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Efficiency for continuous facility location problems with attraction and repulsion

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Abstract The paper deals with the problem of locating new facilities in presence of attracting and repulsive demand points in a continuous location space. When an arbitrary norm is used to measure distances and with closed convex constraints, we develop necessary conditions of efficiency. In the unconstrained case and if the norm derives from a scalar product, we completely characterize strict and weak efficiency and prove that the efficient set coincides with the strictly efficient set and/or coincides with the weakly efficient set. When the convex hulls of the attracting and repulsive demand points do not meet, we show that the three sets coincide with a closed convex set for which we give a complete geometrical description. We establish that the convex hulls of the attracting and repulsive demand points overlap iff the weakly efficient set is the whole space and a similar result holds for the efficient set when we replace the convex hulls by their relative interiors. We also provide a procedure which computes, in the plane and with a finite number of demand points, the efficient sets in polynomial time. Concerning constrained efficiency, we show that the process of projecting unconstrained weakly efficient points on the feasible set provides constrained weakly efficient points.

Keywords Continuous location · Efficient solutions · Attracting and repulsive demand points

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1 Introduction

The problem of locating desirable facilities such as schools, hospitals, fire stations has been extensively studied. A large overview on this question can be found in Plastria (1993, 1995), Love et al. (1988).

However with people getting more and more concerned about their living environment and its impact on health and safety, the undesirable effects of certain types of facilities cannot be left aside. Placing a factory, an airport, hazardous facilities, power plants, chemical factories, dump sites, etc, close by users may cause damages to the quality of their life due to noise, traffic, risk and pollution. At the same time one cannot afford to select certain sites too far away from the population zones. Unfortunately, the objectives of locating a facility close to certain demand points and far from others are conflicting. The difficulties in selecting unanimously approved sites are commonly raised (see Erkut and Neuman 1989) and leads to consider models which combine attraction and repulsion forces. For the last twenty years a lot of such models have been studied in a variety of metric spaces. Concerning one-objective continuous location problems, see e.g. Brimberg and Juel (1998a, 1998b), Carrizosa and Plastria (1999), Chen et al. (1992), Drezner and Wesolowsky (1990), Hansen et al. (1981), Melachrinoudis and Cullinane (1985), Melachrinoudis and Xanthopulos (2003), Muñoz-Pérez and Saameno-Rodriguez (1999), Plastria (1991, 1996), Plastria and Carrizosa (1999), Ratick and White (1988), Romero-Morales et al. (1997), Saameño-Rodríguez et al. (to appear), Tamir (2006), Tellier and Polanski (1989).

Only a limited number of papers deal with multiple criteria in a continuous setting. Most of them addressed a bi-criteria problem which is to locate a semi-obnoxious facility with the two objectives of maximizing a utility function which measures the benefits provided by the facility and of minimizing the undesirable effects induced, see Blanquero and Carrizosa (2002), Brimberg and Juel (1998c), Carrizosa et al. (1997), Carrizosa and Plastria (2000), Melachrinoudis (1999), Melachrinoudis and Xanthopulos (2003), Ohsawa (2000), Ohsawa et al. (to appear), Ohsawa and Tamura (2003), Skriver and Andersen (2003), Yapicioglu et al. (2004, 2006). These authors propose different solution methods as heuristic and/or branch and bound based approaches, solutions methods using Voronoi diagrams, or scalarization based techniques which consist in transforming the problem in a single-objective one. All the approaches followed exploit the bi-criteria aspect of the problem.

In this paper we focus on the problem of locating a facility in presence of attracting and repulsive demand points. We consider a multiple criteria framework where the objectives consist in simultaneously minimizing the distance to the attracting demand points and maximizing the distance to the repulsive ones. We assume that the distances to the demand points are measured by a unique norm function. The paper addresses the case of a regional demand. We are thus, a priori, faced with an infinite number of criteria except if the demand is concentrated on a finite number of points. Regional constraints can also be imposed on the location of the facility. We study the properties of the classical concepts of non dominated solutions, referred to in the literature as weakly efficient, efficient and strictly efficient solutions. Our approach is not based on scalarization techniques. As in Durier (1987, 1990), Durier and Michelot (1986), Ndiaye and Michelot (1998), Ndiaye (1996), we follow a completely different way which consists in exploiting in depth the geometrical properties of the norm.

With an arbitrary norm and in presence of closed convex constraints, we give necessary conditions of strict efficiency, efficiency and weak efficiency. These conditions involve the geometry of the unit ball of the norm and clearly show that for a facility to be efficient it is required that the attracting and repulsive demand points are conveniently dispatched around it in a certain sense revealed by the geometry of the unit ball of the norm.

In the unconstrained case and if the norm derives from a scalar product, we develop specific and additional properties. In particular, we completely characterize strict and weak efficiency. This result was somewhat unexpected because, in general, the non convexities induced by the repulsive demand points prevent from obtaining necessary and sufficient optimality conditions. We show that the efficient set coincides with the strictly efficient set and/or coincides with the weakly efficient set. As a consequence, we obtain that the three sets of efficiency are closed and convex. When the convex hulls of the attracting and repulsive demand points do not meet, we show that, as in the pure attracting setting, the three sets of efficiency coincide with a closed convex set for which we give a complete geometrical description. This common set is unbounded provided that there is at least one repulsive demand point. In the pure attractive setting, we rediscover that the strictly efficient, efficient, and weakly efficient sets coincide with the convex hull of the demand points. We establish that the convex hulls of the attracting and repulsive demand points overlap iff the weakly efficient set is the whole space and that a similar result holds for the efficient set when we replace the convex hulls by their relative interiors. We also show that the process of projecting (unconstrained) weakly efficient points on the feasible set provides constrained weakly efficient points. We terminate by providing a procedure which generates, for a finite set of demand points in the plane, the efficient sets in polynomial time.

Definitions, notations and first properties of efficiency are given in Sect. 2. In Sect. 3, we consider the general case where distances are measured by an arbitrary norm. We provide necessary conditions for efficiency in presence of (closed convex) regional constraints. Section 4 is devoted to the particular case where the norm used to measure the distances derives from a scalar product. The paper ends with a conclusion in Sect. 5.

2 Notations, definitions and first properties

Throughout the paper, (X, γ) is a real normed space. Though a part of the results obtained remain valid in infinite dimension, we assume that X is a finite dimensional space. For convenience the reader may assume that $X = \mathbb{R}^n$.

For a subset $\Omega \subset X$ we denote respectively by $\mathrm{int}(\Omega)$, $\mathrm{ri}(\Omega)$, $\mathrm{cl}(\Omega)$ and by $\mathrm{co}(\Omega)$, the interior, the relative interior, the closure and the convex hull of Ω. The cone generated by the origin and Ω is defined by

$$\mathrm{cone}(\Omega) := \{\lambda x\colon\ x \in \Omega, \lambda \geq 0\}.$$

If Ω is closed and convex and if $x \in \Omega$, we denote by

$$F_\Omega(x) := \{\delta \neq 0\colon\ x + \lambda\delta \in \Omega \text{ for some } \lambda > 0\}$$

the set of feasible directions at $x \in \Omega$. For $\delta \in F_\Omega(x)$, we set

$$\lambda_\delta = \sup\{\lambda > 0\colon\ x + \lambda\delta \in \Omega\}.$$

In the sequel we will consider two sets A^+ and A^- of attracting and repulsive demand points respectively. For technical convenience we will assume that A^+ and A^- are compact and non simultaneously empty sets.

According to our aim which is to simultaneously minimize distances to attracting demand points and maximize the distances to the repulsive demand points, we introduce the following concepts of dominance.

Definition 2.1 We say that a point $y \neq x$ dominates a point x when $\gamma(y-a^+) \leq \gamma(x-a^+)$ for all $a^+ \in A^+$ and $\gamma(y-a^-) \geq \gamma(x-a^-)$ for all $a^- \in A^-$. In case where at least one of the previous inequalities is strict the dominance is called strict. If all the inequalities involved are strict, we say that y strongly dominates x.

Each of these dominance notions induces a set of non dominated solutions known as the set of weakly efficient, efficient and strictly efficient points. We recall the formal definitions.

Definition 2.2 Let $\Omega \subset X$ be a nonempty set of constraints. A point $x \in \Omega$ is said to be strictly efficient (with respect to A^+, A^- and Ω) if there is no feasible point $y \neq x$ which dominates x. Similarly we say that a feasible point x is efficient (resp. weakly efficient) if there is no $y \in \Omega$ which strictly (resp. strongly) dominates x.

The sets of strictly efficient, efficient and weakly efficient points will be respectively denoted by $SE(A^+, A^-, \Omega)$, $E(A^+, A^-, \Omega)$ and $WE(A^+, A^-, \Omega)$.

In absence of constraint, we use the simplified notation $SE(A^+, A^-)$, $E(A^+, A^-)$ and $WE(A^+, A^-)$. We also set $SE(A^+) := SE(A^+, \emptyset)$, $SE(A^-) := SE(\emptyset, A^-)$, $E(A^+) := E(A^+, \emptyset)$, $E(A^-) := E(\emptyset, A^-)$, $WE(A^+) := WE(A^+, \emptyset)$ and $WE(A^-) := WE(\emptyset, A^-)$.

Below we give without proof a list of straightforward properties, direct consequence of the definitions.

Proposition 2.1 *Let $\Omega \subset \Omega' \subset X$ be two nonempty sets of constraints. Then the following properties hold*:

1. $SE(A^+) \subset SE(A^+, A^-)$, $E(A^+) \subset E(A^+, A^-)$ $WE(A^+) \subset WE(A^+, A^-)$.
2. *If* $A^+ \cap A^- \neq \emptyset$ *then* $WE(A^+, A^-, \Omega) = \Omega$.
3. $A^+ \cap \Omega \subset SE(A^+, A^-, \Omega) \subset E(A^+, A^-, \Omega) \subset WE(A^+, A^-, \Omega)$.
4. $SE(A^+, A^-, \Omega') \cap \Omega \subset SE(A^+, A^-, \Omega)$, $E(A^+, A^-, \Omega') \cap \Omega \subset E(A^+, A^-, \Omega)$, $WE(A^+, A^-, \Omega') \cap \Omega \subset WE(A^+, A^-, \Omega)$.
5. $SE(A^+, A^-) \cap \Omega \subset SE(A^+, A^-, \Omega)$, $E(A^+, A^-) \cap \Omega \subset E(A^+, A^-, \Omega)$ $WE(A^+, A^-) \cap \Omega \subset WE(A^+, A^-, \Omega)$.

Note that in a pure repulsive setting without constraint, i.e. if $A^+ = \emptyset$ and $\Omega = X$, the efficient points are, let say, "pushed to infinity" so that we have $SE(A^-) = E(A^-) = WE(A^-) = \emptyset$. In presence of constraints, the repulsive demand points, contrary to the feasible attracting ones, may not be strictly efficient, efficient or weakly efficient. As illustration, consider one attracting demand point a, one repulsive demand point b and the set of constraint $\Omega := [a, b]$.

3 Necessary conditions for efficiency in terms of Q_δ sets

In pure attracting models, characterizations of strict efficient, efficient and weak efficient solutions have been obtained in terms of Q_δ sets (Durier 1990; Durier and Michelot 1986; Ndiaye and Michelot 1998; Ndiaye 1996). These sets, induced by a norm (see Durier and Michelot 1986) and closely related to the geometry of the unit ball $B := \{x \in X\colon \gamma(x) \leq 1\}$, play also an important role to give necessary conditions for efficiency in presence of repulsive demand points.

Definition 3.1 (Durier and Michelot 1986) For any $\delta \in X$, $\delta \neq 0$, we consider the two complementary sets

$$Q_\delta := \{z \in X\colon \forall \lambda > 0,\ \gamma(z - \lambda\delta) > \gamma(z)\},$$
$$P_\delta := \{z \in X\colon \exists \lambda > 0,\ \gamma(z - \lambda\delta) \leq \gamma(z)\}.$$

It can be easily shown that Q_δ and P_δ are cones. For a given direction $\delta \neq 0$, Q_δ is exactly the cone generated by the points x of the unit sphere $S := \{x \in X\colon \gamma(x) = 1\}$ such that we leave the unit ball B when moving from x in the direction $-\delta$. Note that several sets Q_δ may overlap and that the whole family of Q_δ sets cover the whole space. Topological and geometrical properties of the sets Q_δ have been studied in detail in Durier and Michelot (1986). Let us recall the main results.

When the norm derives from a scalar product denoted by $\langle \cdot, \cdot \rangle$, we have $Q_\delta = \mathrm{cl}(P_{-\delta}) = \{x;\ \langle x, \delta \rangle \leq 0\}$ and $P_\delta = \{x;\ \langle x, \delta \rangle > 0\}$. Thus the family of Q_δ sets (resp. P_δ sets) is made up of the closed (resp. open) half-spaces passing through the origin.

In dimension two, the Q_δ sets are closed convex cones:

- If the norm is polyhedral, its unit ball generates two types of Q_δ sets. If the direction δ is not parallel to a one-dimensional face (a one-face in short) of the unit ball, Q_δ is a half-plane generated by two opposite extreme points. If the unit ball has $2p$ extreme points, the pairs of two opposite extreme points generate $2p$ sets Q_δ. When the direction δ is parallel to a one-face of the unit ball, Q_δ is a closed pointed cone generated by the origin and $p - 1$ consecutive one-faces of the unit ball. We have thus $2p$ such sets Q_δ.
- With the rectilinear norm we get eight Q_δ sets, the four quarter of planes $\mathbb{R}^+ \times \mathbb{R}^+$, $\mathbb{R}^+ \times \mathbb{R}^-$, $\mathbb{R}^- \times \mathbb{R}^-$, and $\mathbb{R}^- \times \mathbb{R}^+$, and the four half-planes $\mathbb{R} \times \mathbb{R}^+$, $\mathbb{R} \times \mathbb{R}^-$, $\mathbb{R}^+ \times \mathbb{R}$ and $\mathbb{R}^- \times \mathbb{R}$.
- With the ℓ^∞-norm we get again four quarter of plane and four half-planes obtained by rotating the Q_δ sets of the ℓ^1 norm counterclockwise through a $\pi/4$ angle.

In dimension $n > 2$, some Q_δ sets may not be convex as those associated to the ℓ^1 and ℓ^∞ norms for which a complete description can be found in Durier and Michelot (1986). With exotic norms, the Q_δ sets may also not be closed. An example is given in Durier and Michelot (1986). The closure is guaranteed when the norm is B-regular.

Definition 3.2 A norm γ is called B-regular (i.e. regular in the sense of Brown 1964) if for each direction $\delta \neq 0$ and for each x such that $\gamma(x - \delta) \leq \gamma(x)$, there exists some $\lambda > 0$ and a neighborhood W of x such that $\gamma(y - \lambda\delta) \leq \gamma(y)$ for every $y \in W$.

The B-regularity property is not very restrictive. It is satisfied by strictly convex norms, by polyhedral norms in any dimension, and by any norm in dimension two (Durier and Michelot 1986).

We terminate by several topological properties. Due to the symmetry of the unit ball of a norm we have a relationship between each Q_δ set and the set $P_{-\delta}$. More precisely we have $\mathrm{int}(Q_\delta) \subset P_{-\delta}$, and the inclusion may be strict. For a given direction $\delta \neq 0$, if the unit sphere S does not contain a line segment parallel to δ, as for strictly convex norms, then $Q_\delta = \mathrm{cl}(P_{-\delta})$. The following result provides several useful characterizations of the interior of a Q_δ set. A proof can be found in Durier (1987, 1990), Durier and Michelot (1986) or Ndiaye and Michelot (1998).

Proposition 3.1 *For each $\delta \in X$, $\delta \neq 0$, we have the following equivalences*:

(i) $x \in \mathrm{int}(Q_{-\delta})$;
(ii) $x \notin \mathrm{cl}(P_{-\delta})$;
(iii) $\exists\lambda > 0,\ \gamma(x - \lambda\delta) < \gamma(x)$.

Let us now give necessary conditions for weak efficiency, efficiency and strict efficiency.

Proposition 3.2 *Let $\Omega \subset X$ be a nonempty closed convex set of constraints and $x \in WE(A^+, A^-, \Omega)$. Then, for each feasible direction $\delta \in F_\Omega(x)$ at least one of the two following properties is satisfied*:

(i) $A^+ \cap (x + \mathrm{cl}(P_{-\delta})) \neq \emptyset$;
(ii) $A^- \cap (x + P_\delta) \neq \emptyset$.

Proof Suppose that there exists a feasible direction $\delta \in F_\Omega(x)$ such that $A^+ \cap (x + \mathrm{cl}(P_{-\delta})) = \emptyset$ and $A^- \cap (x + P_\delta) = \emptyset$. By Proposition 3.1, for each attracting demand point $a \in A^+$ there exists some $\lambda_a > 0$ such that $\gamma(a - x) > \gamma(a - x - \lambda_a\delta)$. The norm γ being continuous, this strict inequality still holds in a neighborhood, say $V(a)$, of a. Using the convexity of the norm, we deduce that $\gamma(a' - x) > \gamma(a' - x - \lambda\delta)$ for each $a' \in V(a)$ and each λ such that $0 < \lambda \leq \lambda_a$. The set A^+ being compact, one can select a finite subset $A_+ \subset A^+$ of attracting demand points such that A^+ is covered by the neighborhoods $V(a),\ a \in A_+$. Putting $\overline{\lambda} = \min\{\min\{\lambda_a,\ a \in A_+\},\ \lambda_\delta\}$, it follows that

$$\forall a \in A^+, \quad \gamma(a - x) > \gamma(a - x - \overline{\lambda}\delta).$$

Since it is assumed that the set A^- of repulsive demand points does not meet $x + P_\delta$, by definition of P_δ we also have

$$\forall a \in A^-, \quad \gamma(a - x - \overline{\lambda}\delta) > \gamma(a - x).$$

Thus, x is strongly dominated by the point $y := x + \overline{\lambda}\delta$ which is feasible by construction, and we conclude that $x \notin WE(A^+, A^-, \Omega)$. □

Proposition 3.2 shows that, for a point $x \in \Omega$ to be weakly efficient, it is necessary that the attracting and repulsive demand points are conveniently dispatched around x relative to the constraints in the sense that each $\mathrm{cl}(P_{-\delta})$ generated by a feasible direction should contain at least one attractive demand point and each P_δ should contain at least one repulsive demand point. Unfortunately, and contrary to the pure attracting setting, the conditions are not sufficient.

When the norm γ is strictly convex, the sets $\mathrm{cl}(P_{-\delta})$ and Q_δ coincide as already mentioned (see Durier and Michelot 1986, Corollary 1.2). So Proposition 3.2 can be rewritten in a more simple way as follows.

Corollary 3.1 *Let γ be a strictly convex norm, $\Omega \subset X$ be a nonempty closed convex set of constraints and $x \in WE(A^+, A^-, \Omega)$. Then, for each feasible direction $\delta \in F_\Omega(x)$ at least one of the two following properties is satisfied*:

(i) $A^+ \cap (x + Q_\delta) \neq \emptyset$;
(ii) $A^- \cap (x + P_\delta) \neq \emptyset$.

In the case of non constrained problems involving the Euclidean norm, and as a consequence of Proposition 3.2, we see that the points belonging to an hyperplane which strictly separates A^+ and A^- cannot be weakly efficient because the attracting (resp. repulsive) demand points are all located in a half-space passing through x, and thus are not conveniently dispatched around x. This observation will be developed in the next section to give a complete characterization of weak efficiency when distances are measured by the norm induced by a scalar product.

Proposition 3.2 provides necessary conditions which are not sufficient as illustrated by the following example. Consider in the plane $X = \mathbb{R}^2$ the sets $A^+ := \{a^+\}$ and $A^- := \{a^-\}$ with $a^+ = (-1, 0)$, $a^- = (1, 0)$ and assume that the distances are measured by the Euclidean norm. The Q_δ sets are thus the half-planes limited by a line passing through the origin. If no constraint is involved, the point $x_0 = (2, 0)$ is strongly dominated by $y_0 = (-2, 0)$. Then $x_0 \notin WE(A^+, A^-, \Omega)$. However any half-plane limited by a line passing through x_0 cannot strictly separate a^+ and a^- because x_0, a^+ and a^- are aligned. Note that a point x which is located outside the x-axis is not weakly efficient because the half-planes containing a^- and limited by a line passing through x and a point y situated between a^+ and a^- neither satisfy condition (i) nor condition (ii).

The two next results give similar necessary conditions for efficiency and strict efficiency. Their justifications being very close, we only give a proof for efficient points. According to the characterizations of efficiency and strict efficiency in a pure attracting setting, as developed in Durier and Michelot (1986), Ndiaye and Michelot (1998), Ndiaye (1996), these results require the norm be B-regular.

Proposition 3.3 *Let $\Omega \subset X$ be a nonempty closed convex set of constraints and assume that the norm γ is B-regular. If $x \in SE(A^+, A^-, \Omega)$ and $\delta \in F_\Omega(x)$, at least one of the two following properties is satisfied*:

(i) $A^+ \cap (x + Q_\delta) \neq \emptyset$;
(ii) $A^- \cap (x + \operatorname{int}(Q_{-\delta})) \neq \emptyset$.

Again, these conditions mean that the attracting and repulsive demand points should be well dispatched around x. Note however that the conditions are more demanding than the conditions for weak efficiency because for a given direction $\delta \neq 0$ we have $Q_\delta \subset \operatorname{cl}(P_{-\delta})$ and $\operatorname{int}(Q_{-\delta}) \subset P_{-\delta}$.

Proposition 3.4 *Let $\Omega \subset X$ be a nonempty closed convex set of constraints and assume that the norm γ is B-regular. For $x \in E(A^+, A^-, \Omega)$ and a feasible direction $\delta \in F_\Omega(x)$ at least one of the three following properties is satisfied*:

(i) $A^+ \cap (x + Q_\delta) \neq \emptyset$;
(ii) $A^- \cap (x + \operatorname{int}(Q_{-\delta})) \neq \emptyset$;
(iii) $A^+ \cap (x + \operatorname{int}(Q_{-\delta})) = \emptyset$ *and* $A^- \cap (x + Q_\delta) = \emptyset$.

Proof Suppose there exists a feasible direction $\delta \in F_\Omega(x)$ for which none of the three conditions (i), (ii) and (iii) are satisfied and let us prove that x is strictly dominated by some feasible point of the form $y_\lambda := x + \lambda\delta$ with $\lambda > 0$.

Condition (ii) being not satisfied, by Proposition 3.1 we already have

$$\forall a \in A^-, \ \forall \lambda > 0, \quad \gamma(a - x - \lambda\delta) \geq \gamma(a - x).$$

Since (i) does not hold, for each $a \in A^+$, there exists some $\mu_a > 0$ such that

$$\gamma(a - x - \mu_a\delta) \leq \gamma(a - x).$$

By the B-regularity of the norm , for each $a \in A^+$ it follows that $\gamma(a' - x - \lambda_a \delta) \leq \gamma(a' - x)$ for some $\lambda_a > 0$ and all a' in a neighborhood $V(a)$ of a. Then, due to the convexity the norm, we have $\gamma(a' - x - \lambda\delta) \leq \gamma(a' - x)$ for any $a' \in V(a)$ and any λ such that $0 < \lambda \leq \lambda_a$. Since A^+ is compact, one can select a finite subset $A_+ \subset A^+$ such that A^+ is covered by the neighborhoods $V(a),\ a \in A_+$. With $\overline{\lambda} := \min\{\lambda_a,\ a \in A_+\}$, one gets

$$\gamma(a - x - \lambda\delta) \leq \gamma(a - x) \quad \forall a \in A^+,\ \forall\lambda \in]0, \overline{\lambda}[.$$

Thus x is already dominated by all the y_λ for which $0 < \lambda \leq \overline{\lambda}$.

Since condition (iii) is not satisfied, either $\gamma(a - x - \lambda\delta) > \gamma(a - x)$ for some $a \in A^-$ and any $\lambda > 0$ or $\gamma(a - x - \mu\delta) < \gamma(a - x)$ for some $a \in A^+$ and some $\mu > 0$. In the first case y_λ strictly dominates x for any λ such that $0 < \lambda \leq \overline{\lambda}$. In the second case, due to the convexity of the norm we also have $\gamma(a - x - \lambda\delta) < \gamma(a - x)$ for $0 < \lambda \leq \mu$, and thus y_λ strictly dominates x as soon as $0 < \lambda \leq \min\{\overline{\lambda}, \mu\}$. Since the y_λ becomes feasible when $\lambda \leq \lambda_\delta$, the result follows. □

4 The case of a norm induced by a scalar product

In this section we assume that the norm γ derives from a scalar product denoted by $\langle\cdot,\cdot\rangle$. The (bi)linearity properties of the scalar product provide an extra tool which will be deeply exploited.

4.1 Unconstrained strict efficiency

Theorem 4.1 *The following properties hold*:

(i) $\mathrm{co}(A^+) \cap \mathrm{ri}(\mathrm{co}(\{x\} \cup A^-)) \neq \emptyset \implies x \in SE(A^+, A^-)$;
(ii) $SE(A^+, A^-) = \mathrm{co}(A^+) + \mathrm{cl}[\mathrm{cone}[\mathrm{co}(A^+) - \mathrm{co}(A^-)]]$.

Proof Let K be the closed convex set defined by

$$K := \mathrm{co}(A^+) + \mathrm{cl}\big[\mathrm{cone}[\mathrm{co}(A^+) - \mathrm{co}(A^-)]\big].$$

Suppose that $x \notin SE(A^+, A^-)$. Then, x is dominated, i.e. there exists $y \in X$, $y \neq x$, such that $A^+ \subset H^{\geq}$ and $A^- \subset H^{\leq}$ with $H^{\geq} = \{z \in X;\ \langle p, z\rangle \geq \alpha\}$, $H^{\leq} = \{z \in X;\ \langle p, z\rangle \leq \alpha\}$, $p = y - x$ and $\alpha = (\|y\|^2 - \|x\|^2)/2$. It follows that

$$K \subset H^{\geq} \quad \text{and} \quad \mathrm{co}(A^-) \subset H^{\leq}.$$

Indeed, the first inclusion occurs because a point $z \in \mathrm{co}(A^+) + \mathrm{cone}\big[\mathrm{co}(A^+) - \mathrm{co}(A^-)\big]$ can be written as $z = \widehat{a}^+ + \lambda(a^+ - a^-)$ with $\widehat{a}^+, a^+ \in \mathrm{co}(A^+)$, $a^- \in \mathrm{co}(A^-)$ and $\lambda \geq 0$. It follows that

$$\langle p, z\rangle = \langle p, \widehat{a}^+\rangle + \lambda\langle p, a^+\rangle - \lambda\langle p, a^-\rangle \geq \alpha + \lambda\alpha - \lambda\alpha = \alpha$$

and thus $z \in H^{\geq}$. The second inclusion is clear. Next, since $x \in H^{<} := \{z \in X;\ \langle p, z\rangle < \alpha\}$ we deduce that $x \notin K$ and that the sets $\mathrm{co}(A^+)$ and $\mathrm{ri}(\mathrm{co}(\{x\} \cup A^-))$ are disjoint. This simultaneously proves (i) and the inclusion $K \subset SE(A^+, A^-)$.

Now let us establish the reverse inclusion $SE(A^+, A^-) \subset K$. Let $x \notin K$. We can separate strictly x and K. Thus there exist $p \in X$, $p \neq 0$, and $\alpha \in \mathbb{R}$ such that

$$\begin{cases} \langle p, x\rangle < \alpha \\ \lambda\langle p, a^+ - a^-\rangle + \langle p, \widehat{a}^+\rangle \geq \alpha \quad \forall a^+,\ \widehat{a}^+ \in \mathrm{co}(A^+),\ \forall a^- \in \mathrm{co}(A^-),\ \forall \lambda \geq 0. \end{cases}$$

Dividing the last inequality by λ and making $\lambda \to +\infty$, we obtain

$$\langle p, a^+ - a^-\rangle \geq 0 \quad \forall a^+ \in \mathrm{co}(A^+),\ \forall a^- \in \mathrm{co}(A^-).$$

Now, observe that we can take $\alpha = \inf\{\langle p, a^+\rangle;\ a^+ \in \mathrm{co}(A^+)\}$ so that

$$\begin{cases} \langle p, x\rangle < \alpha, \\ \langle p, a^+\rangle \geq \alpha \quad \forall a^+ \in \mathrm{co}(A^+), \\ \langle p, a^-\rangle \leq \alpha \quad \forall a^- \in \mathrm{co}(A^-). \end{cases}$$

It is then clear that the symmetric y of x with respect to $H := \{z \in X;\ \langle p, z\rangle = \alpha\}$ dominates x and thus x is not strictly efficient. □

The following example shows that the sufficient condition of strict efficiency (i) is, unfortunately, not necessary. Consider in the Euclidean space $\mathbb{R}^3$ the sets $A^+ := \{a_1^+, a_2^+, a_3^+\}$, and $A^- := \{a_1^-, a_2^-\}$ with $a_1^+ = (-1, 0, 0)$, $a_2^+ = (0, 0, -1)$, $a_3^+ = (0, 0, 1)$, $a_1^- = (1, 0, 0)$, $a_2^- = (0, 1, 0)$ and look at $\overline{x} = (0, -1, 0)$. The relative interior of $\mathrm{co}(\{\overline{x}\} \cup A^-)$ does not meet $\mathrm{co}(A^+)$. However x is strictly efficient. If $\overline{x}$ was not strictly efficient it would be dominated, i.e. there would exist a point $\overline{y} \in \mathbb{R}^3$, $\overline{y} \neq \overline{x}$ such that

$$\begin{cases} \gamma(a_i^+ - \overline{x}) \geq \gamma(a_i^+ - \overline{y}) \quad \forall i = 1, 2, 3, \\ \gamma(a_i^- - \overline{x}) \leq \gamma(a_i^- - \overline{y}) \quad \forall i = 1, 2. \end{cases}$$

The plane $H(\overline{x}, \overline{y})$ made up of the points which are equidistant to $\overline{x}$ and $\overline{y}$ would then separate $\mathrm{co}(A^+)$ and $\mathrm{co}(\{\overline{x}\} \cup A^-)$. Observing that $\overline{x} \notin H(\overline{x}, \overline{y})$, that there is a unique plane defined by $H := \{(x, y, z);\ x = 0\}$ which separates $\mathrm{co}(A^+)$ and $\mathrm{co}(\{\overline{x}\} \cup A^-)$ and that $\overline{x} \in H$, we get a contradiction.

Corollary 4.1 *The set $SE(A^+, A^-)$ is closed and convex. Moreover if A^+ and A^- are finite sets or if $\mathrm{co}(A^+)$ and $\mathrm{co}(A^-)$ are polyhedral sets, then*

$$SE(A^+, A^-) = \mathrm{co}(A^+) + \mathrm{cone}\left[\mathrm{co}(A^+) - \mathrm{co}(A^-)\right].$$

Corollary 4.2 *If $\mathrm{co}(A^+) \cap \mathrm{ri}(\mathrm{co}(A^-)) \neq \emptyset$ then $\mathrm{co}(A^-) \subset SE(A^+, A^-)$.*

Proof A direct consequence of Theorem 4.1(i). □

The following result yields another characterization of strict efficiency.

Theorem 4.2 *We have the following equivalence*:

$$x \in \mathrm{ri}(SE(A^+, A^-)) \iff \mathrm{ri}(\mathrm{co}(A^+)) \cap \mathrm{ri}(\mathrm{co}(\{x\} \cup A^-)) \neq \emptyset.$$

Proof Since the relative interior of a convex set coincides with the relative interior of its closure and the relative interior of a sum of convex sets is the sum of their relative interiors (Rockafellar 1970), the result is a direct consequence of Theorem 4.1 and of the following technical result. □

Lemma 4.1 *Let $C \subset X$ and $D \subset X$ be two convex sets. Then*

$$\mathrm{ri}(C) \cap \mathrm{ri}[\mathrm{co}(D \cup \{x\})] \neq \emptyset \quad \Longleftrightarrow \quad x \in \mathrm{ri}(C) + \mathrm{ri}[\mathrm{cone}(C - D)].$$

Proof Assume that $\mathrm{ri}(C) \cap \mathrm{ri}[\mathrm{co}(D \cup \{x\})] \neq \emptyset$. According to Theorem 6.9 of Rockafellar (1970), there exist $c \in \mathrm{ri}(C)$, $d \in \mathrm{ri}(D)$ and $0 < \lambda < 1$ such that $c = (1-\lambda)d + \lambda x$. Since

$$x = c + \mu(c - d) \quad \text{with } \mu := \frac{1-\lambda}{\lambda},$$

we conclude by Corollary 6.6.2 and the remark following Corollary 6.8.1 of Rockafellar (1970) that $\mu(c-d) \in \mathrm{ri}(\mathrm{cone}(C-D))$ and hence $x \in \mathrm{ri}(C) + \mathrm{ri}[\mathrm{cone}(C-D)]$.

Conversely, if $x \in \mathrm{ri}(C) + \mathrm{ri}(\mathrm{cone}(C-D))$, there exist $c_1 \in \mathrm{ri}(C)$ and $\xi \in \mathrm{ri}(\mathrm{cone}(C-D))$ such that $x = c_1 + \xi$. Again, according to Corollary 6.6.2 and the remark following Corollary 6.8.1 of Rockafellar (1970), we see that ξ is of the form $\xi = \lambda(c_2 - d)$ with $c_2 \in \mathrm{ri}(C)$, $d \in \mathrm{ri}(D)$ and $\lambda > 0$. Then

$$\frac{\lambda}{\lambda+1} d + \frac{1}{\lambda+1} x = \frac{1}{\lambda+1} c_1 + \frac{\lambda}{\lambda+1} c_2.$$

By Theorem 6.9 of Rockafellar (1970) the left-hand side of this equality belongs to $\mathrm{ri}(\mathrm{co}(D \cup \{x\}))$ while the right-hand side is in $\mathrm{ri}(C)$ because the relative interior of C is convex. Thus $\mathrm{ri}(C) \cap \mathrm{ri}(\mathrm{co}(D \cup \{x\})) \neq \emptyset$ and the proof is complete. □

4.2 Unconstrained efficiency

Proposition 4.1 *A condition under which the efficient set coincides with the whole space is given by*:

$$\mathrm{ri}(\mathrm{co}(A^+)) \cap \mathrm{ri}(\mathrm{co}(A^-)) \neq \emptyset \Longleftrightarrow E(A^+, A^-) = X.$$

Proof Let x be a point which is not efficient. Then x is strictly dominated by some $y \neq x$. Set $H^{\geq} := \{z; \gamma(z-x) \geq \gamma(z-y)\}$ and $H^{\leq} := \{z; \gamma(z-x) \leq \gamma(z-y)\}$. We have $A^+ \subset H^{\geq}$, $A^- \subset H^{\leq}$ and, as the dominance is strict, there exists $a_0^+ \in \mathrm{int}(H^{\geq})$ or $a_0^- \in \mathrm{int}(H^{\leq})$. It follows that $\mathrm{ri}(\mathrm{co}(A^+)) \cap \mathrm{ri}(\mathrm{co}(A^-)) = \emptyset$ and the implication $\Longrightarrow$ is proved.

Conversely, if $\mathrm{ri}(\mathrm{co}(A^+)) \cap \mathrm{ri}(\mathrm{co}(A^-)) = \emptyset$ then the convex sets $\mathrm{co}(A^+)$ and $\mathrm{co}(A^-)$ can be properly separated (Rockafellar 1970). That implies there exist some $p \in X$, $p \neq 0$, and $\alpha \in \mathbb{R}$ such that

$$\begin{cases} \mathrm{co}(A^+) \subset H^{\geq} := \{z \in X\colon \langle p, z\rangle \geq \alpha\}, \\ \mathrm{co}(A^-) \subset H^{\leq} := \{z \in X\colon \langle p, z\rangle \leq \alpha\}, \\ \sup\{\langle p, z\rangle\colon z \in \mathrm{co}(A^+)\} > \inf\{\langle p, z\rangle\colon z \in \mathrm{co}(A^-)\}. \end{cases}$$

The last condition implies the existence of some $a_0^+ \in A^+$ such that $a_0^+ \in \mathrm{int}(H^{\geq})$ or the existence of some $a_0^- \in A^-$ such that $a_0^- \in \mathrm{int}(H^{\leq})$. Then, consider the point x which is symmetric to a_0^+ (resp. to a_0^-) with respect to the hyperplane $H = \{z, \langle p, z\rangle = \alpha\}$. The point x

(resp. a_0^-)is not efficient because strictly dominated by a_0^+ (resp. x). Thus there exists a point which is not efficient and the proof is complete. □

Theorem 4.3 *We have the following properties*:

(i) *If both sets A^+ and A^- are not contained in a same hyperplane then $SE(A^+, A^-) = E(A^+, A^-)$.*
(ii) *If A^+ and A^- are contained in a same hyperplane H, then*

$$\mathrm{ri}(\mathrm{co}(A^+)) \cap \mathrm{ri}(\mathrm{co}(A^-)) = \emptyset \iff E(A^+, A^-) \subset H.$$

(iii) $\mathrm{ri}(\mathrm{co}(A^+)) \cap \mathrm{ri}(\mathrm{co}(A^-)) = \emptyset \iff SE(A^+, A^-) = E(A^+, A^-) \neq X.$

Proof (i) Assume that there exists a point x which is efficient without being strictly efficient. This point is then dominated but not strictly dominated. Since the norm is induced by a scalar product, that exactly means all the attracting and repulsive demand points are contained in a same hyperplane and the result follows.

(ii) Now, assume that $\mathrm{ri}(\mathrm{co}(A^+)) \cap \mathrm{ri}(\mathrm{co}(A^-)) = \emptyset$ and let us start by proving that, if $x \notin H$, then there exist $\widehat{p} \in X$, $\widehat{p} \neq 0$, and $\widehat{\alpha} \in \mathbb{R}$

$$\begin{cases} A^+ \subset \widehat{H}^{\geq} := \{z \in X;\ \langle \widehat{p}, z\rangle \geq \widehat{\alpha}\}, \\ A^- \subset \widehat{H}^{\leq} := \{z \in X;\ \langle \widehat{p}, z\rangle \leq \widehat{\alpha}\}, \\ A^+ \cup A^- \not\subset \widehat{H}, \\ \langle \widehat{p}, x\rangle < \widehat{\alpha}. \end{cases}$$

Indeed, since $\mathrm{ri}(\mathrm{co}(A^+)) \cap \mathrm{ri}(\mathrm{co}(A^-)) = \emptyset$, the sets $\mathrm{co}(A^+)$ and $\mathrm{co}(A^-)$ can be properly separated (Rockafellar 1970). That means there exist some $q \in X$, $q \neq 0$, and $\beta \in \mathbb{R}$ such that

$$\begin{cases} \langle q, a^+\rangle \geq \beta & \forall a^+ \in \mathrm{co}(A^+), \\ \langle q, a^-\rangle \leq \beta & \forall a^- \in \mathrm{co}(A^-), \\ \exists a \in A^+ \cup A^- & \text{such that } \langle q, a\rangle \neq \beta. \end{cases}$$

If $\langle q, x\rangle < \beta$ then we can take $\widehat{p} := p$ and $\widehat{\alpha} := \alpha$. If $\langle q, x\rangle > \beta$ then, any $\widehat{p}$ and $\widehat{\alpha}$ of the form $\widehat{p} = q + \lambda p$ and $\widehat{\alpha} = \beta + \lambda\alpha$, with $H := \{z \in X;\ \langle p, z\rangle = \alpha\}$, and λ satisfying

$$\lambda\big[\langle p, x\rangle - \alpha\big] < \langle q, x\rangle - \beta,$$

work because we have

$$\begin{cases} \langle \widehat{p}, a^+\rangle = \langle q, a^+\rangle + \lambda\langle p, a^+\rangle \geq \beta + \lambda\alpha = \widehat{\alpha} & \forall a^+ \in A^+, \\ \langle \widehat{p}, a^-\rangle = \langle q, a^-\rangle + \lambda\langle p, a^-\rangle \leq \beta + \lambda\alpha = \widehat{\alpha} & \forall a^- \in A^-, \\ \langle \widehat{p}, a\rangle = \langle q, a\rangle + \lambda\langle p, a\rangle \neq \widehat{\alpha}, \\ \langle \widehat{p}, x\rangle = \langle q, x\rangle + \lambda\langle p, x\rangle < \beta + \lambda\alpha = \widehat{\alpha}. \end{cases}$$

Note that the sign of λ depends on the position of x relative to H. Next, consider the point y which is symmetric to x with respect to $\widehat{H}$. It is clear that y strictly dominates x and thus $x \notin E(A^+, A^-)$. We finally have proved that $E(A^+, A^-) \subset H$. The reverse implication directly follows from Proposition 4.1.

(iii) Assume that $x \notin SE(A^+, A^-)$. Then x is dominated by some $y \neq x$. Set $H^{\geq} := \{z; \gamma(z-x) \geq \gamma(z-y)\}$ and $H^{\geq} := \{z; \gamma(z-x) \geq \gamma(z-y)\}$. We have $A^+ \subset H^{\geq}$, $A^- \subset H^{\leq}$. If $A^+ \cup A^- \not\subset H := \{z; \gamma(z-x) = \gamma(z-y)\}$ then y strictly dominates x and thus $x \notin E(A^+, A^-)$. If $A^+ \cup A^- \subset H$ then assertion (ii) shows that $E(A^+, A^-) \subset H$. However, since $x \notin H$, we deduce that $x \notin E(A^+, A^-)$. Consequently $E(A^+, A^-) \subset SE(A^+, A^-)$ and these two sets coincide. The proof ends by observing that Proposition 4.1 implies that $E(A^+, A^-) \neq X$. □

4.3 Unconstrained weak efficiency

Theorem 4.4 *A characterization of weakly efficient points is given by*

$$x \in WE(A^+, A^-) \quad \Longleftrightarrow \quad \mathrm{co}(A^+) \cap \mathrm{co}(\{x\} \cup A^-) \neq \emptyset.$$

Proof Suppose that $\mathrm{co}(A^+) \cap \mathrm{co}(\{x\} \cup A^-) = \emptyset$. The sets $\mathrm{co}(A^+)$ and $\mathrm{co}(\{x\} \cup A^-)$ are convex and compact. Thus, by a classical separation theorem they can be strictly separated by an hyperplane. Thus there exists some $p \in X$, $p \neq 0$, and $\alpha \in \mathbb{R}$ such that

$$\begin{cases} \mathrm{co}(A^+) \subset H^> := \{z \in X;\ \langle p, z\rangle > \alpha\} \quad \text{and} \\ \mathrm{co}(\{x\} \cup A^-) \subset H^< := \{z \in X;\ \langle p, z\rangle < \alpha\}. \end{cases}$$

Now consider the point y which is symmetric to x with respect to $H := \{z \in X,\ \langle p, z\rangle = \alpha\}$. Since γ derives from a scalar product, $H^>$ and $H^<$ can be equivalently redefined via x and y as $H^> = \{z \in X;\ \gamma(z-x) > \gamma(z-y)\}$ and $H^< = \{z \in X;\ \gamma(z-x) < \gamma(z-y)\}$. It immediately follows that y strongly dominates x and thus $x \notin WE(A^+, A^-)$.

Conversely suppose that $x \notin WE(A^+, A^-)$. Then x is strongly dominated, i.e., there exists $y \in X$, $y \neq x$, such that

$$\begin{cases} \forall a^+ \in A^+,\ \gamma(a^+ - x) > \gamma(a^+ - y) \quad \text{and} \\ \forall a^- \in A^-,\ \gamma(a^- - x) < \gamma(a^- - y). \end{cases}$$

Thus $A^+ \subset H^> := \{z \in X;\ \gamma(z-x) > \gamma(z-y)\}$ and $A^- \subset H^< := \{z \in X;\ \gamma(z-x) < \gamma(z-y)\}$. It follows that $\mathrm{co}(A^+) \subset H^>$, $\mathrm{co}(\{x\} \cup A^-) \subset H^<$ and consequently $\mathrm{co}(A^+)$ and $\mathrm{co}(\{x\} \cup A^-)$ are disjoint. □

As already mentioned in the introduction, this very simple characterization of weak efficiency is somewhat surprising due to the non-convexity induced by the presence of repulsive demand points. Unfortunately this result, the proof of which is completely based on the specific properties induced by the scalar product, cannot be extended to arbitrary norms or in presence of constraints. A major difficulty consists in finding a convenient substitute for $\mathrm{co}(A^+)$ and $\mathrm{co}(\{x\} \cup A^-)$ due to the lack of nice properties of the set of points equidistant to two points (which is the bisector in the Euclidean plane).

Note that in the pure attracting case, i.e. if $A^- = \emptyset$, the efficiency condition merely reduces to $\mathrm{co}(A^+) \cap \{x\} \neq \emptyset$. We rediscover the well known property asserting that the set of weakly efficient points coincides with the convex hull of the demand points.

We also should indicate that, in presence of repulsive demand points, the set of weakly efficient points is never bounded. More precisely, we have the following results which are direct consequences of Theorem 4.4 and the proofs of which are omitted.

Corollary 4.3 *The following assertions are equivalent*:

(i) $WE(A^+, A^-)$ *is compact*;
(ii) A^- *is empty*;
(iii) $WE(A^+, A^-) = \mathrm{co}(A^+)$.

Proposition 4.2 *A condition under which the weakly efficient set coincides with the whole space is given by*

$$\mathrm{co}(A^+) \cap \mathrm{co}(A^-) \neq \emptyset \iff WE(A^+, A^-) = X.$$

Now, we can give a complete geometrical description of the efficient sets.

Theorem 4.5 *Assume that* A^+ *and* A^- *are both non empty and consider the set* $K := \mathrm{co}(A^+) + \mathrm{cone}[\mathrm{co}(A^+) - \mathrm{co}(A^-)]$. *Then*:

(i) $E(A^+, A^-) = WE(A^+, A^-)$ *or* $E(A^+, A^-) = SE(A^+, A^-)$.
(ii) *If* $\mathrm{co}(A^+) \cap \mathrm{co}(A^-) = \emptyset$ *then*

$$SE(A^+, A^-) = E(A^+, A^-) = WE(A^+, A^-) = K.$$

The proof of Theorem 4.5 requires the following technical result.

Lemma 4.2 *Let* $C \subset X$ *and* $D \subset X$ *be two nonempty convex sets such that* $C \cap D = \emptyset$. *Then*

$$C \cap \mathrm{co}(D \cup \{x\}) \neq \emptyset \iff x \in C + \mathrm{cone}(C - D).$$

Proof The justification is very similar to the proof of Lemma 4.1. Assume that $C \cap \mathrm{co}(D \cup \{x\}) \neq \emptyset$. Then there exist $c \in C$, $d \in D$ and $0 \leq \lambda \leq 1$ such that $c = (1 - \lambda)d + \lambda x$. Since $C \cap D = \emptyset$, we have $\lambda > 0$. Then

$$x = c + \frac{1 - \lambda}{\lambda}(c - d)$$

and this asserts that $x \in C + \mathrm{cone}(C - D)$.

Conversely, if $x \in C + \mathrm{cone}(C - D)$, there exist $c_1, c_2 \in C$, $d \in D$ and $\lambda \geq 0$ such that $x = c_1 + \lambda(c_2 - d)$. Then

$$\frac{\lambda}{\lambda + 1} d + \frac{1}{\lambda + 1} x = \frac{1}{\lambda + 1} c_1 + \frac{\lambda}{\lambda + 1} c_2.$$

The left-hand side of this equality is in $\mathrm{co}(D \cup \{x\})$ while the right-hand side is in C. Thus $C \cap \mathrm{co}(D \cup \{x\}) \neq \emptyset$ and the proof is complete. □

Proof of Theorem 4.5 Let us start by proving (i) and assume that $E(A^+, A^-) \neq WE(A^+, A^-)$. According to Proposition 4.1, we have

$$\mathrm{ri}(\mathrm{co}(A^+)) \cap \mathrm{ri}(\mathrm{co}(A^-)) = \emptyset$$

and we conclude by Theorem 4.3(iii).

Now let us prove (ii). Consider some $x \notin SE(A^+, A^-)$. Then, by Theorem 4.1, $x \notin K$. Since $\mathrm{co}(A^+) \cap \mathrm{co}(A^-) = \emptyset$, Lemma 4.2 shows that $\mathrm{co}(A^+) \cap \mathrm{co}(\{x\} \cup A^-) = \emptyset$. By Theorem 4.4 we conclude that $x \notin WE(A^+, A^-)$. Thus $WE(A^+, A^-) \subset SE(A^+, A^-)$ and finally the sets $SE(A^+, A^-)$, $E(A^+, A^-)$ and $WE(A^+, A^-)$ coincide with K. □

In presence of attracting and repulsive demand points it should be noted that the sets $SE(A^+, A^-)$, $E(A^+, A^-)$ and $WE(A^+, A^-)$ may no longer coincide as in the pure attracting case. As example consider in the plane $X = \mathbb{R}^2$ the sets $A^+ = \{a_1^+, a_2^+\}$ and $A^- = \{a_1^-, a_2^-, a_3^-\}$ with $a_1^+ = (0, 0)$, $a_2^+ = (0, -1)$, $a_1^- = (-1, 0)$, $a_2^- = (1, 0)$ and $a_3^- = (-1, 1)$ and take $x = (1, 1)$. We have $\mathrm{co}(A^+) \cap \mathrm{co}(A^-) \neq \emptyset$ and thus $WE(A^+, A^-) = \mathbb{R}^2$. However, x, which is weakly efficient, is strictly dominated by $y = (1, -1)$ and hence is not efficient.

Corollary 4.4 *The sets $E(A^+, A^-)$ and $WE(A^+, A^-)$ are closed and convex.*

4.4 Constrained efficiency

The following very simple example gives an idea of the difficulty to obtain sufficient conditions for efficiency in presence of constraints. Consider in the plane $X = \mathbb{R}^2$ the attracting demand point $a^+ = (0, 1)$ and the repulsive demand point $a^- = (0, 0)$. If we consider as set of constraints the set $\Omega := \mathrm{co}\{b_1, b_2, b_3\}$, with $b_1 = (-2, 0)$, $b_2 = (-1, \frac{1}{2})$ and $b_3 = (0, -\frac{1}{2})$, the feasible point $x = (-2, 0)$ if weakly efficient. However if we slightly move b_3 in the direction $\delta = (0, -1)$, e.g. if we consider the set of constraint $\Omega := \mathrm{co}\{b_1, b_2, b_4\}$ with $b_4 = (0, -1)$, the point x becomes not weakly efficient because strongly dominated by b_4. Note that b_4 is, among the feasible points, the one which is the farthest from x. The condition of weak efficiency hence does not only depend on the geometry of Ω at x. This is due to the non convexity induced by the repulsive demand point. Note also that x remains locally weakly efficient.

The following relationship between $WE(A^+, A^-)$ and $WE(A^+, A^-, \Omega)$ provides a sufficient condition for weak efficiency. However this condition just allows to reach a small part of the constrained weakly efficient points as will be illustrated.

Proposition 4.3 *If $\Omega \subset X$ is a nonempty closed convex set of constraints, then*

$$\mathrm{Proj}_\Omega\, WE(A^+, A^-) \subset WE(A^+, A^-, \Omega).$$

Proof Consider some x of the form $x = \mathrm{Proj}_\Omega(\overline{x})$ with $\overline{x} \in WE(A^+, A^-)$ and assume that x is not weakly efficient. Then, there exists $y \in \Omega$ such that

$$A^+ \subset H^> := \{z;\ \gamma(z - x) > \gamma(z - y)\} \quad \text{and}$$
$$A^- \subset H^< := \{z;\ \gamma(z - x) < \gamma(z - y)\}.$$

The norm being strictly convex, x is the unique feasible point minimizing the distance between $\overline{x}$ and Ω. It follows that $\overline{x} \in H^<$ and $\mathrm{co}(\{\overline{x}\} \cup A^-) \subset H^<$. Since $\mathrm{co}(A^+) \subset H^>$ we deduce that

$$\mathrm{co}(\{\overline{x}\} \cup A^-) \cap \mathrm{co}(A^+) = \emptyset$$

and we get a contradiction with Theorem 4.4. □

The following example shows that, in general, there may exist constrained weakly efficient points which cannot be obtained as projection of an unconstrained weakly efficient point. Consider in the plane $X=\mathbb{R}^2$ the sets $A^+=\{a^+=(0,0)\}$ and $A^-=\{a^-=(0,1)\}$ and take as set of constraints Ω the line segment $[b,c]$ with $b=(-1,0)$ and $c=(-3,0)$. The set of weakly efficient points is the negative half line on the y-axis. The projection of the set $WE(A^+,A^-)$ onto Ω is then reduced to the single point b. But one can easily see that all points in Ω are weakly efficient.

Note that all the points of Ω satisfy the necessary conditions of Corollary 3.1. At a point $x\in \mathrm{ri}(\Omega)$ we have two feasible directions, $\delta_1=(1,0)$ and $\delta_2=(-1,0)$ and these directions generate the sets $Q_{\delta_1}=\mathbb{R}^-\times\mathbb{R}$ and $Q_{\delta_2}=\mathbb{R}^+\times\mathbb{R}$. One can easily verify that $a^+\in x+Q_{\delta_2}$ and $a^-\in x+P_{\delta_1}$. At the point c (resp. b) we have one feasible direction δ_1 (resp. δ_2) and $a^-\in c+P_{\delta_1}$ (resp. $a^+\in b+Q_{\delta_2}$).

4.5 Computing the efficient sets in the plane

In this section we show how to compute the efficient sets in dimension two. According to the results of the previous section, we just need to consider the case $\mathrm{co}(A^+)\cap\mathrm{co}(A^-)=\emptyset$ for which all the efficient sets $SE(A^+,A^-,E(A^+,A^-)$ and $WE(A^+,A^-)$ coincide.

Theorem 4.6 *Assume that $A^+\subset\mathbb{R}^2$ and $A^-\subset\mathbb{R}^2$ are finite sets containing n and m demand points respectively, and that* $\mathrm{co}(A^+)\cap\mathrm{co}(A^-)=\emptyset$. *Then, $SE(A^+,A^-)$ can be computed in $O(nm)+O(n\log n)$ time.*

Proof According to Theorem 4.5 we have

$$SE(A^+,A^-)=\mathrm{co}(A^+)+\mathrm{cone}[\mathrm{co}(B)]\quad\text{with } B=A^+-A^-.$$

It is well known that we can compute $\mathrm{co}(A^+)$ in $O(n\log n)$ time. A procedure which provides a list $\mathcal{L}$ containing the extreme points of $\mathrm{co}(A^+)$ in clockwise order in $O(n\log n)$ time can be found e.g. in De Berg et al. (1998). This procedure needs to test whether a point r lies left or right of the directed line through two other points p and q. Let $p=(p_x,p_y)$, $q=(q_x,q_y)$ and $r=(r_x,r_y)$,. This can be done by testing the sign of the scalar product

$$\Delta(p,q,r):=\langle u,r\rangle\quad\text{with } u:=(p_y-q_y,q_x-p_x).$$

Clearly, r lies left of the line when $\Delta(p,q,r)>0$ and lies right of the line when $\Delta(p,q,r)>0$. This primitive operation required in most geometrical algorithms will also be used below.

Consider the computation of the convex cone $C:=\mathrm{cone}[\mathrm{co}(B)]$ generated by the set B. The sets A^+ and A^- are compact and we have $\mathrm{co}(A^+)\cap\mathrm{co}(A^-)=\emptyset$. Thus C is closed and pointed. Since we are in dimension two, C is polyhedral. Moreover either C has a nonempty interior, in which case it has two extreme rays, or C has an empty interior, in which case C is a half-line and it has only one extreme ray. The problem is then to determine, among the elements of B a pair of them, say b^i and b^j, for which the angle $\angle b^i 0 b^j$ is maximal. This can be done by the following procedure described in a style of pseudocode

Algorithm 1

Input. A set $B = \{b^1, b^2, \dots, b^\ell\}$ of ℓ points contained in an open halfplane passing through the origin.

Output. Two extreme rays r^+ and r^-.

1. $r^+ \leftarrow b^1$ and $r^- \leftarrow b^1$
2. **for** all points b^i, $i \neq 1$
3.   **do** compute $\Delta^+ := \Delta(0, r^+, b^i)$
4.   **if** $\Delta^+ > 0$ then $r^+ \leftarrow b^i$
5.   **do** compute $\Delta^- := \Delta(0, r^-, b^i)$
6.   **if** $\Delta^+ < 0$ then $r^- \leftarrow b^i$
7. **return** r^+ and r^-.

Applying the procedure with $B = \{e - a^-; e \in \text{Ext}(\text{co}(A^+)), a^- \in A^-\}$ where $\text{Ext}(\text{co}(A^+)) = \{e^1, e^2, \dots, e^k\}$ is the set of $k \leq n$ extreme points of $\text{co}(A^+)$ and $\ell = km$, clearly the total time required for computing the extreme rays r^+ and r^- is $O(nm)$. Once these rays are obtained it remains to generate the extreme points of $SE(A^+, A^-)$. The procedure used to compute the extreme rays can be adapted for computing the two extreme points $e(r^+)$ and $e(r^-)$ which are adjacent to the two extreme directions $\{e(r^+) + \lambda r^+; \lambda \geq 0\}$ and $\{e(r^-) + \lambda r^-; \lambda \geq 0\}$ of $SE(A^+, A^-)$. We recall that an extreme direction of a convex set is a half-line face. We can proceed as follows.

Algorithm 2

Input. The list $\mathcal{L} = \{e^1, e^2, \dots, e^k\}$ of the extreme points of $\text{co}(A^+)$ in clockwise order and the two extreme rays r^+ and r^-.

Output. The list $\mathcal{L}_{we}$ containing the extreme points of $WE(A^+, A^-)$.

1. $k^- \leftarrow 1, k^+ \leftarrow 1, e^- \leftarrow e^1, e^+ \leftarrow e^1$.
2. **for** all points $e^i \in \mathcal{L}$, $i \neq 1$
3. **do** compute $\Delta^+ := \Delta[e^+, e^+ + r^+, e^j]$
4.   **if** $\Delta^+ > 0$ then $e^+ \leftarrow e^i$ and $k^+ \leftarrow i$
5. **do** compute $\Delta^- := \Delta[e^-, e^- + r^-, e^j]$
6.   **if** $\Delta^+ < 0$ then $e^- \leftarrow e^i$ and $k^- \leftarrow i$
7. **return** the list $\mathcal{L}_{we} = \{e^{k^+}, e^{k^+ + 1}, \dots, e^{k^-}\}$.

Clearly the total time required for computing e^{k^+} and e^{k^-} is $O(k)$. Finally the total time to compute $SE(A^+, A^-)$ is $O(n \log n) + O(nm)$. □

5 Conclusion

In this paper, we have developed sufficient and/or necessary conditions of efficiency for the problem of locating a facility in presence of attracting and repulsive interactions.

In the absence of constraint, we have completely clarified the case of a norm induced by a scalar product. The efficient set coincides with the strictly efficient set and/or coincides with the weakly efficient set. When the convex hull of the attracting demand points does not meet the convex hull of the repulsive demand points, the three sets coincide. The weakly efficient set is the whole space iff these convex hulls overlap and a similar result holds for the efficient set when we replace the convex hulls by their relative interiors. We also have shown that, in the plane and with a finite set of demand points, the efficient sets can be computed in polynomial time.

With closed convex constraints and when an arbitrary norm is used, we have provided necessary conditions for efficiency in terms of Q_δ sets. An open question is clearly to characterize efficiency with an ℓ_p norm, the ℓ_1 norm, or more generally with a polyhedral norm. Some results could be expected in the polyhedral case in presence of only one repulsive demand point.

It may also be possible to derive properties of efficiency via duality arguments as in Durier and Michelot (1986), Ndiaye and Michelot (1998). All these questions are presently under investigation.

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