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A differential equation approach to implicit sweeping processes ☆

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Abstract

In this paper, we study an implicit version of the sweeping process. Based on methods of convex analysis, we prove the equivalence of the implicit sweeping process with a differential equation, which enables us to show the existence and uniqueness of the solution to the implicit sweeping process in a very general framework. Moreover, this equivalence allows us to give a characterization of nonsmooth Lyapunov pairs and invariance for implicit sweeping processes. The results of the paper are illustrated with two applications to quasistatic evolution variational inequalities and electrical circuits.
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1. Introduction

The Moreau's sweeping process is the following first-order differential inclusion involving normal cones to time-dependent moving sets:

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$$\begin{cases} -\dot{x}(t) \in N(C(t); x(t)) & \text{a.e. } t \in [0,T], \\ x(0) = x_0 \in H, \end{cases}$$

where $C\colon [0,T] \rightrightarrows H$ is a set-valued map defined from $[0,T]$ ($T > 0$) to a separable Hilbert space H with nonempty, closed and convex values. Here $N(C(t); \cdot)$ denotes the outward normal cone, in the sense of convex analysis, to the moving set $C(t)$. Roughly speaking, a point is swept by a moving closed set.

The sweeping process was introduced and deeply studied by J.-J. Moreau in a series of papers (see [1–4]) to model an elasto-plastic mechanical system. Since then, many other applications have been given, such as applications in switched electrical circuits [5], nonsmooth mechanics [6,7], crowd motion [8], hysteresis in elasto-plastic models [9], among others. Moreover, due to the development of new techniques to deal with differential inclusions involving normal cones, new variants of the sweeping process have been introduced. We can mention the state-dependent sweeping process, the second sweeping process, and some other variants (see [10] and the references therein).

This paper aims to study the following variant of the sweeping process:

$$\begin{cases} \dot{x}(t) \in f(t, x(t)) - N(C(t); A\dot{x}(t) + h(t, x(t))) & \text{a.e. } t \in [0,T], \\ x(0) = x_0 \in H, \end{cases} \tag{1}$$

where $C\colon [0,T] \rightrightarrows H$ is a set-valued map defined from $[0,T]$ ($T > 0$) to a separable Hilbert space H with nonempty, closed and convex values, $N(C(t); \cdot)$ denotes the outward normal cone, in the sense of convex analysis, to the set $C(t)$, $A\colon H \to H$ is a bounded, linear and symmetric operator and h and f are two given functions. We called the differential inclusion (1) as Implicit Sweeping Process because the velocity is implicitly defined in the dynamical system.

An early version of the implicit sweeping process was studied in [11], where the author proved the existence and uniqueness of solutions for the following implicit differential inclusion:

$$\begin{cases} -\dot{x}(t) \in N(C(t); \dot{x}(t)) & \text{a.e. } t \in [0,T], \\ x(0) = x_0 \in H. \end{cases} \tag{2}$$

Here the moving sets are assumed to be r-uniformly prox-regular. However, the author did not realize that problem (2) has an explicit solution (as long as, 0 belongs to the r prox-regularity neighborhood of the moving set $C(t)$ and the map $t \mapsto \operatorname{proj}_{C(t)}(0)$ is integrable) given by

$$x(t) = x_0 + \int_0^t \operatorname{proj}_{C(s)}(0)\,ds \quad t \in [0,T].$$

Next, the implicit sweeping process was considered (in a simpler way than (1)) in [12], where the authors proved the existence and uniqueness of solutions by using an adapted version of Moreau's catching-up algorithm. Moreover, in [12], the authors give applications of the implicit sweeping process to quasistatic variational inequalities.

In this paper, we aim to extend the results of [12] to a more general framework by using tools from differential equations and convex analysis. Indeed, under general assumptions, we show the equivalence of the implicit sweeping process (1) with a differential equation for which the

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existence can be obtained by using classical methods. This equivalence result is central in this work because it allows, on the one hand, to apply classical existence theorems from differential equations to study the dynamical system (1) and, on the other hand, to apply the Lyapunov method to the differential inclusion (1).

The Lyapunov method is an important approach to deal with the stability of dynamical systems. This indirect method allows addressing several stability properties of dynamical systems as finite or asymptotic stability, the existence of equilibria, stabilization, among others (see [13–15]). The idea behind the Lyapunov method consists in constructing a pair of functions, called Lyapunov pair, which constitute a kind of energy of the system which decreases along its solutions. Moreover, since in complex real-world applications, it is not possible to find explicit solutions, it is imperative to have explicit characterizations of Lyapunov pairs for dynamical systems. While the initial Lyapunov method was developed for smooth functions, it became clear the necessity of considering nonsmooth Lyapunov pairs. This is mainly due to the flexibility of working with nonsmooth functions and to the unfortunate fact that some dynamical systems do not admit smooth Lyapunov functions (see [15]). To pass from smooth to nonsmooth Lyapunov functions several notions of directional derivatives where used in the past (see [16, Chapter 6]). Since directional derivatives are naturally associated with subdifferentials, other authors started to use subgradients and subdifferentials. In this context, among all these subdifferentials, the use of the proximal subdifferential became a benchmark because it allows characterizing nonsmooth Lyapunov pairs for differential inclusions (see [13]). In fact, it is recognized that the proximal subdifferential is the smallest reasonable subdifferential that allows characterizations of nonsmooth Lyapunov pairs. In this work, we follow this path and give an explicit characterization, involving proximal subdifferentials, of Lyapunov pairs for the dynamical systems (1). This characterization is used to give a criterion for invariance of closed sets for the implicit sweeping process.

The paper is organized as follows. After some preliminaries, in Section 3 we gather the hypotheses used along the paper. Then, in Section 4, we prove the equivalence of the implicit sweeping process with a differential equation, and then we prove existence and uniqueness of solutions for the implicit sweeping process. In Section 5, we prove the existence of solutions for a state-dependent version of the implicit sweeping process. In Section 6, we give a characterization of nonsmooth Lyapunov pairs for the implicit sweeping process. Sections 7 and 8 are dedicated to giving applications in quasistatic evolution variational inequalities and nonsmooth electrical circuits, respectively. The paper ends with conclusions and final remarks.

2. Notation and preliminaries

Let H be a separable Hilbert space endowed with a scalar product $\langle\cdot,\cdot\rangle$ and unit ball $\mathbb{B}$. Given a closed and convex set $S\subset H$ we define the convex normal cone to S at $x\in S$ as

$$N(S;x):=\{\zeta\in H:\ \langle\zeta,y-x\rangle\leq 0\ \text{ for all } y\in S\}.$$

For a closed and convex set $S\subset H$, we consider the distance function d_S and the projection over S as the maps

$$d_S(x):=\inf_{y\in S}\|y-x\| \quad \text{and} \quad \operatorname{proj}_S(x):=\{y\in H:\, d_S(x)=\|x-y\|\}.$$

It is not difficult to prove that for S convex the map $x\mapsto d_S^2(x)$ is differentiable with

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$$\nabla d_S^2(x) = 2\left(x - \text{proj}_S(x)\right) \quad \text{for all } x \in H.$$

Moreover, for S convex the following inclusion holds

$$x - \text{proj}_S(x) \in N\left(S; \text{proj}_S(x)\right) \quad \text{for all } x \in H. \tag{3}$$

The last inclusion shows that the map $x \mapsto (I + N(S;\cdot))^{-1}(x)$ is single-valued and coincides with the projection onto the closed convex set S, that is,

$$(I + N(S;\cdot))^{-1}(x) = \text{proj}_S(x) \quad \text{for all } x \in H.$$

Given two closed set $A, B \subset H$, we define its Hausdorff distance as

$$d_H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{x\in B} d(x, A)\}.$$

It is well known (see [17, Chapter 3]) that the Hausdorff distance between two sets can be represented in the following way:

$$d_H(A, B) = \sup_{x\in H} |d(x, A) - d(x, B)|.$$

Moreover, for two closed and convex sets $A, B \subset H$ the following inequality holds (see [3, Formula 2.17]):

$$\left\|\text{proj}_A(x) - \text{proj}_B(y)\right\|^2 \le \|x - y\|^2 + 2\left[d(x, A) + d(y, B)\right] d_H(A, B). \tag{4}$$

The next proposition will be used in the proof of Theorem 4.2.

Lemma 2.1. *Assume that the following conditions hold:*

1. *For all $(t, x) \in [0, T] \times H$ the set $C(t, x)$ is closed and convex.*
2. *For all $x \in H$, the set-valued map $t \rightrightarrows C(t, x)$ is measurable.*
3. *For all $t \in [0, T]$ there exists a continuous function $g_t\colon H \to \mathbb{R}$ such that*

$$\sup_{z\in H} |d(z, C(t, x)) - d(z, C(t, y))| \le |g_t(x) - g_t(y)| \quad x, y \in H.$$

Then the following assertions hold:

(i) *For all $x, y \in H$, the map $t \mapsto \text{proj}_{C(t,x)}(y)$ is measurable.*
(ii) *For all $t \in [0, T]$ and $y \in H$, the map $x \mapsto \text{proj}_{C(t,y)}(x)$ is Lipschitz of constant 1.*
(iii) *For all $t \in [0, T]$ and $y \in H$, the map $x \mapsto \text{proj}_{C(t,x)}(y)$ is continuous.*

Proof. Since the sets $C(t, x)$ are convex, the projection is single-valued and well defined. The first assertion follows from [18, Theorem III.41]. The second and third assertion are a consequence of formula (4). □

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In Section 5, we will establish an existence result for an implicit state-dependent sweeping process under some compactness conditions on the moving sets. These conditions are quantified by using the notion of measure of noncompactness. Let A be a bounded subset of H. We define the Kuratowski measure of noncompactness of A, $\alpha(A)$, as

$$\alpha(A) = \inf\{d > 0\colon A \text{ admits a finite cover by sets of diameter } \leq d\},$$

and the Hausdorff measure of non-compactness of A, $\beta(A)$, as

$$\beta(A) = \inf\{r > 0\colon A \text{ can be covered by finitely many balls of radius } r\}.$$

The following result gives the main properties of the Kuratowski and Hausdorff measure of non-compactness (see [19, Proposition 9.1 from Section 9.2]).

Proposition 2.2. *Let H be a Hilbert space and B, B_1, B_2 be bounded subsets of H. Let γ be the Kuratowski or the Hausdorff measure of non-compactness. Then,*

(i) $\gamma(B) = 0$ *if and only if* $\mathrm{cl}(B)$ *is compact.*
(ii) $\gamma(\lambda B) = |\lambda|\gamma(B)$ *for every* $\lambda \in \mathbb{R}$.
(iii) $\gamma(B_1 + B_2) \leq \gamma(B_1) + \gamma(B_2)$.
(iv) $B_1 \subset B_2$ *implies* $\gamma(B_1) \leq \gamma(B_2)$.
(v) $\gamma(\mathrm{conv}\, B) = \gamma(B)$.
(vi) $\gamma(\mathrm{cl}(B)) = \gamma(B)$.

We end this section by recalling the notion of the proximal subdifferential needed in Section 6. Let $f\colon H \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and $x \in \mathrm{dom}\, f$. An element ζ belongs to the *proximal subdifferential* $\partial^P f(x)$ of f at x (see [13, Chapter 1]) if there exist two positive numbers σ and η such that

$$f(y) \geq f(x) + \langle \zeta, y - x\rangle - \sigma\|y - x\|^2 \quad \forall y \in B(x; \eta).$$

We define the *proximal normal cone* to a closed set S, denoted by $N^P(S; \cdot)$, as the proximal subdifferential of the indicator function I_S, that is,

$$N^P(S; x) := \partial_P I_S(x) \quad \text{for all } x \in S.$$

Finally, we recall the following property (see [13, Chapter 1]):

$$\zeta \in \partial^P f(x) \quad \Leftrightarrow \quad (\zeta, -1) \in N^P(\mathrm{epi}\, f; (x, f(x))).$$

3. Technical assumptions

For the sake of readability, in this section, we collect the hypotheses used in the paper.

Hypotheses on the operator $A\colon H \to H$**:** We consider the following conditions on the operator A:

$(\mathcal{H}_A)$ $A\colon H \to H$ is a linear, bounded and symmetric operator such that $A = P^*P$ for some invertible operator P (P^* is the adjoint operator of P).

Hypotheses on the set-valued map $C\colon [0,T] \rightrightarrows H$**:** C is a set-valued map with nonempty, closed and convex values. Moreover, we consider the following conditions:

$(\mathcal{H}_C)$ The set-valued map $C\colon [0,T] \rightrightarrows H$ is measurable and the function $t \mapsto d_{C(t)}(0)$ is integrable.

Hypotheses on the set-valued map $C\colon [0,T] \times H \rightrightarrows H$**:** C is a set-valued map with nonempty, closed and convex values. Moreover, we consider the following conditions:

$(\mathcal{H}_{Cx})$ (a) For all $x \in H$, the set-valued map $t \rightrightarrows C(t,x)$ is measurable and there exists $\mu \in L^1(0,T)$ such that for a.e. $t \in [0,T]$ and all $x \in H$

$$d_{C(t,x)}(0) \leq \mu(t)\left(\|x\| + 1\right).$$

(b) For all $t \in [0,T]$, there exists a continuous function $g_t\colon H \to \mathbb{R}$ such that for all $x, y \in H$

$$d_H(C(t,x), C(t,y)) := \sup_{z \in H} |d(z, C(t,x)) - d(z, C(t,y))| \leq |g_t(x) - g_t(y)|.$$

(c) There exists $k \in L^1(0,T)$ such that for every $t \in [0,T]$, every $r > 0$ and every bounded set $A \subset H$

$$\gamma\left(C(t,A) \cap r\mathbb{B}\right) \leq k(t)\gamma(A),$$

where $\gamma = \alpha$ or $\gamma = \beta$ is either the Kuratowski of the Hausdorff measure of non-compactness (see Proposition 2.2).

Hypotheses on the maps $h, f\colon [0,T] \times H \to H$**:** We consider the following conditions on the mappings h, f.

$(\mathcal{H}_{h,f})$ (a) For all $x \in H$, the maps $t \mapsto h(t,x)$, $t \mapsto f(t,x)$ are measurable.
(b) For a.e. $t \in [0,T]$ and all $x, y \in H$

$$\max\{\|h(t,x) - h(t,y)\|, \|f(t,x) - f(t,y)\|\} \leq \kappa(t)\|x - y\|,$$

where $\kappa \in L^1(0,T)$.
(c) For a.e. $t \in [0,T]$ and all $x, y \in H$

$$\max\{\|h(t,x)\|, \|f(t,x)\|\} \leq a(t)\|x\| + b(t),$$

where $a, b \in L^1(0,T)$.

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4. Existence and uniqueness of solutions for implicit sweeping process

In this section, we present an existence and uniqueness result for the implicit sweeping process (1). Our approach consists in transforming the differential inclusion (1) into a differential equation. Indeed, we show that the differential inclusion (1) is equivalent to the following differential equation:

$$\begin{cases} \dot{x}(t) = -P^{-1}Qh(t,x(t)) \\ \qquad\qquad + P^{-1}\operatorname{proj}_{QC(t)}(Pf(t,x(t)) + Qh(t,x(t))) \quad \text{a.e. } t \in [0,T], \\ x(0) = x_0 \in H, \end{cases} \tag{5}$$

where $A = P^*P$ and $Q := (P^*)^{-1}$. The following result states this equivalence.

Proposition 4.1. *Assume, in addition to $(\mathcal{H}_A)$, that $C(t)$ is closed and convex for all $t \in [0,T]$. Then x is a solution of the differential inclusion (1) if and only if it is a solution of the differential equation (5).*

Proof. Before starting the proof, we observe that under hypothesis $(\mathcal{H}_A)$, the following formula holds:

$$\zeta \in N(C(t); A\dot{x}(t) + h(t,x(t))) \Leftrightarrow P\zeta \in N(QC(t); P\dot{x}(t) + Qh(t,x(t))), \tag{6}$$

where $Q := (P^*)^{-1}$ is well defined. Let x be a solution of (1). Then, according to formula (6),

$$Pf(t,x(t)) + Qh(t,x(t)) \in (I + N(QC(t);\cdot))(P\dot{x}(t) + Qh(t,x(t))) \quad \text{a.e. } t \in [0,T].$$

Moreover, since $QC(t)$ is convex, we have that $(I + N(QC(t);\cdot))^{-1} = \operatorname{proj}_{QC(t)}$. Therefore,

$$P\dot{x}(t) + Qh(t,x(t)) = \operatorname{proj}_{QC(t)}(Pf(t,x(t)) + Qh(t,x(t))) \quad \text{a.e. } t \in [0,T],$$

which proves that x is a solution of (5). Reciprocally, let x be a solution of (5). Then,

$$\begin{aligned} P\dot{x}(t) &= -Qh(t,x(t)) + \operatorname{proj}_{QC(t)}(Pf(t,x(t)) + Qh(t,x(t))) \\ &\in -N\left(QC(t); \operatorname{proj}_{QC(t)}(Pf(t,x(t)) + Qh(t,x(t)))\right) + Pf(t,x(t)) \\ &= -N(QC(t); P\dot{x}(t) + Qh(t,x(t))) + Pf(t,x(t)), \end{aligned}$$

where we have used the inclusion (3). Therefore,

$$P(-\dot{x}(t) + f(t,x(t))) \in N(QC(t); P\dot{x}(t) + Qh(t,x(t))) \quad \text{a.e. } t \in [0.T],$$

which proves, according to formula (6), that x is a solution of (1). □

From Proposition 4.1, we are able to state the main result of this section, that is, the existence of solutions for the implicit sweeping process. It is worth noting that, contrary to the existence results for the sweeping process, it is not assumed that the variation of the moving sets is Lipschitz with respect to the Hausdorff distance.

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Theorem 4.2. *Assume that* $(\mathcal{H}_A)$, $(\mathcal{H}_C)$ *and* $(\mathcal{H}_{h,f})$ *hold. Then for any initial point* $x_0 \in H$ *there exists a unique absolutely continuous mapping* $x\colon [0,T] \to H$ *satisfying (1), i.e.,*

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); A\dot{x}(t) + h(t,x(t))\right) + f(t,x(t)) & a.e.\ t \in [0,T], \\ x(0) = x_0. \end{cases}$$

Proof. Let us consider the map

$$F(t,x) := -P^{-1}Qh(t,x) + P^{-1}\operatorname{proj}_{QC(t)}(Pf(t,x) + Qh(t,x))(t,x) \in [0,T] \times H.$$

First, as a consequence of $(\mathcal{H}_{h,f})$, $(\mathcal{H}_C)$ and Lemma 2.1, for all $x \in H$, the map $t \mapsto F(t,x)$ is measurable. Second, there exists $\alpha \in L^1(0,T)$ such that for a.e. $t \in [0,T]$ and all $x, y \in H$

$$\|F(t,x) - F(t,y)\| \le \alpha(t)\|x - y\|.$$

Indeed, for a.e. $t \in [0,T]$ and $x, y \in H$,

$$\begin{aligned} \|F(t,x) - F(t,y)\| &\le \|P^{-1}Q\| \cdot \|h(t,x) - h(t,y)\| \\ &\quad + \|P^{-1}\| \cdot (\|Pf(t,x) - Pf(t,y)\| + \|Qh(t,x) - Qh(t,y)\|) \\ &\le \alpha(t)\|x - y\|, \end{aligned}$$

where α is the integrable function defined by $\alpha(t) := \|P^{-1}\|\,(\|P\| + 2\|Q\|\|)\,\kappa(t)$.

Third, there exist $c, d \in L^1(0,T)$ such that for a.e. $t \in [0,T]$ and all $x \in H$

$$\|F(t,x)\| \le c(t)\|x\| + d(t).$$

Indeed, according to $(\mathcal{H}_{h,f})$ and $(\mathcal{H}_C)$,

$$\begin{aligned} \|F(t,x)\| &\le \|P^{-1}\| d_{QC(t)}(Pf(t,x) + Qh(t,x)) + \|f(t,x)\| \\ &\le \|P^{-1}\| \cdot \|Pf(t,x) + Qh(t,x)\| + \|P^{-1}\| d_{QC(t)}(0) + \|f(t,x)\| \\ &\le \|P^{-1}\| \cdot (\|P\| + \|Q\|)\,(a(t)\|x\| + b(t)) + \|P^{-1}\| \cdot \|Q\| d_{C(t)}(0) \\ &\quad + a(t)\|x\| + b(t) \\ &\le c(t)\|x\| + d(t), \end{aligned}$$

where c and d are the integrable functions defined by:

$$\begin{aligned} c(t) &:= \left(\|P^{-1}\|\|P\| + \|P^{-1}\|Q\| + 1\right) a(t), \\ d(t) &:= \|P^{-1}\|\left(\|P\| + \|Q\| + \|Q\| d_{C(t)}(0)\right) + b(t). \end{aligned}$$

Therefore, according to [19, Theorem 10.5], the differential equation (5) has a unique solution defined on $[0,T]$. Finally, the existence and uniqueness of solutions for the implicit sweeping process (1) comes from Proposition 4.1. □

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Remark 4.1.

1. Theorem 4.2 holds for a time-dependent, linear, bounded and symmetric operator $A(t)$ as long the operator $t \mapsto A(t)$ is measurable and hypothesis $(\mathcal{H}_A)$ holds with uniformly bounded matrices $P(t)$ and $P^{-1}(t)$.
2. Assume that $A = \lambda I$ and $h = I$, where I denotes the identity operator. Then, (5) is equivalent to

$$\dot{x}(t) = \frac{1}{\lambda}\left(-x(t) + \text{proj}_{C(t)}\left(x(t) + \lambda f(t, x(t))\right)\right), \tag{7}$$

which gives an alternative regularization scheme of Moreau–Yosida type for the sweeping process. Moreover, by the methods developed in [20], it is possible to prove that the unique solution x_λ of (7) converges, as $\lambda \to 0^+$, to the unique solution of the sweeping process

$$\dot{x}(t) \in -N(C(t); x(t)) + f(t, x(t)) \quad \text{a.e. } t \in [0, T].$$

Therefore, the sweeping process can be seen as a limit case of the implicit sweeping process.

5. Existence for implicit state-dependent sweeping process

In this section, we prove the existence of solutions for a state-dependent version of the differential inclusion (1). The interest in considering a state-dependent version of (1) comes from the fact that it can be useful to model quasistatic frictional contact problem where the surface and traction forces also depend on the displacements (see Section 4 from [12] and the references therein).

The analysis in the state-dependent case is more complicated and, such as in the state-dependent sweeping process, we need to assume a certain compactness condition in the moving sets, namely, $(\mathcal{H}_{Cx})$(c) (see, e.g., [21,22]).

Theorem 5.1. *Assume that $(\mathcal{H}_A)$, $(\mathcal{H}_{Cx})$ and $(\mathcal{H}_{h,f})$ hold. Then, for any initial point $x_0 \in H$, there exists at least one absolutely continuous mapping $x\colon [0, T] \to H$ such that*

$$\begin{cases} \dot{x}(t) \in f(t, x(t)) - N(C(t, x(t)); A\dot{x}(t) + h(t, x(t))) & \textit{a.e. } t \in [0, T], \\ x(0) = x_0. \end{cases} \tag{8}$$

Proof. The proof consists of two steps:

1. Prove the existence of solutions for the differential equation

$$\dot{x}(t) = F(t, x(t)), \tag{9}$$

where F is the mapping defined for $(t, x) \in [0, T] \times H$ by the formula

$$F(t, x) := -P^{-1}Qh(t, x) + P^{-1}\,\text{proj}_{QC(t,x)}\left(Pf(t, x) + Qh(t, x)\right)$$

2. Show that any solution of (9) is also a solution of (8).

Step 1.: First, as a consequence of $(\mathcal{H}_{h,f})$, $(\mathcal{H}_{Cx})$ and Lemma 2.1, for all $x \in H$, the map $t \mapsto F(t,x)$ is measurable and for every $t \in [0,T]$, $F(t,\cdot)$ is continuous. Second, from hypotheses $(\mathcal{H}_A)$, $(\mathcal{H}_{Cx})$ and $(\mathcal{H}_{h,f})$, it is not difficult to prove that for a.e. $t \in [0,T]$ and all $x \in H$

$$\| \operatorname{proj}_{QC(t,x)}(Pf(t,x) + Qh(t,x))\| \leq \tilde{a}(t)\|x\| + \tilde{b}(t), \tag{10}$$

for some $\tilde{a}, \tilde{b} \in L^1(0,T)$. Third, for a.e. $t \in [0,T]$ and every $A \subset H$ bounded,

$$\gamma(F(t,A)) \leq 2\left(\kappa(t) + k(t)\right)\|P^{-1}\| \cdot \|Q\| \cdot \gamma(A),$$

where $\gamma = \alpha$ or $\gamma = \beta$ is either the Kuratowski or the Hausdorff measure of non-compactness. Indeed, fix $A \subset H$ bounded and $r > 0$ such that $A \subset r\mathbb{B}$. Then, by virtue of Proposition 2.2,

$$\begin{aligned}
\gamma(F(t,A)) &\leq \gamma\left(P^{-1}Qh(t,A)\right) + \gamma\left(P^{-1}\operatorname{proj}_{QC(t,A)}\left(Pf(t,A) + Qh(t,A)\right)\right) \\
&\leq 2\|P^{-1}\| \cdot \|Q\|\gamma(h(t,A)) \\
&\quad + 2\|P^{-1}\|\gamma\left(\operatorname{proj}_{QC(t,A)}\left(Pf(t,A) + Qh(t,A)\right) \cap \left(\tilde{a}(t)r + \tilde{b}(t)\right)\mathbb{B}\right) \\
&\leq 2\|P^{-1}\| \cdot \|Q\|\kappa(t)\gamma(A) + 2\|P^{-1}\|\gamma\left(QC(t,A) \cap \left(\tilde{a}(t)r + \tilde{b}(t)\right)\mathbb{B}\right) \\
&\leq 2\|P^{-1}\| \cdot \|Q\|\kappa(t)\gamma(A) + 2\|P^{-1}\| \cdot \|Q\|\gamma\left(C(t,A)\right) \\
&\leq 2\left(\kappa(t) + k(t)\right)\|P^{-1}\| \cdot \|Q\| \cdot \gamma\left(A\right),
\end{aligned}$$

where we have used (10), $(\mathcal{H}_{h,f})$, $(\mathcal{H}_{Cx})$, and the following property: if g is a Lipschitz function of constant L and $A \subset H$, then

$$\gamma\left(g(A)\right) \leq 2L\gamma(A).$$

Finally, all the conditions of [23, Theorem 2] are satisfied. Therefore, the differential equation $\dot{x}(t) \in F(t,x(t))$ has at least one absolutely continuous solution x.

Step 2.: It remains to prove that x is a solution of (8). Indeed, for a.e. $t \in [0,T]$

$$\begin{aligned}
P\dot{x}(t) &= PF(t,x(t)) \\
&= -Qh(t,x(t)) + \operatorname{proj}_{QC(t,x(t))}\left(Pf(t,x(t)) + Qh(t,x(t))\right) \\
&\in -N\left(QC(t,x(t)); \operatorname{proj}_{QC(t,x(t))}\left(Pf(t,x(t)) + Qh(t,x(t))\right)\right) + Pf(t,x(t)) \\
&= -N\left(QC(t,x(t)); P\dot{x}(t) + Qh(t,x(t))\right) + Pf(t,x(t))
\end{aligned}$$

where we have used the inclusion (3). Therefore,

$$P\dot{x}(t) \in -N\left(QC(t,x(t)); P\dot{x}(t) + Qh(t,x(t))\right) + Pf(t,x(t)) \quad \text{a.e. } t \in [0.T].$$

Thus, according to formula (6), x is a solution of (8), which ends the proof. □

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Remark 5.1.

1. Proposition 4.1 also holds for the implicit state-dependent sweeping process, that is, any solution of (8) is also a solution of (9) and viceversa.
2. Theorem 5.1 holds for a time-dependent, linear, bounded and symmetric operator $A(t)$ as long the operator $t \mapsto A(t)$ is measurable and hypothesis $(\mathcal{H}_A)$ holds with uniformly bounded matrices $P(t)$ and $P^{-1}(t)$.
3. Assume that $A = \lambda I$ and $h = I$, where I denotes the identity operator. Then, (8) corresponds to

$$\dot{x}(t) = \frac{1}{\lambda}\left(-x(t) + \text{proj}_{C(t,x(t))}\left(x(t) + \lambda f(t, x(t))\right)\right), \tag{11}$$

which gives an alternative regularization scheme of Moreau–Yosida type for the state-dependent sweeping process. Moreover, by the methods developed in [24], it is possible to prove that a family of solutions x_λ of (11) converges (up to a subsequence) to a solution of the state-dependent sweeping process

$$\dot{x}(t) \in -N\left(C(t, x(t)); x(t)\right) + f(t, x(t)) \quad \text{a.e. } t \in [0, T].$$

Therefore, the state-dependent sweeping process can be seen as a limit case of the implicit state-dependent sweeping process.

6. Lyapunov stability and applications

In this section, we give a characterization of nonsmooth Lyapunov pairs for the dynamical system (1). This characterization is possible due to the equivalence of the implicit sweeping process (1) with the differential equation (5).

We start by giving the notion of Lyapunov pairs (see [13, Chapter 4]) for the implicit sweeping process (1).

Definition 6.1. Let $V : \mathbb{R}_+ \times H \to \mathbb{R} \cup \{+\infty\}$ and $W : [0, T] \times H \to \mathbb{R}$ be two proper and lower semicontinuous functions. We say that (V, W) forms a *Lyapunov pair* for the implicit sweeping process (1) if for every $t_0 \geq 0$ and $x_0 \in H$ the solution x of (1) is such that

$$V(t, x(t)) + \int_{t_0}^{t} W(s, x(s))ds \leq V(t_0, x_0) \quad \text{for all } t \geq t_0.$$

Moreover, we say that V is a *Lyapunov function* for (1) if $(V, 0)$ is a Lyapunov pair for (1).

The next theorem gives a characterization of Lyapunov pairs for the implicit sweeping process (1).

Theorem 6.1. *Let $V : \mathbb{R}_+ \times H \to \mathbb{R} \cup \{+\infty\}$ be a proper and lower semicontinuous function and $W : \mathbb{R}_+ \times H \to \mathbb{R} \cup \{+\infty\}$ be a Lipschitz function. Assume, in addition to $(\mathcal{H}_A)$ and $(\mathcal{H}_{h,f})$, that there exists $k \geq 0$ such that*

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$$d_H(C(t), C(s)) \leq k|t-s| \quad \text{for all } s, t \in [0, T].$$

Then, the following assertions are equivalent:

(i) *For all* $(\theta, \zeta) \in \partial^P V(t,x)$

$$\theta + \left\langle \zeta, -P^{-1}Qh(t,x) + P^{-1}\operatorname{proj}_{QC(t)}(Pf(t,x) + Qh(t,x)) \right\rangle + W(t,x) \leq 0.$$

(ii) (V, W) *forms a Lyapunov pair for the implicit sweeping process (1).*

Proof. It is a consequence of Proposition 4.1 and the characterizations of Lyapunov pairs for differential equations (see [13, Chapter 4]). □

As a direct consequence, we obtain the following characterization of invariance for (1).

Corollary 6.2. *Let* $K : [0, T] \rightrightarrows H$ *be a set-valued map with nonempty and closed values. Assume, in addition to* $(\mathcal{H}_A)$ *and* $(\mathcal{H}_{h,f})$, *that there exists* $k \geq 0$ *such that*

$$d_H(C(t), C(s)) \leq k|t-s| \quad \text{for all } s, t \in [0, T].$$

Then, the following conditions are equivalent:

- *For all* $(\theta, \zeta) \in N^P(\operatorname{graph} K, (t,x))$

$$\theta + \left\langle \zeta, -P^{-1}Qh(t,x) + P^{-1}\operatorname{proj}_{QC(t)}(Pf(t,x) + Qh(t,x)) \right\rangle \leq 0.$$

- *For every* $(t_0, x_0) \in \operatorname{graph} K$ *there exists a unique solution* x *of (1) such that* $x(t) \in K(t)$ *for all* $t \geq t_0$.

Proof. It is enough to apply Theorem 6.1 to the functions $V(t,x) := I_{\operatorname{graph} K}(t,x)$ and $W \equiv 0$. □

7. Quasistatic evolution variational inequalities

In this section, as a consequence of Theorem 4.2, we obtain the existence and uniqueness of solutions for quasistatic evolution variational inequalities. Our results generalize those from [12]. Indeed, contrary to [12], we do not assume any compatibility condition on the initial data and we only need the measurability on the map $t \rightrightarrows C(t)$ (see Remark 7.1). Moreover, our approach enables us to transform the quasistatic evolution variational inequality into a differential equation. This is of great value because it allows applying all the machinery of differential equations instead more complicated methods for differential inclusions.

The following result, which is a direct consequence of Theorem 4.2, improves the main result of [12].

Theorem 7.1. *Assume, in addition to* $(\mathcal{H}_A)$ *and* $(\mathcal{H}_C)$, *that* $B : H \to H$ *is a linear and bounded operator. Then for any initial point* $x_0 \in H$ *there exists a unique absolutely continuous mapping* $x : [0, T] \to H$ *satisfying*

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); A\dot{x}(t) + Bx(t)\right) & a.e.\ t \in [0,T], \\ x(0) = x_0. \end{cases} \tag{12}$$

Moreover, if the map $C\colon [0,T] \rightrightarrows H$ is continuous with respect to the Hausdorff distance, then the solution x is continuously differentiable.

Remark 7.1. It is worth emphasizing some features of Theorem 7.1 that contrast with the results of [12].

1. In Theorem 7.1 we do not need to assume the compatibility condition $Bx_0 \in C(0)$. Moreover, if this compatibility condition holds and the map $t \mapsto C(t)$ is continuous then, it is possible to show from the differential equation formulation that $\dot{x}(0) = 0$, which is a restrictive condition.
2. In Theorem 7.1 we do not need to assume that the variation of the moving sets $C(t)$ is Lipschitz. Moreover, if $C(t) := f(t) - C$ for some function $f\colon [0,T] \to H$ and a closed convex set C (which appears in the study of some quasistatic variational inequalities; see Proposition 7.2), then $(\mathcal{H}_C)$ holds if and only if $f \in L^1([0,T];H)$.

As an application of Theorem 7.1, we study the following evolution variational inequality: Find $x\colon [0,T] \to H$ such that $\dot{x}(t) \in K$ a.e. $t \in [0,T]$ and

$$\begin{cases} a(\dot{x}(t), y - \dot{x}(t)) + b(x(t), y - \dot{x}(t)) + j(y) - j(\dot{x}(t)) \\ \qquad\qquad \geq \langle f(t), y - \dot{x}(t)\rangle \quad \text{for all } y \in K, \\ x(0) = x_0 \in H. \end{cases} \tag{13}$$

The problem (13) was studied in [12], where the authors transform, for the first time, the evolution variational inequality (13) into an implicit sweeping process. We extend this approach by transforming (13) into a differential equation. We assume that the following conditions are satisfied:

- $(\mathcal{HV}_1)$ $K \subset H$ is a nonempty, closed, and convex cone.
- $(\mathcal{HV}_2)$ $a(\cdot,\cdot), b(\cdot,\cdot)\colon H \times H \to \mathbb{R}$ are two real continuous bilinear and symmetric forms such that for all $u \in H$ the condition $a(u,u) \geq \alpha\|u\|^2$ holds, for some $\alpha > 0$.
- $(\mathcal{HV}_3)$ $j\colon K \to \mathbb{R}$ is a convex, positively homogeneous of degree 1 (i.e., $j(\lambda x) = \lambda j(x)$ for all $\lambda > 0$) and Lipschitz continuous with $j(0) = 0$.
- $(\mathcal{HV}_4)$ The function f belong to $L^1([0,T];H)$.

As mentioned in [12], the evolution variational inequality (13) is of great interest in the modeling of quasistatic frictional contact problems. In a mechanical language, the bilinear form $a(\cdot,\cdot)$ represents the viscosity term, the bilinear form $b(\cdot,\cdot)$ represents the elasticity term, and the functional j represents the friction functional of Tresca type.

In order to convert the evolution variational inequality (13) into the differential inclusion (12), let us first extend the function j from K to the whole space H by introducing the functional $J\colon H \to \mathbb{R} \cup \{+\infty\}$ defined by

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$$J(z)=\begin{cases} j(z), & z\in K,\\ +\infty, & z\notin K.\end{cases} \tag{14}$$

Since K is a nonempty, closed, and convex cone, and j is convex, positively homogeneous of degree 1 and Lipschitz continuous on K, the extended functional $J\colon H\to\mathbb{R}\cup\{+\infty\}$ is proper, positively homogeneous of degree 1, convex and lower semicontinuous with $J(0)=0$. With this extension, (13) is equivalent to: Find $x\colon[0,T]\to H$ such that for a.e. $t\in[0,T]$ we have

$$\begin{cases} a(\dot x(t),y-\dot x(t))+b(x(t),y-\dot x(t))+J(y)-J(\dot x(t))\\ \qquad\qquad \geq\langle f(t),y-\dot x(t)\rangle \quad \text{for all } y\in H,\\ x(0)=x_0\in H.\end{cases} \tag{15}$$

Let A and B be the linear bounded and symmetric operators associated, respectively, with the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, that is,

$$a(x,y)=\langle Ax,y\rangle \text{ and } b(x,y)=\langle Bx,y\rangle \quad \text{for all } x,y\in H.$$

Using the definition of convex subdifferential, we can rewrite (15) in the following form:

$$\begin{cases} f(t)-A\dot x(t)-Bx(t)\in\partial J(\dot x(t)) \quad \text{a.e. } t\in[0,T],\\ x(0)=x_0\in H.\end{cases}$$

The following result, based on [12, Proposition 4.2], shows the equivalence between (12) and the quasistatic variational inequality (13).

Proposition 7.2. *Assume that* $(\mathcal{HV}_1)$–$(\mathcal{HV}_4)$ *hold. The function* $x\colon[0,T]\to H$ *is a solution of (13) if and only if it is a solution of the differential inclusion (12), where* A *and* B *are the linear bounded and symmetric operators associated with* $a(\cdot,\cdot)$ *and* $b(\cdot,\cdot)$*, and* $C(t)=f(t)-\partial J(0)$*,* $t\in[0,T]$ *with* J *defined in (14).*

Thus, as a consequence of Theorem 7.1, we have the following existence and uniqueness result for the quasistatic variational inequality (13), which improves [12, Proposition 4.3] and [25, Theorem 4.1].

Corollary 7.3. *Assume that* $(\mathcal{HV}_1)$–$(\mathcal{HV}_4)$ *hold. Then for each* $x_0\in H$*, the evolution variational inequality (13) has a unique solution* x.

Proof. It is clear that $(\mathcal{HV}_2)$ implies $(\mathcal{H}_A)$. Moreover, since $C(t)=f(t)-\partial J(0)$, hypothesis $(\mathcal{HV}_4)$ is equivalent to $(\mathcal{H}_C)$. Therefore, all the assumptions of Theorem 7.1 hold. □

Remark 7.2. Corollary 7.3 greatly extends [12, Corollary 4.3]. Indeed, here f is only integrable and we do not need the compatibility condition

$$b(x_0,v)+j(y)\geq\langle f(0),y\rangle \quad \text{for all } y\in K$$

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on the initial state. Moreover, as a direct application of Corollary 7.3, it is possible to improve [12, Corollary 4.5] related to the existence and uniqueness for a contact problem involving viscoelastic materials with short memory. We refer to Example 4.1 from [12] for the details.

8. Application to nonsmooth electrical circuits

In this section, we give an application of Theorem 4.2 to nonsmooth electrical circuits. The following presentation is strongly based on [26].

Let us consider the electric circuit shown in Fig. 1 that is composed of two resistors $R_1 > 0$, $R_2 > 0$ with voltage/current laws $V_{R_k} = R_k x_k$ $(k = 1, 2)$, three capacitors C_1, C_2 with voltage/current laws $V_{C_k} = \frac{1}{C_k}\int x_k(t)dt$, $k = 1, 2$ and two ideal diodes with characteristics $0 \le -V_{D_k} \perp i_k \ge 0$. Using Kirchhoff's laws, we have

$$V_{R_1} + V_{C_1} + V_{C_2} = -V_{D_1} \in -N(\mathbb{R}_+; x_1 - c)$$
$$V_{R_2} + V_{C_1} - V_{C_2} = -V_{D_2} \in -N(\mathbb{R}_+; x_2).$$

Therefore the dynamics of this circuit is given by

$$\overbrace{\begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}}^{A_1}\overbrace{\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}}^{\dot{q}} + \overbrace{\begin{pmatrix} \frac{1}{C_1} + \frac{1}{C_2} & -\frac{1}{C_2} \\ -\frac{1}{C_2} & \frac{1}{C_1} + \frac{1}{C_2} \end{pmatrix}}^{A_0}\overbrace{\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}}^{q} \in -N(C(t); \dot{q}(t)), \tag{16}$$

with $C(t) = [c(t), +\infty) \times [0, +\infty)$ and $\dot{q}(t) = x_i(t)$, $i = 1, 2$.

In order to apply Theorem 4.2, let us consider P be a invertible matrix such that $A_1 = P^t P$. Then, (16) is equivalent to

$$\dot{q}(t) = P^{-1}\operatorname{proj}_{PC(t)}(-QA_0 q(t)),$$

where $Q = \left(P^t\right)^{-1}$ (this equivalence was already noted in [26]).

The following result is a variant of [26, Theorem 5.1], where the existence of solutions for (17) is addressed for bounded moving sets with continuous variation, a continuous mapping $f : [0, T] \to H$ and a positive semi-definite operator A_1.

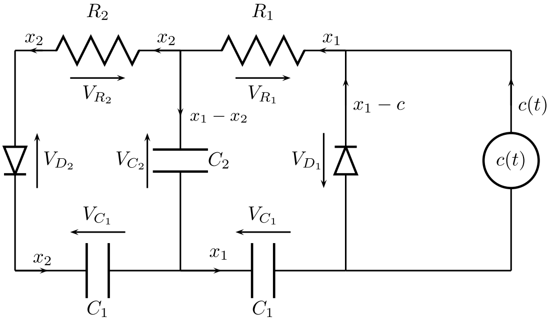

Fig. 1. Electrical circuit with resistors, capacitors and ideal diodes (RCD).

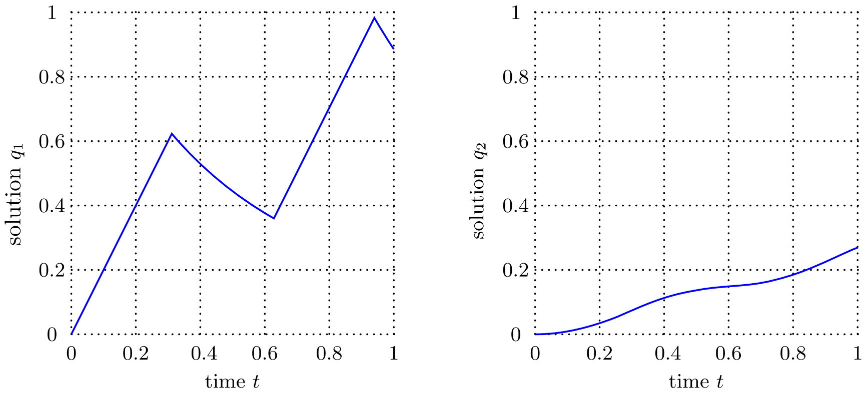

Fig. 2. Numerical solutions (q_1, q_2) for the system (16) with $R_1 = R_2 = C_1 = C_2 = 1$ and a discontinuous current source $c(t) = 2\,\text{sign}\,(\sin(10t))$.

Theorem 8.1. *Let $A_0, A_1 \colon H \to H$ be two bounded symmetric linear operator with A_1 symmetric and positive definite and let $f \colon [0, T] \to H$ be a integrable mapping. Assume that $C \colon [0, T] \rightrightarrows H$ is a measurable set-valued map with nonempty, closed and convex values. Then, for any initial point $x_0 \in H$, the evolution variational inequality*

$$\begin{cases} A_1\dot{x}(t) + A_0 x(t) - f(t) \in -N\left(C(t); \dot{x}(t)\right) & \text{a.e. } t \in [0, T], \\ x(0) = x_0 \in H, \end{cases} \tag{17}$$

admits a unique absolutely continuous solution $x \colon [0, T] \to H$.

Remark 8.1. It is worth to emphasize that in Theorem 8.1 the set-valued map $C \colon [0, T] \rightrightarrows H$ is only assumed to be measurable with $t \mapsto d_{C(t)}(0)$ integrable. To illustrate this, let us consider the discontinuous current source given by $c(t) = 2\,\text{sign}\,(\sin(10t))$, then it is observed that the solution (q_1, q_2) is absolutely continuous (see Fig. 2).

9. Conclusions and final remarks

In this paper, we have studied the well-posedness (existence and uniqueness of solutions) for an implicit version of the sweeping process. Our approach consists in transforming the implicit sweeping process into a differential equation for which the existence and uniqueness of solutions can be obtained through classical results of differential equations in Hilbert spaces. This approach is quite practical because, in general, it is easier to deal with differential equations than with implicit differential inclusions governed by normal cones. Moreover, we have used this equivalence to study Lyapunov pairs for the implicit sweeping process, which show the efficiency of this approach. Then, we give two applications of implicit sweeping processes to quasistatic evolution variational inequalities (Section 7) and electrical circuits (Section 8), respectively. This paper improves known results in the literature and, in particular, proposes a method to deal with some differential variational inequalities which could be used to deal with optimal control, sensitivity, stabilization, and other developments. We hope that the results of this paper shed lights into the study of more complicated differential variational inequalities arising in mechanical problems (for example, with history-dependent operators).

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