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Strategic behavior and partial cost sharing

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Abstract

The main objects here are games in which players mainly compete but nonetheless collaborate on some subsidiary activities. Play assumes a two-stage nature in that first-stage moves presume coordination of some subsequent tasks. Specifically, we consider instances where second-stage coordination amounts to partial cost sharing, anticipated and sustained as a core solution. Examples include regional Cournot oligopolies with joint transportation. We define and characterize equilibria, and inquire about their existence.

JEL classification: C71; C72

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1. Introduction

For motivation consider two firms which compete in as many markets, supplying these with one—or maybe several—homogeneous commodities:

$$
\begin{array}{lcl}
\text{firm 1} & \rightarrow & \text{market 1} \\
 & \searrow & \\
 & \nearrow & \\
\text{firm 2} & \rightarrow & \text{market 2.}
\end{array}
$$

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The said firms interact over *two* stages. *First*, each decides independently how much to produce and bring to every market. *Second*, having produced their quantities, there are gains to be had in coordinating the subsequent transportation from factories to markets. In fact, cost reductions obtain if each firm, fully or partly, serves the nearest market on behalf of his rival.

Similar examples include: coordinated distribution of competing newspapers, or a common shuttle bus serving rival air companies. More generally, one may think of "noncooperative" producers who maintain shared inventories, or organize internal exchange of scarce resources, or outsource some subsidiary tasks jointly. We ask: *ex post, may such secondary activities be coordinated to mutual advantage*? *If so, is it possible to share the associated costs fairly*? *And ex ante, if players anticipate the subsequent cost sharing, can they reach an overall equilibrium*?

Indeed, they can. Under broad and natural assumptions all these questions have positive and intimately related answers. For illustration Section 2 elaborates on the above figure so as to have a running example. Section 3 builds a rather general model, including the duopoly already depicted, and going well beyond it. It is defined there what is meant by an equilibrium.

Since second-stage collaboration constitutes a key part of the overall setting, Section 4 digresses to study *transferable-cost cooperative games*. All our instances concern *cost sharing,* and they fit the form of so-called *production games* (*or production economies*) in which technologies, tasks and endowments are pooled (Dubey and Shapley, 1984; Granot, 1986; Kalai and Zemel, 1982a, 1982b; Samet and Zemel, 1994; Shapley and Shubik, 1969, 1972; Sondermann, 1974).

Section 5 brings out some simple, novel properties of cost-sharing games, extending the results of Owen (1975) to nonlinear instances; see also (Evstigneev and Flåm, 2001) and (Sandsmark, 1999). One desirable property is that *infimal convolution* of convex cost functions yields a nonempty *core*. Another, more useful and practical property is that Lagrange multipliers constitute a (shadow) *price regime* that decentralizes cooperative planning and defines a core imputation. Using such prices, each agent is charged, at the second stage, for his "quantity" less a competitive profit, computed as though he were a price-taker.

Section 6 concludes by briefly mentioning how equilibrium could be learned or approached.

2. A regional oligopoly

As running example consider *a regional oligopoly*—already motivated in the introduction. Finitely many firms $i \in I$ produce the same homogenous good[1] to be shipped from *origins* $o \in O$ to *destinations* $d \in D$. Both sets O, D are finite and—without loss—regarded as disjoint. Denote by quantity q_{io} the output of firm i at o, and let q_{id} be how much it delivers at d. We tacitly assume that $\sum_{o \in O} q_{io} = \sum_{d \in D} q_{id}$ for every i. The non-

[1] More than one good could easily be accommodated—at the expense of more complex notations.

negative vector $q_i \in \mathbb{E} := \mathbb{R}^{O \cup D}$ describes i's overall plan. Upon ignoring transportation expenses for a moment, the profile $q = (q_i)$ yields firm i a payoff $\pi_i(q)$ to be fleshed out later.

For now, consider the subsequent transportation problem. Suppose transportation along the route $o \to d$ comes at fixed unit tariff $t_{od} \geqslant 0$. Firms had better meet customers' demand at minimum expense while not exceeding their supply capacities. Since customers presumably are indifferent about the origins of the goods they receive, firms may save outlays by pooling individual supply and demand. That is, well situated firms may supply nearby customers on behalf of other firms. Formally, given a (first-stage) feasible profile $q = (q_i)$, generating an aggregate $q_I := \sum_{i \in I} q_i$, the most efficient (second-stage) overall transportation cost equals

$$c_I(q_I) := \min \left\{ \sum_{o \in O, d \in D} t_{od} x_{od} \;\middle|\; \forall o \sum_{d \in D} x_{od} = \sum_{i \in I} q_{io}; \right.$$
$$\left. \forall d \sum_{o \in O} x_{od} = \sum_{i \in I} q_{id},\ \forall x_{od} \geqslant 0 \right\}. \tag{1}$$

The minimal value $c_I(q_I)$ of this linear program is finite and attained. Can that value be equitably shared? Yes it can! Section 4 spells out that $c_I(q_I)$ can be split to constitute a *core solution* denoted $co_i(q)$, $i \in I$, of a suitably defined, well motivated cooperative game. So, while anticipating such cost sharing, firm i faces overall profit $\pi_i(q) - co_i(q)$.

3. The game

Consider henceforth a finite set I of economic agents (decision makers) who interact over *two* stages. *First*, each individual $i \in I$ makes, without cooperation, a choice q_i within some nonempty convex subset Q_i of an Euclidean space $\mathbb{E}$ (the same space for all players).[2] That choice q_i, and the profile $q_{-i} := (q_j)_{j \neq i}$ implemented by his rivals, determines i's first-stage *payoff* $\pi_i(q) = \pi_i(q_i, q_{-i}) \in \mathbb{R} \cup \{-\infty\}$. For interpretation one may construe the vectors q_i, $i \in I$, as quantity bundles, each comprising specified amounts of various commodities.

At the *second* stage, after having committed q_i, player i faces a potential *cost* $c_i(q_i) \in \mathbb{R} \cup \{+\infty\}$ to be deducted from his preceding payoff.[3] However, instead of incurring that

[2] All vector spaces mentioned in the sequel are Euclidean. Generalizations to infinite-dimensional Banach spaces are possible; see (Evstigneev and Flåm, 2001).

[3] Infinite values $\pi_i(q) = -\infty$ or $c_i(q_i) = +\infty$ are used here as "death penalties" for violating implicit constraints. This modelling device focuses on essentials and saves repeated mention of (evident) restrictions (e.g., nonnegativity).

(supposedly substantial) cost right away, he may rather join a *coalition* $S \subseteq I$ which, if formed,[4] would pay *stand-alone cost*

$$c_S(q_S) := \inf\left\{ \sum_{i \in S} c_i(x_i) \,\middle|\, \sum_{i \in S} x_i = \sum_{i \in S} q_i =: q_S \right\}. \tag{2}$$

Infimal convolution (2) models exchange of perfectly divisible goods, freely transferable among the concerned parties, q_i being the production plan that member i brings to the joint enterprise.[5] It is tacitly assumed here that no individual i misrepresents his cost function c_i to own advantage. Thus, at the second stage emerges a cooperative game with player set I, *characteristic function* $S \mapsto c_S(q_S)$, and *side-payments*. For such games we let efficiency and stability be encapsulated by *core* solutions (Peleg, 1992): a *cost allocation* $co = (co_i) \in \mathbb{R}^I$, where i pays co_i, belongs to the *core*, and we write $co \in core(q)$, iff it entails

$$\begin{aligned} &\textit{Pareto efficiency}: \quad \sum_{i \in I} co_i = c_I(q_I), \quad \text{and} \\ &\textit{social stability}: \quad \sum_{i \in S} co_i \leqslant c_S(q_S) \quad \text{for all coalitions } S \subset I. \end{aligned} \tag{3}$$

Social stability means that no singleton or set $S \subset I$ of players could improve their outcome by splitting away from the society. Clearly, stability can be achieved by charging so small costs that $\sum_{i \in S} co_i \leqslant c_S(q_S)$, $\forall S \subseteq I$. Therefore, the biting requirement is that total cost be Pareto efficient. Now, do core allocations exist? And if so, can such an allocation be found? Assuming convex cost functions, Section 3 provides positive, constructive answers. Here simply *posit a well defined mapping* $q \mapsto co(q) = [co_i(q)] \in core(q)$, which specifies how the aggregate efficient cost should be split. Then the overall solution concept is the customary one:

Definition 1 (Equilibrium). Given a core-compatible rule for cost allocation $q \mapsto co(q) = [co_i(q)]_{i \in I} \in core(q)$ the vector $q = (q_i) \in \mathbb{E}^I$ constitutes a *Nash equilibrium* iff

$$q_i \in \arg\max\{\pi_i(\cdot, q_{-i}) - co_i(\cdot, q_{-i})\} \quad \text{for all } i \in I. \tag{4}$$

As usual, equilibrium must be upheld individual incentives—and be confirmed by correct beliefs. Clearly, each requirement is rather demanding. So, Definition 1 begs many questions: can second-stage, efficient costs be equitably shared? Does equilibrium exist? Can it be implemented or reached by repeated play? We shall address all these questions, beginning with the first one in the next section. Before that we return briefly to our running example:

[4] Whenever we speak about a coalition S of players, it is tacitly understood that S be nonempty. Alternatively, one may use the convention that any empty sum equals 0.

[5] Equal *convex* $c_i = c, i \in S$, yields $c_S(q_S) = |S|c(q_S/|S|)$, with uniform distribution of the aggregate quantity q_S. If a common cost function c also is 1-homogeneous, then $c_S(q_S) = c(q_S)$.

3.1. *The regional oligopoly continued*

On the right-hand side of (1) replace $\sum_{i\in I}$ by $\sum_{i\in S}$ to obtain thereby the stand-alone cost $c_S(q_S)$ of coalition $S \subset I$ as the optimal value of a linear (transportation) program in which S have aggregate supply-demand schedule $q_S := \sum_{i\in S} q_i$. As noted earlier, the cost $c_S(q_S)$ is finite and attained. We proceed to elaborate on the payoff functions $\pi_i(q_i, q_{-i})$. Following Cournot let

$$\pi_i(q_i, q_{-i}) := \sum_{d\in D} P_d(\mathcal{S}_d) q_{id} - \sum_{o\in O} f_{io}(q_{io}). \tag{5}$$

Here the *inverse demand curve* $P_d(\mathcal{S}_d)$ states the unit price at which demand at destination d equals *supply* $\mathcal{S}_d := \sum_{i\in I} q_{id}$ there. The function $f_{io}(q_{io})$ accounts for the production cost incurred by i at site o.

4. Cost-sharing games

Given stand-alone cost (2) of each coalition S, we ask here: *will the core of the resulting* (*production*) *game be nonempty*? The following result underscores the convenience of having convex costs.

Proposition 1 (Nonempty core (Sandsmark, 1999)). *Suppose that all functions c_i, $i \in I$, are convex and that $c_I(q_I)$ is finite-valued. Then the cost-sharing game has a nonempty core.*

When moreover, all $c_S(q_S)$, $S \subseteq I$, are finite, Proposition 1 shows that cost-sharing is a *totally balanced game*, meaning that all subgames have nonempty cores. Particular instances, called *additive* games, emerge if each function $c_i(\cdot)$ assumes a constant finite value on some convex set, and $+\infty$ elsewhere. Shapley and Shubik (1969) showed that totally balanced games can be generated by so-called market (exchange) games.[6] The present construction completely parallels theirs. Kalai and Zemel (1982a, 1982b) related such games to flow problems in (generalized) networks where various arcs are owned by different individuals. They also observed that a *profit-sharing game* will be totally balanced if its characteristic function equals the minimum of corresponding functions stemming from finitely many additive games. The cost-sharing games considered here can similarly be seen as the upper envelope of finitely many additive games.

Our task is to find a core element—not merely ensure existence. To that end write xy for the standard inner product between two vectors $x, y \in \mathbb{E}$ and let $f^*(p) := \sup_x\{px - f(x)\}$ denote the *Fenchel conjugate* (Rockafellar, 1970) of any function $f : \mathbb{E} \to \mathbb{R} \cup \{\pm\infty\}$. Given a profile $q \in \mathbb{E}^I$, introduce

$$L_S(x, p) := \sum_{i\in S} \big[c_i(x_i) + p(q_i - x_i)\big]$$

[6] The idea of modeling exchange as coalitional games stems from von Neumann and Morgenstern (1944, pp. 583–584).

as the standard Lagrangian $L_S : \mathbb{E}^S \times \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ of coalition S. Note that

$$\inf_x L_S(x, p) = \sum_{i \in S} \big[p q_i - c_i^*(p) \big].$$

We declare $p \in \mathbb{E}$ *a Lagrange multiplier or a shadow price* iff $c_I(q_I) \leqslant \inf_x L_I(x, p)$.

Theorem 1 (Lagrange multipliers yield core solutions). *Given $q \in \mathbb{E}^I$, any shadow price p generates a cost allocation*

$$co(q) := \big[p q_i - c_i^*(p) \big]_{i \in I} \in core(q). \tag{6}$$

Proof. Social stability obtains because

$$\sum_{i \in S} co_i(q) = \inf_x L_S(x, p) \leqslant \sup_{p'} \inf_x L_S(x, p') \leqslant \inf_x \sup_{p'} L_S(x, p') = c_S(q_S)$$

for all $S \subseteq I$. The second inequality in this string is called *weak duality*. The assumption $c_I(q_I) \leqslant \inf_x L_I(x, p)$ takes care of *strong duality*. To wit,

$$\sum_{i \in I} co_i(q) = \inf_x L_I(x, p) \geqslant c_I(q_I).$$

Since we already know that $\sum_{i \in I} co_i(q) \leqslant c_I(q_I)$, Pareto efficiency now follows. □

Core element (6) has a nice and natural interpretation: each agent i pays the price times his quantity less the profit $c_i^*(p)$ he could contribute as price-taker. Clearly, agents with relatively low marginal cost are at advantage; they will produce on behalf of others and be reimbursed. The arrangement is decentralized and voluntary in that every individual i freely minimizes his modified cost $c_i(x_i) + p(q_i - x_i)$. If $c_I(q_I)$ is attained by production pattern x; that is, if $\sum_i [x_i, c_i(x_i)] = [q_I, c_I(q_I)]$, then $co_i(q) = c_i(x_i) + p(q_i - x_i)$.

For illustration of Theorem 1 suppose that individual cost is a marginal function $c_i(q_i) := \inf_{y_i} C_i(q_i, y_i)$, stemming from a bivariate proper objective C_i. Then

$$c_S(q_S) = \inf_{x,y} \left\{ \sum_{i \in S} C_i(x_i, y_i) \;\middle|\; \sum_{i \in S} x_i = q_S \right\}.$$

Let here $L_S(x, y, p) := \sum_{i \in S} [C_i(x_i, y_i) + p(q_i - x_i)]$ and note that $\inf_{x,y} L_S(x, y, p) = \sum_{i \in S} [p q_i - C_i^*(p, 0)]$. In the proof of Theorem 1 replace x with (x, y) and correspondingly $\inf_x$ with $\inf_{x,y}$ to obtain:

Proposition 2 (Core solutions for inf-convolutions of marginal functions (Evstigneev and Flåm, 2001)). *Suppose $c_I(q_I) \leqslant \inf_{x,y} L_I(x, y, p)$ for some shadow price p. Then*

$$co(q) := \big[p q_i - C_i^*(p, 0) \big]_{i \in I} \in core(q).$$

Example 1 (*An assignment game*). Assume that each player $i \in I$, *not* collaborating at the second stage, must "dump" all q_i at *one* particular destination, "owned" by him, that site also being baptized i. Then $c_i(q_i) = C_{ii}(q_i)$ whereas if he could bring his entire produce to some other destination $j \in I$, he would incur cost $C_{ij}(q_i)$. Since coalition S owns the destinations S, its stand-alone cost $c_S(q_S)$ equals the optimal value of the following assignment problem:

$$\min\left\{\sum_{i\in S,\, j\in S} C_{ij}(x_i)y_{ij} \;\middle|\; \sum_{j\in S} y_{ij} = \sum_{i\in S} y_{ij} = 1\ \forall i, j \in S,\ \sum_{i\in S} x_i = q_S,\ \text{and } y \geqslant 0\right\}.$$

For each fixed x this problem has integer solutions y.

Example 2 (*Assignment games continued*). Another scenario, more fitting to (Crawford and Knoer, 1981; Kaneko, 1982; Shapley and Shubik, 1972), is the following. At the second stage enters another finite set P of new players. It is assumed that each agent can sign an exclusive, bilateral contract with at most one agent of the other type, each of the contracting parties i, p then enjoying profit $-c_{ip}(q_i) \geqslant 0$. Suppose $|I| \leqslant |P|$. The potential second-stage profit of coalition $S \subseteq I$ equals

$$\max_{x,\mathbf{p}}\left\{-\sum_{i\in S} c_{i\mathbf{p}(i)}(x_i) \;\middle|\; \sum_{i\in S} x_i = q_S\right\}$$

where $\mathbf{p}$ denotes an injective mapping from I to P, associating to individual i a partner $\mathbf{p}(i)$. When $S = I$, the overall profit equals the negative of the optimal value of the problem

$$\min\left\{\sum_{i\in I,p\in P} c_{ip}(x_i)y_{ip} \;\middle|\; \sum_{p\in P} y_{ip} \leqslant 1\ \forall i,\ \sum_{i\in I} y_{ip} \leqslant 1\ \forall p,\ \sum_{i\in I} x_i = q_I \text{ and } y \geqslant 0\right\}.$$

For each fixed x this problem has integer solutions y.

For additional illustration of Theorem 1 consider reduced cost functions

$$c_i(q_i) := \inf\{f_i(y_i) \mid q_i + g_i(y_i) \in K\} \tag{7}$$

with $K \subset \mathbb{E}$ closed under addition, $0 \in K$, and f_i, g_i mapping some Euclidean space $\mathbb{E}_i$ into $\mathbb{R}$ and $\mathbb{E}$, respectively. Note that K is common to all players. Coalition S could now incur stand-alone cost

$$c_S(q_S) = \inf_y\left\{\sum_{i\in S} f_i(y_i) \;\middle|\; q_S + \sum_{i\in S} g_i(y_i) \in K\right\}.$$

Let here $L_S(y, p) := \sum_{i\in S}[f_i(y_i) + p(q_i + g_i(y_i))]$ and note that

$$\inf_y L_S(y, p) = \sum_{i\in S}\left[pq_i - (f_i + pg_i)^*(0)\right].$$

In the proof of Theorem 1, after replacing x by y one gets:

Proposition 3 (Core solutions for inf-convoluted programs (Evstigneev and Flåm, 2001)). *Suppose $c_I(q_I) \leqslant \inf_y L_I(y, p)$ for some shadow price p belonging to the cone*

$$K^* := \{p \in \mathbb{E}: px \leqslant 0 \text{ for all } x \in K\}.$$

Then $co(q) := [pq_i - (f_i + pg_i)^*(0)] \in core(q)$.

Example 3. *Linear production games* (Granot, 1986; Kalai and Zemel, 1982a, 1982b; Samet and Zemel, 1994; Owen, 1975) fit (7) and deserve special mention. After slight generalization these assume the following canonical form: $c_i(q_i) := \inf\{c_i y_i \mid A_i y_i = q_i, y_i \geqslant 0\}$. Here $c_i \in \mathbb{R}^{n_i}$ is the only row vector, $q_i \in \mathbb{R}^m$, and the matrix A_i is of size $m \times n_i$. Then coalition S also faces a linear program

$$c_S(q_S) := \inf\left\{\sum_{i\in S} c_i y_i \;\middle|\; \sum_{i\in S} A_i y_i = q_S,\ \text{all } y_i \geqslant 0\right\}.$$

If $c_I(q_I)$ is finite, it equals the optimal value of the associated dual:

$$\max\{pq_I \mid c_i \geqslant pA_i \text{ for all } i\}.$$

For any dual optimal solution $p \in \mathbb{R}^m$, the allocation $co_i(q) := pq_i, i \in I$, belong to the core. On a purely technical note, this example illustrates that one should hesitate in assuming $c_i(q_i)$ smooth (differentiable).

4.1. The regional oligopoly continued

In Example 3 let $c_i := t$ and $A_i := A$ be independent of i. Then clearly $c_I(q_I) = \min\{tx \mid Ax = q_I,\ x \geqslant 0\}$. Obvious specification of A yields the transportation instance (1). Thus any optimal solution p to the corresponding dual problem $\max\{pq_I \mid t \geqslant pA\}$ produces a core solution $co_i(q) := pq_i, i \in I$, for the joint undertaking of regional transport. Inside the brackets of (1) one may safely replace the first equality sign with $\leqslant$, and the second with $\geqslant$, so as to obtain the more customary format $co_i(q) = \sum_{d\in D} p_d q_{id} - \sum_{o\in O} p_o q_{io}$ for an optimal dual solution $p \geqslant 0$.

The above results depend on the additive separability of objectives and constraints. Otherwise efficient, stable outcomes can hardly be implemented by linear prices; see (Moulin, 1990, 1996) and (Young, 1985) and references therein. Note that neither Theorem 1 nor Propositions 2, 3 required convexity. These results hinge however, on *strong duality*; that is, on the equality

$$\underline{v} := \sup_p \inf_x L_I(x, p) = \inf_x \sup_p L_I(x, p) =: \bar{v}$$

and on the existence of a *dual optimal solution* p which realizes value $\underline{v}$. Generally, a nonnegative *duality gap* $d := \bar{v} - \underline{v}$ prevails. Any dual optimal p defines a cost allocation $pq_i - c_i^*(p), i \in I$, which is socially stable, but possibly does not pay the entire bill: it may leave an uncovered *deficit* $d \geqslant 0$ because $\sum_{i\in I}\{pq_i - c_i^*(p)\} = c_I(q_I) - d$. For estimation of this deficit see (Evstigneev and Flåm, 2001).

The Lagrangian approach circumvents troublesome computation of $c_S(q_S), S \subseteq I$, sometimes needed to solve (3). Instead, by invoking dual optimal solutions, and presuming no duality gap, it rather displays core elements directly. But clearly, conditions are called for which ensure existence of shadow prices (alias Lagrange multipliers). For that purpose we shall invoke the (quite reasonable) *constraint qualification* that

$$Q_I := \sum_{i\in I} Q_i \subset \operatorname{int}\left\{\sum_{i\in I} \operatorname{dom} c_i\right\}, \tag{8}$$

where $\operatorname{dom} c_i := c_i^{-1}(\mathbb{R}) = \{x_i\colon\ c_i(x_i) \in \mathbb{R}\}$. Clearly, (8) holds if $Q_i \subset \operatorname{int}\operatorname{dom} c_i$ for at least one i, and $Q_i \subseteq \operatorname{dom} c_i$ for all other indices. In other words: it suffices for (8) that each cost function c_i be finite-valued on Q_i and at least one also slightly beyond that set. The following result is derived from Theorem 4.1 in (Ekeland and Temam, 1974).

Proposition 4 (Existence of shadow prices). *Let all c_i be convex and suppose* (8) *holds. Then, provided $q_I \in Q_I$, there exists a shadow price.*

Alternatively, whenever $\{x = (x_i)\colon\ x_i \in ri \operatorname{dom} c_i \text{ and } \sum_i x_i = q_I\}$ is nonempty and $q_I \in Q_I$, existence of a shadow price follows from Fenchel's duality theorem (see Rockafellar (1970, Theorem 31.1)). It is informative to relate multipliers to marginal costs. Let ∂ denote the subdifferential operator of convex analysis (Rockafellar, 1970). As mentioned above (see Example 3), it is natural and important to accommodate nonsmooth functions.

Proposition 5 (Subgradients yield core solutions). *Let all c_i be convex. Then p is a shadow price if and only if it is a subgradient $p \in \partial c_I(q_I)$. Given $p \in \partial c_I(q_I)$, every profile x, satisfying $\sum_{i\in I} x_i = q_I$ and $\sum_{i\in I} c_i(x_i) = c_I(q_I)$, yields equal marginal costs in the sense that $p \in \partial c_i(x_i)$ for all i.*

Proof. Note that $c_I(\cdot)$ is convex. Let p be a shadow price. This means that

$$c_I(q_I) = \sup_{p'} \inf_x L_I(x, p') = \sup_{p'} \left\{ p' q_I - \sum_{i\in I} c_i^*(p') \right\} = p q_I - \sum_{i\in I} c_i^*(p),$$

and $p \in \partial c_I(q_I)$ now follows from Danskin's envelope theorem. Conversely, pick any $p \in \partial c_I(q_I)$. Then $c_I^*(p) + c_I(q_I) = p q_I$. Since c_I is an inf-convolution, we get $c_I^* = \sum_{i\in I} c_i^*$; see (Laurent, 1972). Combining the last two equations we obtain

$$c_I(q_I) = p q_I - c_I^*(p) = p q_I - \sum_{i\in I} c_i^*(p) = \inf_x L_I(x, p).$$

This shows that the subgradient p is a Lagrange multiplier of the desired sort. For the last assertion of the proposition observe that

$$\sum_{i\in I} \left[c_i(x_i) + c_i^*(p) \right] = c_I(q_I) + c_I^*(p) = p q_I = \sum_{i\in I} p x_i.$$

In general $c_i(x_i) + c_i^*(p) \geqslant p x_i$. Thus, to preserve equality throughout the last string, we must have $c_i(x_i) + c_i^*(p) = p x_i$ whence $p \in \partial c_i(x_i)$ for all i. $\square$

To interpret Proposition 5 suppose $c_I(q_I)$ is attained; that is, suppose $+\infty > c_I(q_I) = \sum_{i\in I} c_i(x_i)$ with $q_I = \sum_{i\in I} x_i$. Care is taken then that marginal cost be uniform across players: $p \in \partial c_i(x_i)$ for all i. Otherwise production should reallocated away from marginally inefficient agents. If moreover, all c_i are convex, two things hold: *first*, if some function c_i is strictly convex, the corresponding component x_i becomes unique; *second*, if all c_i are continuous at x_i, except maybe one, then c_I becomes continuous whence subdifferentiable at q_I.

Cost sharing may presume specific agreement on how production should be implemented. Reflecting this concern we record:

Proposition 6 (On attainment of second-stage cost (Evstigneev and Flåm, 2001)). *Let all c_i be lower semicontinuous proper, and suppose*

$$\left\{x \in \mathbb{E}^I \;\middle|\; \sum_{i \in I} x_i \in \mathcal{K},\; \sum_{i \in I} c_i(x_i) \leqslant r\right\} \quad \textit{is compact} \tag{9}$$

for every compact $\mathcal{K} \subset \mathbb{E}$ and $r \in \mathbb{R}$. Then c_I also becomes lower semicontinuous proper, and the value $c_I(q_I)$ will be attained by some x. If moreover, $\sum_{i \in I} c_i(x_i)$ is upper semicontinuous at the said x, then c_I becomes continuous at q_I.

5. Characterization and existence of equilibrium

Suppose, just for the argument (not as a description of play), that agent i were the last to join the grand coalition. Hypothetically he would then

$$\text{maximize } \pi_i(q_i, q_{-i}) - \left[c_I(q_i + q_{I\setminus i}) - c_{I\setminus i}(q_{Ii})\right] \quad \text{with respect to } q_i,$$

the justification being that upon joining the others with his production task, he must cover the resulting cost increment. This scenario admittedly suffers from an asymmetry in the role assigned to i, but it facilitates discussion of existence. Suppose henceforth that each set Q_i *is nonempty convex compact, and that $\pi_i(q_i, q_{-i})$ is concave in q_i and jointly continuous over $Q := \prod_i Q_i$.*

Proposition 7 (Concerning existence of equilibrium). *Suppose that $c_I(q_I)$ is convex continuous on $Q_I = \sum_i Q_i$. Then there exists at least one profile q which satisfies*

$$q_i \in B_i(q_{-i}) := \arg\max_{Q_i}\left\{\pi_i(\cdot, q_{-i}) - \left[c_I(\cdot + q_{I\setminus i}) - c_{I\setminus i}(q_{I\setminus i})\right]\right\} \quad \textit{for every } i. \tag{10}$$

Proof. Observe, under the standing assumption, that each best response $B_i(q_{-i})$—whence the product $B(q) := (B_i(q_{-i}))_{i \in I}$—is nonempty convex when $q \in Q$. Moreover, the presumed continuity ensures that the correspondence $B : Q \rightsquigarrow Q$ has closed graph. It follows from Kakutani's theorem (Aubin and Frankowska, 1990) that B admits a fixed point $q \in B(q)$. □

Note that q satisfies (10) iff

$$0 \in \arg\max_{d_i}\left\{\pi_i(q_i + d_i, q_{-i}) - \left[c_I(d_i + q_I) - c_I(q_I)\right] \;\middle|\; q_i + d_i \in Q_i\right\} \quad \forall i \in I. \tag{11}$$

Interpret d_i here as a feasible *deviation*, undertaken unilaterally by i, that generates additional cost $\Delta c_I(q_I) = c_I(q_I + d_i) - c_I(q_I)$ to be fully born by him. (11) says that no player wants to deviate. This observation points to the convenience of describing equilibrium in terms of marginal payoffs and costs. To simplify the statement we posit that

$\pi_i(q) = -\infty$ whenever $q_i \notin Q_i$. We assume that the partial superdifferential $(\partial/\partial q_i)\pi_i(q)$ is nonempty on Q.

Theorem 2 (Existence and characterization of equilibrium). *Suppose all cost functions c_i are convex and that $c_I(q_I)$ is continuous at any $q_I \in Q_I = \sum_i Q_i$. Then there exists at least one Nash equilibrium. Moreover, any such profile q is characterized by all marginal payoffs "being equal." More precisely, there exists a subgradient $p \in \partial c_I(q_I)$ such that*

$$p \in \frac{\partial}{\partial q_i}\pi_i(q) \quad \textit{for every } i. \tag{12}$$

If the second-stage cost is realized by x; that is, if $\sum_i [x_i, c_i(x_i)] = [q_I, c_I(q_I)]$, then $p \in \partial c_i(x_i)$ for all i.

Proof. Since all c_i are convex, so is the function c_I. The continuity assumption on the latter guarantees its subdifferentiability. That is, $\partial c_I(q_I)$ is nonempty whenever $q_I \in Q_I$. A standard result in convex analysis (Ekeland and Temam, 1974) says that any $p \in \partial c_I(q_I)$ entails absence of duality gap; that is

$$c_I(q_I) = \inf_x \sum_{i \in I}\big[c_i(x_i) + p(q_i - x_i)\big] \quad \text{provided all } q_i \in Q_i \text{ and } \sum_{i \in I} q_i = q_I.$$

In other words, the subgradient p is indeed a shadow price associated to q_I. So, by Theorem 1 any feasible first-stage profile $q \in Q$ gives rise to a second stage cooperative game in which cost is split according to the core-compatible rule (6). Under these conditions the profile q is Nash iff

$$0 \in \frac{\partial}{\partial q_i}\big\{\pi_i(q_i, q_{-i}) - co_i(q)\big\} \quad \text{for every } i,$$

or equivalently iff (12) holds. The question which remains is whether system (12) can be solved? In fact, it can: take any solution q to fixed point system (10). Such a solution exists by Proposition 7. Note that (10) can be restated on the compact form

$$q \in \arg\max \sum_{i \in I}\big(\pi_i(\cdot, q_{-i}) - \big[c_I(\cdot + q_{I\setminus i}) - c_{I\setminus i}(q_{I\setminus i})\big]\big).$$

Take the total differential of the last sum to get the desired conclusion. The final assertion repeats part of Proposition 6. □

Condition (12) emphasizes a most natural principle, namely: there should always be *one* marginal cost in the cooperative business. To wit, when (12) cannot be satisfied, at least two agents might improve their outcomes by trading. It is illuminating that in equilibrium each agent acts as though he pays the aggregate marginal cost:

Proposition 8 (Equilibrium conditions). *In any equilibrium q it holds that*

$$0 \in \frac{\partial}{\partial q_i}\left\{\pi_i(\cdot, q_{-i}) - c_I\left(\cdot + \sum_{j \neq i} q_j\right)\right\}(q_i) \quad \textit{for all } i. \tag{13}$$

In particular, this happens when $0 \in (\partial/\partial q_i)\pi_i(q) - \partial c_I(q_I)\ \forall i$, *and a fortiori when there exists a common subgradient* $p \in \partial c_I(q_I)$ *verifying* (12). *Conversely, if each function* $\pi_i(q_i, q_{-i}) - c_I(q_i + \sum_{j \neq i} q_j)$ *is concave in* q_i, *then under* (13) *the profile* q *must be an equilibrium.*

5.1. The regional oligopoly continued

In (5) the function $f_{io}(q_{io})$ accounts for the production cost incurred by i at site o. Such cost typically increases at the margin; that is, f_{io} is convex. For conditions ensuring concavity of $P_d(\mathcal{S}_d)q_{id}$ in q_{id} see (Murphy et al., 1982). A particularly convenient instance has each P_d affine and sloping downwards. Provided each firm i has bounded production capacity $\bar{q}_{io} \geqslant 0$ at every origin o, posit $\bar{q}_{id} := \sum_{o \in O} \bar{q}_{io}$ for each destination d and set $Q_i := \prod_{o \in O}[0, \bar{q}_{io}] \times \prod_{d \in d}[0, \bar{q}_{id}]$. Existence of equilibrium now follows from Theorem 2.

6. Conclusion

Game theory currently undergoes some re-orientation, away from strategic-form, one-shot interaction between hyper-rational, omniscient players. Many recent studies allow repeated play of the same game and accommodate agents with bounded rationality, competence, or knowledge. A most natural question emerges then: *will the concerned parties learn Nash equilibrium over time*? That question seems particularly pressing in our setting. To complicate further, suppose no individual i knows, or gets to know, the payoff functions π_j of his co-players $j \neq i$. Then: *how can these noninformed agents ultimately come to solve system* (12)? Concerning that issue the following mode of repeated play appears promising.

Let $\{s_k\}$ be a sequence of nonnegative stepsizes $s_k \searrow 0$ such that $\sum s_k = \infty$. *Start with a feasible strategy profile* q^0 *determined by history or accident. Iteratively, at stage* $k = 0, 1, \ldots,$ *select an updated profile* q^{k+1} *by the rule*

$$q_i^{k+1} = P_{Q_i}\left[q_i^k + s_k\left\{\frac{\partial \pi_i(q^k)}{\partial q_i} - p^k\right\}\right] \quad \text{for all } i. \tag{14}$$

Here P_{Q_i} denotes the orthogonal projection onto Q_i, and $p^k \in \partial c_I(q_I^k)$ is a common marginal cost. The decomposed, decentralized nature of (14) makes it a natural and attractive candidate for moving towards a solution of (12). Process (14) demands merely that each agent forms a local perspective (as well as a linear approximation) and persistently moves in direction of (projected) payoff ascent. By approximation theory, as described in (Benaim, 1996, Proposition 3.1) convergence is likely to obtain; see (Flåm, 1996, 1998, 1999). For the regional oligopoly, when inverse demand is affine, the mapping $q \mapsto [(\partial/\partial q_i)\pi_i(q)]_{i \in I}$ becomes maximal monotone, a property most conducive to stability and computation of equilibrium (Aubin and Frankowska, 1990; Rockafellar and Wets, 1998). To explore possible convergence of (14) is left for a subsequent study.

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