$ and $(v_i, v_i^*) \in \partial_{\beta_i} g$ such that $u_i^* + v_i^* \xrightarrow{\tau} x^*$, $u_i \xrightarrow{f} x$, $v_i \xrightarrow{g} x$, and $\lim_i \langle u_i^*, u_i - x\rangle = \lim_i \langle v_i^*, v_i - x\rangle = 0$, where $\alpha_i, \beta_i \geq 0$ and $\alpha_i + \beta_i = \varepsilon$. Let $\delta > \varepsilon$ and denote

$$\delta_{1,i} := \alpha_i + f(x) - f(u_i) + \langle u_i^*, u_i - x\rangle \ \text{ and } \delta_{2,i} := \delta - \delta_{1,i};$$

we may suppose that for all i it holds $\delta_{2,i} \geq \beta_i + g(x) - g(v_i) + \langle v_i^*, v_i - x\rangle$. So, since $u_i^* \in \partial_{\alpha_i} f(u_i)$ and $v_i^* \in \partial_{\beta_i} g(v_i)$ we have, for every $y \in X$,

$$\langle u_i^*, y - x\rangle \leq f(y) - f(x) + \alpha_i + f(x) - f(u_i) + \langle u_i^*, u_i - x\rangle \leq f(y) - f(x) + \delta_{1,i},$$

$$\langle v_i^*, y - x\rangle \leq g(y) - g(x) + \beta_i + g(x) - g(u_i) + \langle v_i^*, v_i - x\rangle \leq g(y) - g(x) + \delta_{2,i}.$$

Whence, $u_i^* \in \partial_{\delta_{1,i}} f(x)$ and $v_i^* \in \partial_{\delta_{2,i}} g(x)$, showing that x^* belongs to the right-hand side of the desired inclusion.

For the opposite inclusion we consider a net of positive integer numbers $(n_V)_{V\in\mathcal{N}(\theta)}$ converging to ∞, where $\mathcal{N}_\tau(\theta)$ is the set of closed convex symmetric neighborhoods of θ endowed with the usual partial order given by the reverse inclusion. We take x^* on the right-hand side and, for a given $V\in\mathcal{N}_\tau(\theta)$, choose n_V such that for all $n\geq n_V$ it holds

$$2\sqrt{\frac{1}{n}}B_{X^*}\subset\frac{1}{3}V,\ \sqrt{\frac{1}{n}}\lambda_n(x^*+\frac{1}{n}V)\subset\frac{1}{3}V,\ \left(1+\sqrt{\frac{1}{n}}\right)\frac{1}{n}\leq\frac{1}{3}.$$

We fix $n\geq\max\{n_V,3\}$. Then, we find $\delta_{1,n},\delta_{2,n}\geq 0$, with $\delta_{1,n}+\delta_{2,n}=\frac{1}{n}+\varepsilon$, $x_n^*\in\partial_{\delta_{1,n}}f(x)$, and $y_n^*\in\partial_{\delta_{2,n}}g(x)$ such that

$$x^*-x_n^*-y_n^*\in\frac{1}{n}V. \tag{25}$$

So, $(x_n^*,y_n^*)\in\partial_{\delta_{1,n}}f(x)\times\partial_{\delta_{2,n}}g(x)\subset\partial_{\varepsilon+\frac{1}{n}}\varphi(x,x)$, where $\varphi: X\times X\to\mathbb{R}\cup\{+\infty\}$ is the proper lsc convex function defined by $\varphi(u,v):=f(u)+g(v)$. We endow $X\times X$ with the norm $\|(u,v)\|:=\|u\|+\|v\|$ whose dual norm is $\|(u^*,v^*)\|=\max\{\|u^*\|,\|v^*\|\}$. Then, by Lemma 13, there exist $((u_n,v_n),(u_n^*,v_n^*))\in\partial_\varepsilon\varphi$ and $\lambda_n\in[-1,1]$ such that

$$\left\|u_n^*-\left(1+\sqrt{\frac{1}{n}}\lambda_n\right)x_n^*\right\|\leq\sqrt{\frac{1}{n}},\ \left\|v_n^*-\left(1+\sqrt{\frac{1}{n_n}}\lambda_n\right)y_n^*\right\|\leq\sqrt{\frac{1}{n}}, \tag{26}$$

$$\|u_n-x\|+|\langle u_n-x,x_n^*\rangle|\leq\sqrt{\frac{1}{n}},\ \|v_n-x\|+|\langle v_n-x,y_n^*\rangle|\leq\sqrt{\frac{1}{n}},$$

$$|\langle u_n-x,u_n^*\rangle|\leq\frac{1}{n}+\sqrt{\frac{1}{n}},\ |\langle v_n-x,v_n^*\rangle|\leq\frac{1}{n}+\sqrt{\frac{1}{n}},$$

$$f(u_n)+g(v_n)-f(x)-g(x)\leq\frac{1}{n}+\sqrt{\frac{1}{n}}.$$

Consequently, in view of the lsc of f and g, the last inequality implies that

$$\lim_V f(u_{n_V})=f(x),\ \lim_V g(v_{n_V})=g(x).$$

Moreover, by using the triangle inequality we get

$$\begin{aligned}u_n^*+v_n^*&=x_n^*+y_n^*+u_n^*-\left(1+\sqrt{\frac{1}{n}}\lambda_n\right)x_n^*+v_n^*-\left(1+\sqrt{\frac{1}{n}}\lambda_n\right)y_n^*+\sqrt{\frac{1}{n}}\lambda_n(x_n^*+y_n^*)\\&\in x_n^*+y_n^*+2\sqrt{\frac{1}{n}}B_{X^*}+\sqrt{\frac{1}{n}}\lambda_n(x_n^*+y_n^*)\\&\in x^*+2\sqrt{\frac{1}{n}}B_{X^*}+\sqrt{\frac{1}{n}}\lambda_n(x^*+\frac{1}{n}V)+\frac{1}{n}V\subset x^*+V.\end{aligned}$$

This shows that $u_{n_V}^*+v_{n_V}^*\to_\tau x^*$. To conclude the proof, it suffices to observe that

$$(u_{n_V}^*,v_{n_V}^*)\in\partial_\varepsilon\varphi(u_{n_V},v_{n_V})=\bigcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon\\ \varepsilon_1,\varepsilon_2\geq 0}}\partial_{\varepsilon_1}f(u_{n_V})\times\partial_{\varepsilon_2}g(v_{n_V}).$$

■

The following combines the last theorem and Corollary 2.

Corollary 16 *Let $f, g : X \to \mathbb{R}\cup\{+\infty\}$ be two lsc convex functions such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. *If X is a Banach space such that the closed unit ball of its dual is w^*-sequentially compact, then for every $x \in X$ and $\varepsilon \geq 0$ we have that*

$$\partial_\varepsilon(f+g)(x) = \underset{\substack{u \xrightarrow{f} x,\ v \xrightarrow{g} x \\ \langle \cdot, u-x\rangle,\ \langle \cdot, v-x\rangle \to 0}}{\sigma\text{-}\limsup} \left(\bigcup_{\substack{\alpha+\beta=\varepsilon \\ \alpha,\beta\geq 0}} \partial_\alpha f(u) + \partial_\beta g(v) \right);$$

that is, $x^* \in \partial_\varepsilon(f+g)(x)$ if and only if there are sequences $(u_n^*)_n$, $(v_n^*)_n \subset X^*$ such that $x^* = w^*\text{-}\lim_i u_n^* + v_n^*$ for $u_n^* \in \partial_{\alpha_n} f(u_n)$ and $v_n^* \in \partial_{\beta_n} g(v_n)$, where $u_n, v_n \in X$, $\alpha_n, \beta_n \geq 0$ satisfy $\alpha_n + \beta_n = \varepsilon$,

$$u_n \xrightarrow{f} x,\ v_n \xrightarrow{g} x,\ \lim_n \langle u_n^*, u_n - x\rangle = \lim_n \langle v_n^*, v_n - x\rangle = 0.$$

The following result gives the counterpart of the previous theorem in the bidual setting, without requiring the closure condition (7). Its proof follows from Theorem 7 and Lemma 15.

Theorem 17 *Let $f, g : X \to \mathbb{R}\cup\{+\infty\}$ be two lsc convex functions such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. *Then, for all $x^{**} \in X^{**}$ and $\varepsilon \geq 0$ we have*

$$X^* \cap \partial_\varepsilon(f^{**}+g^{**})(x^{**}) = X^* \bigcap \limsup_{\substack{u \xrightarrow{f^{**}} x^{**},\ v \xrightarrow{g^{**}} x^{**} \\ \langle \cdot, u-x^{**}\rangle,\ \langle \cdot, v-x^{**}\rangle \to 0}} \left(\bigcup_{\substack{\alpha+\beta=\varepsilon \\ \alpha,\beta\geq 0}} \partial_\alpha f^{**}(u) + \partial_\beta g^{**}(v) \right).$$

Moreover, when $x \in X$ and $\varepsilon > 0$ (resp., $\varepsilon \geq 0$ if X is a Banach space) this formula reads

$$X^* \cap \partial_\varepsilon(f^{**}+g^{**})(x) = \limsup_{\substack{u \xrightarrow{f} x,\ v \xrightarrow{g} x \\ \langle \cdot, u-x\rangle,\ \langle \cdot, v-x\rangle \to 0}} \left(\bigcup_{\substack{\alpha+\beta=\varepsilon \\ \alpha,\beta\geq 0}} \partial_\alpha f(u) + \partial_\beta g(v) \right).$$

In what follows, we use the set

$$\tau\text{-}\limsup_{y^*} \left\{ A^* y^* \mid z^* - y^* \to \theta,\ z^* \in \partial_\varepsilon h(v),\ v \ \xrightarrow{h}\ Ax,\ \langle z^*, v - Ax\rangle \to 0 \right\},$$

of elements of the form $\tau\text{-}\lim_i A^* y_i^*$ where $y_i^* - z_i^* \to \theta$ for some $z_i^* \in \partial_\varepsilon h(v_i)$, $v_i \xrightarrow{h} Ax$, and $\langle z_i^*, v_i - Ax\rangle \to 0$. When τ is the norm topology, we omit the reference to τ.

Corollary 18 *Let X be a reflexive Banach space, Y a Banach space, $A : X \to Y$ a continuous linear operator with adjoint A^*, and $h : Y \to \mathbb{R}\cup\{+\infty\}$ a lsc convex function such that $A^{-1}(\operatorname{dom} h) \neq \emptyset$. Then, for every $x \in X$ and $\varepsilon \geq 0$ it holds*

$$\partial_\varepsilon(h \circ A)(x) = \limsup_{y^*} \left\{ A^* y^* \mid z^* - y^* \to \theta,\ z^* \in \partial_\varepsilon h(v),\ v \ \xrightarrow{h}\ Ax,\ \langle z^*, v - Ax\rangle \to 0 \right\}.$$

Proof. We endow the product space $X \times Y$ with the box norm and define the functions $f, g :$

$X \times Y \to \mathbb{R} \cup \{+\infty\}$ by

$$f(x,y) := h(y), \quad g(x,y) := \mathrm{I}_{\mathrm{Gr}\, A}(x,y).$$

According to Example 1, we have that $\overline{f^* \Box g^*}^{w^*} = \overline{f^* \Box g^*}$, and so the desired conclusion follows by Theorem 14, in view of the following straightforward relationships

$$x^* \in \partial_\varepsilon (h \circ A)(x) \Longleftrightarrow (x^*, \theta) \in \partial_\varepsilon (f+g)(x, Ax),$$
$$(u^*, v^*) \in \partial_\delta \mathrm{I}_{\mathrm{Gr}\, A}(u, Au) \Longleftrightarrow u^* = -A^* v^*, \text{ and } \partial_\delta f(u,v) = \{\theta\} \times \partial_\delta h(v).$$

■

Remark 3 The corollary above can be stated without assuming the reflexivity assumption in terms of any topology $\tau := \tau_{X^*} \times \tau_{Y^*}$, being intermediate between the weak* and norm topology on $X^* \times Y^*$. Indeed, it suffices to observe that $\overline{A^* h^*}^{w^*} = \overline{A^* h^*}^{\tau_{X^*}}$ iff $\overline{f^* \Box g^*}^{w^*} = \overline{f^* \Box g^*}^{\tau}$ (see (14) for the definition of $A^* h^*$), thus, according to Theorem 3 and Lemma 15, iff

$$\partial_\varepsilon (h \circ A)(x) = \tau\text{-}\limsup_{y^* \in Y^*} \left\{ A^* y^* \mid z^* - y^* \to \theta,\ z^* \in \partial_\varepsilon h(v),\ v \overset{h}{\to} Ax,\ \langle z^*, v - Ax \rangle \to 0 \right\}.$$

On the other hand, if instead of the reflexivity assumption in the previous corollary we assume that the dual of X has a w*-sequentially compact unit ball, then for every $x \in X$ and $\varepsilon \geq 0$ we obtain that

$$\partial_\varepsilon (h \circ A)(x) = \sigma\text{-}\limsup_{y^* \in Y^*} \left\{ A^* y^* \mid z^* - y^* \to \theta,\ z^* \in \partial_\varepsilon h(v),\ v \overset{h}{\to} Ax,\ \langle z^*, v - Ax \rangle \to 0 \right\}.$$

In general, when no extra assumption is used, according to Theorem 17 the subdifferential formulas are given in the bidual setting:

Corollary 19 *Let X and Y be two Banach spaces, $A : X \to Y$ a continuous linear operator, and $h : Y \to \mathbb{R} \cup \{+\infty\}$ a lsc convex function such that $A^{-1}(\operatorname{dom} h) \neq \emptyset$. Then, for every $x^{**} \in X^{**}$ and $\varepsilon \geq 0$ we have that*

$$X^* \cap \partial_\varepsilon (h^{**} \circ A^{**})(x^{**}) = \limsup_{\xi \in Y^{***}} \{ A^{***}\xi \mid \zeta - \xi \to \theta,\ \zeta \in \partial_\varepsilon h^{**}(v^{**}),$$
$$v^{**} \overset{h^{**}}{\to} A^{**}x^{**},\ \langle \zeta, v^{**} - A^{**}x^{**} \rangle \to 0 \Big\}.$$

In particular, when $x \in X$ we get

$$X^* \cap \partial_\varepsilon (h^{**} \circ A^{**})(x) = \limsup_{y^* \in Y^*} \{ A^* y^* \mid z^* - y^* \to \theta,\ z^* \in \partial_\varepsilon h(v),$$
$$v \overset{h}{\to} Ax,\ \langle z^*, v - Ax \rangle \to 0 \Big\}.$$

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