GALERKIN-LIKE METHOD AND GENERALIZED PERTURBED SWEEPING PROCESS WITH NONREGULAR SETS

ABDERRAHIM JOURANI* AND EMILIO VILCHES *†

Abstract. In this paper we present a new method to solve differential inclusions. This method is a Galerkin-like method where we approach the original problem by projecting the state into a n-dimensional Hilbert space but not the velocity. We prove that the approached problem always has a solution and that under some compactness conditions the approached problems have a subsequence which converges strongly pointwisely to a solution of the original differential inclusion. We apply this method to the following differential inclusion:

$$
\begin{cases} -\dot{u}(t) = Bv(t) & \text{a.e. } t \in [T_0, T]; \\ -\dot{v}(t) \in N\left(C(t, u(t), v(t)); v(t)\right) + F(t, u(t), v(t)) + Au(t) & \text{a.e. } t \in [T_0, T]; \\ u(T_0) = u_0, v(T_0) = v_0 \in C(T_0, u_0, v_0), \end{cases} \tag{1}
$$

where $A\colon U \to V$ and $B\colon V \to U$ are two bounded linear operators, $N(S; \cdot)$ denotes the Clarke normal cone to a closed set $S \subset V$ and $F\colon [T_0, T] \times U \times V \rightrightarrows V$ is a set-valued mapping with nonempty closed and convex values satisfying some appropriate conditions. The sets $C(\cdot, \cdot, \cdot)$ are nonregular (equi-uniformly subsmooth or positively α-far). The differential inclusion (1) includes the Moreau's sweeping process, the state-dependent sweeping process and second-order sweeping process for which we give very general existence results.

Key words. Sweeping process, subsmooth sets, positively α-far sets, differential inclusions, second-order sweeping process, normal cone

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1. Introduction.

Let H, U and V be separable Hilbert spaces, T_0, T be two non-negative real numbers with $T_0 < T$. In this paper we present a new method to solve differential inclusions. This method is a Galerkin-like method where we approach the original problem by projecting the state into a n-dimensional Hilbert space but not the velocity. We prove that the approached problem always has a solution (see Proposition 13) and that under some compactness conditions the approached problems have a subsequence which converges strongly pointwisely to a solution of the original differential inclusion (see Theorem 14).

More explicitly, we consider the following differential inclusion:

$$
\begin{cases} \dot{x}(t) \in F(t, x(t)) & \text{a.e } t \in [T_0, T]; \\ x(T_0) = x_0. \end{cases} \tag{2}
$$

For each $n \in \mathbb{N}$ we approach (2) by the following differential inclusion:

$$
\begin{cases} \dot{x}(t) \in F(t, P_n(x(t))) & \text{a.e } t \in [T_0, T]; \\ x(T_0) = P_n(x_0), \end{cases}
$$

where, given an orthonormal basis $(e_n)_{n\in\mathbb{N}}$, P_n is the projector from H into $\operatorname{span}\{e_1, \ldots, e_n\}$. We will call this method *Galerkin-like method*. We will show how

*Univ. Bourgogne Franche-Comté, Institut de Mathématiques de Bourgogne UMR 5584 CNRS B.P. 47870, 21078 Dijon, France (jourani@u-bourgogne.fr).

†Departamento de Ingeniería Matemática, Universidad de Chile, Beauchef 851, Santiago, Chile (emilio.vilches@u-bourgogne.fr). The research of this author was supported by CONICYT-PCHA/Doctorado Nacional/2013-21130676.

this method is well adapted to deal with constrained differential inclusions by providing existence of solutions to the following differential inclusion:

$$\begin{cases} -\dot{u}(t) = Bv(t) & \text{a.e. } t \in [T_0, T]; \\ -\dot{v}(t) \in N\left(C(t, u(t), v(t)); v(t)\right) + F(t, u(t), v(t)) + Au(t) & \text{a.e. } t \in [T_0, T]; \\ u(T_0) = u_0, v(T_0) = v_0 \in C(T_0, u_0, v_0), \end{cases} \tag{3}$$

where $A\colon U \to V$ and $B\colon V \to U$ are two bounded linear operators, $N(S;\cdot)$ denotes the Clarke normal cone to a closed set $S \subset V$ and $F\colon [T_0, T] \times U \times V \rightrightarrows V$ is a set-valued mapping with nonempty closed and convex values satisfying some appropriate conditions.

We call the differential inclusion (3) *generalized sweeping process* because it includes the perturbed state-dependent sweeping process, the Moreau's sweeping process and the perturbed second-order sweeping process.

Perturbed state-dependent sweeping process. The perturbed state-dependent sweeping process is the following differential inclusion:

$$\begin{cases} -\dot{x}(t) \in N\left(C(t, x(t)); x(t)\right) + F(t, x(t)) & \text{a.e. } t \in [T_0, T]; \\ x(T_0) = x_0 \in C(T_0, x_0), \end{cases} \tag{4}$$

where for any subset S in H the set $N(S;\cdot)$ denotes the Clarke normal cone to S and $F\colon [T_0, T] \times H \rightrightarrows H$ is a set-valued mapping, called perturbation term, with nonempty closed and convex values. This differential inclusion includes the state-dependent sweeping process:

$$\begin{cases} -\dot{x}(t) \in N\left(C(t, x(t)); x(t)\right) & \text{a.e. } t \in [T_0, T]; \\ x(T_0) = x_0 \in C(T_0, x_0), \end{cases} \tag{5}$$

and the perturbed Moreau's sweeping process:

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); x(t)\right) + F(t, x(t)) & \text{a.e. } t \in [T_0, T]; \\ x(T_0) = x_0 \in C(T_0). \end{cases} \tag{6}$$

The study of this kind of differential inclusions was initiated by Moreau [41, 42, 43, 44, 45], for (6), to deal with problems arising in mechanics (see [40] for a general introduction to the subject). Since then, several authors have been interested in the existence and uniqueness of solutions in the convex and nonconvex case (see [19, 25, 9, 29, 50, 16, 28, 16, 51, 33, 38, 37]).

Concerning (5), as far as we know, it has been introduced and studied for the first time, for convex sets $C(t, x)$ in $\mathbb{R}^3$, by Chraibi Kaadoud [23] to model certain mechanical problems and later generalized to (4) in the convex and nonconvex setting.

In the convex setting, Kunze and Monteiro-Marques [39] proved the existence of solutions to (5) when the set-valued satisfies the following Lipschitz condition: There exist $L_1 \geq 0$ and $L_2 \in [0, 1[$ such that

$$|d(x, C(t, u)) - d(x, C(s, v))| \leq L_1|t - s| + L_2\|u - v\|, \tag{7}$$

for $t, s \in [T_0, T]$ and $x, u, v \in H$. Also, they showed that when $L_2 \geq 1$ no solution of (5) can be expected. The authors used Darbo's fixed point theorem to show the convergence of the following semi-implicit discretization scheme:

$$x_{i+1}^n = \operatorname{proj}\left(x_i^n; C(t_{i+1}^n, x_{i+1}^n)\right). \tag{8}$$

The discretization scheme (8) comes from an implicit discretization of (5) and can be seen as a generalization of the well known Moreau's Catching-up algorithm [41, 44, 45]. Next, Haddad and Haddad [32] showed, using an explicit discretization scheme, the existence of solutions to (4) in the particular case $C(t,x) := C(x)$ and $F(t,x) = Ax + f(t)$, where A is a linear bounded operator and f is a continuous and bounded function. This result was used to show the existence of solutions to a superconductivity model. Later, Haddad [31] showed the existence of solutions of (4) with upper semicontinuous perturbation by using the explicit discretization scheme:

$$x_{i+1}^n = \operatorname{proj}(x_i^n - \frac{T-T_0}{n} f_i^n; C(t_{i+1}^n, x_i^n)) \text{ and } f_i^n \in F(t_i^n, x_i^n). \tag{9}$$

Finally, Bounkhel and Castaing [14], by using (9), showed the existence of solutions to (5) in uniformly smooth and uniformly convex Banach spaces.

In the nonconvex case, Chemetov and Monteiro-Marques [21] proved the existence of solutions to (4) for uniformly prox-regular sets $C(t,x)$ with absolutely continuous variation in space and Lipschitz variation in time with a single-valued perturbation. They construct the operator $w = P(v)$ where w is the unique solution of (6) with $C(t) := C(t, v(t))$ and they show the existence of a fixed point of P via Schauder's fixed point theorem. Then, the same authors [22] proved the existence of solutions to (5) by using a fixed point argument in ordered spaces. Next, Castaing, Ibrahim and Yarou [20] used an extended version of Schauder's theorem and the discretization scheme (8) to show the existence of solutions to (5) in the uniformly prox-regular case. Later, Azzam-Laouir, Izza and Thibault [7] and Haddad, Kecis and Thibault [34] showed the existence of solutions to (4) in the finite dimensional and uniformly prox-regular setting with a perturbation term defined as the sum of an u.s.c and a mixed semicontinuous set-valued mapping with closed and convex values satisfying a linear growth condition. They reduce the constrained differential inclusion (5) to the following unconstrained one

$$\begin{cases} -\dot{x}(t) \in \dfrac{|\dot{\zeta}(t)|}{1-L_2} \partial d\left(x(t); C(t, x(t))\right) & \text{a.e. on } [T_0, T]; \\ x(T_0) = x_0 \in C(T_0, x_0), \end{cases}$$

where $L_2 \in [0,1[$ is the constant in (7) and ζ is the variation in time of C. Next, Noel [46] and Noel and Thibault [47] showed, respectively, the existence of solutions of (4) with equi-uniformly subsmooth and uniformly prox-regular sets for scalarly upper semicontinuous set-valued perturbations with closed and convex values. By using an extended Schauder theorem, they showed the convergence of the following semi-implicit discretization scheme:

$$x_{k+1}^n \in \operatorname{Proj}(x_k^n + \frac{T-T_0}{p^n} g(t_k^n, x_k^n); C(t_{k+1}^n, x_{k+1}^n)) \text{ and } g(t,x) = \operatorname{Proj}_{F(t,x)}(0),$$

Finally, Jourani and Vilches [37] showed the existence of solutions to (5) and (6) (with $F \equiv 0$), respectively, for subsmooth and positively α-far sets by using the Moreau-Yosida regularization techniques.

The perturbed state-dependent sweeping process (4) includes, as a special case, the Bensoussan-Lions-Mosco problem (see [48]): Find $v \in [T_0, T] \to H$ with $v(t) \in \Gamma(v(t))$ such that

$$a(v(t), u - v(t)) + \langle \dot{v}(t), u - v(t)\rangle \geq \langle l(t), u - v(t)\rangle, \tag{10}$$

for a.e. $t \in [T_0, T]$ and for all $u \in \Gamma(v(t))$, $v(T_0) = v_0 \in \Gamma(v_0)$. In the above parabolic quasi-variational inequality $a(\cdot, \cdot)$ is a real bilinear, symmetric, bounded and elliptic form on $H \times H$, $l \in L^1([T_0, T]; H)$ and $\Gamma(\cdot) \subset H$ is a convex set of constraints. The interest in the study of (10) arises in connection with quasi-static problems, sandpile growth and superconductivity models, among others (see [48, 49] for more details).

In section 7, we give a very general existence result to (4) (see Theorem 17) where the moving sets are assumed to be nonempty, closed and subsmooth with absolutely continuous variation in time and Lipschitz variation in the state. The perturbation term is supposed to be upper semicontinuous from H into H_w with nonempty, closed and convex values satisfying a weak linear growth condition, namely, the intersection between the perturbation term and the ball with linear growth is nonempty. This enables us to deal with unbounded perturbation terms.

Perturbed second-order sweeping process. The perturbed second-order sweeping process is the following differential inclusion:

$$\begin{cases} -\ddot{u}(t) \in N\left(C(t, u(t), \dot{u}(t)); \dot{u}(t)\right) + F(t, u(t), \dot{u}(t)) & \text{a.e. } t \in [T_0, T], \\ u(T_0) = u_0, \dot{u}(T_0) = v_0 \in C(T_0, u_0, v_0). \end{cases} \tag{11}$$

The study of this kind of differential inclusions was initiated by Castaing [18], where the moving set depends on the state with convex and compacts values. Since then, several works deal with second-order sweeping process with convex/prox-regular sets in Hilbert/Banach spaces (see [19, 10, 13, 5, 15, 6, 3, 12, 1]).

The second-order sweeping process (11) includes the dynamic analogue of the Signorini problem: Find $u\colon [T_0, T] \to H$, $u(T_0) = u_0$, $\dot{u}(T_0) = v_0 \in C(T_0)$ such that $\dot{u}(t) \in C(t)$ for a.e. $t \in [T_0, T]$ and

$$\langle l(t) - \ddot{u}(t), y - \dot{u}(t)\rangle \leq a\left(u(t), y - \dot{u}(t)\right) + J(t, y) - J(t, \dot{u}(t)), \tag{12}$$

for all $y \in C(t)$ and a.e. $t \in [T_0, T]$. Here $a(\cdot, \cdot) := \langle A(\cdot), \cdot\rangle$ is a real bilinear, symmetric, bounded and elliptic form on $H \times H$, $l \in L^1\left([T_0, T]; H\right)$ and $J\colon [T_0, T] \times H \to \mathbb{R}$ is a convex and locally Lipschitz continuous function. We observe that (12) can be written in the following form:

$$-\ddot{u}(t) \in \partial J\left(t, \dot{u}(t)\right) + N\left(C(t); \dot{u}(t)\right) + Au(t) - l(t) \quad \text{a.e. } t \in [T_0, T].$$

This differential inclusion can be studied in a more general context, namely, the convexity of J and $C(\cdot)$ can be removed.

In section 9, we give a very general existence result to (11) (see Theorem 22) where the moving sets are assumed to be nonempty, closed and subsmooth or positively α_0-far with absolutely continuous variation in time and Lipschitz variation in the state. The perturbation term is supposed to be upper semicontinuous from $H \times H$ into H_w with nonempty, closed and convex values satisfying a weak linear growth condition which enables us to deal with unbounded perturbation terms. We emphasize that the novelty of our work resides as much in the method as in the great generality in that the second-order sweeping process is treated. In fact, this is the first time in that the moving set depends jointly on the state and on the velocity.

The paper is organized as follows. After some preliminaries in section 3, we collect the hypotheses used along the paper. In section 4, we give some necessary lemmata that are used along the paper. The Galerkin-like method is studied in section 5 where we prove the existence of solutions to the approached problems section 1 (see Proposition 13) and its convergence (up to a sequence) strongly pointwisely to a solution of (2) (see Theorem 14). In section 6, we established existence of solutions to (3) via the Galerkin-like method. In section 7, section 8 and section 9, we obtain, respectively, existence of solutions to (4), (6) and (11). Finally, in the last section, we give an example proving the necessity of the compactness assumptions on the swept sets.

2. Preliminaries. From now on H, U and V stands for separable Hilbert spaces whose norm is denoted by $\|\cdot\|$. The closed ball centered at x with radius r is defined by $\bar{B}(x,\rho) := \{y \in H \colon \|x-y\| \leq \rho\}$ and the closed unit ball is denoted by $\mathbb{B}$. The notation H_w stands for H equipped with the weak topology and $x_n \rightharpoonup x$ denotes the weak convergence of a sequence $(x_n)_n$ to x (similar notation for U_w and V_w).

Recall that a vector $h \in H$ belongs to the Clarke tangent cone $T(S;x)$ when for every sequence $(x_n)_n$ in S to x and every sequence of positive numbers $(t_n)_n$ converging to 0, there exists some sequence $(h_n)_n$ in H converging to h such that $x_n + t_n h_n \in S$ for all $n \in \mathbb{N}$. This cone is closed and convex and its negative polar $N(S;x)$ is the Clarke normal cone to S at $x \in S$, that is,

$$N\left(S;x\right) = \left\{v \in H \colon \langle v,h\rangle \leq 0 \,\forall h \in T(S;x)\right\}.$$

As usual, $N(S;x) = \emptyset$ if $x \notin S$. Through that normal cone, the Clarke subdifferential of a function $f \colon H \to \mathbb{R}\cup\{+\infty\}$ is defined by

$$\partial f(x) := \left\{v \in H \colon (v,-1) \in N\left(\operatorname{epi} f, (x, f(x))\right)\right\},$$

where $\operatorname{epi} f := \{(y,r) \in H \times \mathbb{R} \colon f(y) \leq r\}$ is the epigraph of f. When the function f is finite and locally Lipschitzian around x, the Clarke subdifferential is characterized (see [24]) in the following simple and amenable way

$$\partial f(x) = \left\{v \in H \colon \langle v,h\rangle \leq f^{\circ}(x;h) \text{ for all } h \in H\right\},$$

where

$$f^{\circ}(x;h) := \limsup_{(t,y)\to(0^+,x)} t^{-1}\left[f(y+th) - f(y)\right],$$

is the *generalized directional derivative* of the locally Lipschitzian function f at x in the direction $h \in H$. The function $f^{\circ}(x;\cdot)$ is in fact the support of $\partial f(x)$. That characterization easily yields that the Clarke subdifferential of any locally Lipschitzian function has the important property of upper semicontinuity from H into H_w.

For $x \in H$ and $S \subset H$ the distance function is defined by $d_S(x) := \inf_{y\in S}\|x-y\|$. We denote $\operatorname{Proj}_S(x)$ the set (possibly empty)

$$\operatorname{Proj}_S(x) := \left\{y \in S \colon d_S(x) = \|x-y\|\right\}.$$

The equality (see [24])

$$N\left(S;x\right) = \overline{\mathbb{R}_+\partial d_S(x)} \quad \text{for } x \in S, \tag{13}$$

gives an expression of the Clarke normal cone in terms of the distance function. As usual, it will be convenient to write $\partial d(x,S)$ in place of $\partial d\left(\cdot,S\right)(x)$.

We denote by $L^1\left([T_0,T];H\right)$ the space of H-valued Lebesgue integrable functions defined over $[T_0,T]$. We write $L^1_w\left([T_0,T];H\right)$ to mean the space $L^1\left([T_0,T];H\right)$ endowed with the weak topology.

We say that $u \in \mathrm{AC}([T_0,T];H)$ if there exists $f \in L^1([T_0,T];H)$ and $u_0 \in H$ such that $u(t) = u_0 + \int_{T_0}^t f(s)ds$ for all $t \in [T_0,T]$. Also, we say that $u \in \mathrm{W}^{2,1}([T_0,T];H)$ if $\dot{u} \in \mathrm{AC}([T_0,T];H)$. Also, for $u\colon [T_0,T] \to H$ we define $\mathrm{Lip}(u) := \sup_{t\neq s} \|u(t) - u(s)\|/|t-s|$ and $\mathrm{Lip}([T_0,T];H) := \{u\colon [T_0,T] \to H\colon \mathrm{Lip}(u) < +\infty\}$.

The following lemma, proved in [37], is a compactness criteria for absolutely continuous functions.

Lemma 1. *Let $(x_n)_n$ be a sequence of absolutely continuous functions from $[T_0,T]$ into H with $x_n(T_0) = x_0^n$. Assume that for all $n \in \mathbb{N}$*

$$\|\dot{x}_n(t)\| \leq \psi(t) \quad a.e\ t \in [T_0,T], \tag{14}$$

where $\psi \in L^1(T_0,T)$ and that $x_0^n \to x_0$ as $n \to +\infty$. Then, there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ and an absolutely continuous function x such that

(i) $x_{n_k}(t) \rightharpoonup x(t)$ in H as $k \to +\infty$ for all $t \in [T_0,T]$;
(ii) $x_{n_k} \rightharpoonup x$ in $L^1([T_0,T];H)$ as $k \to +\infty$;
(iii) $\dot{x}_{n_k} \rightharpoonup \dot{x}$ in $L^1([T_0,T];H)$ as $k \to +\infty$;
(iv) $\|\dot{x}(t)\| \leq \psi(t)$ a.e. $t \in [T_0,T]$.

Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of H. For every $n \in \mathbb{N}$ we consider the linear operator P_n from H into $\mathrm{span}\{e_1,\ldots,e_n\}$ defined as

$$P_n\left(\sum_{k=1}^{\infty} \langle x, e_k\rangle e_k\right) = \sum_{k=1}^{n} \langle x, e_k\rangle e_k.$$

The following lemma summarize the main properties of the linear operator P_n.

Lemma 2. *(i) $\|P_n(x)\| \leq \|x\|$ for all $x \in H$;*
(ii) $\langle P_n(x), x - P_n(x)\rangle = 0$ for all $x \in H$;
(iii) $P_n(x) \to x$ as $n \to +\infty$ for all $x \in H$;
(iv) if $(x_n)_n$ is a bounded sequence with $x_n \rightharpoonup x$ as $n \to +\infty$ then $P_n(x_n) \rightharpoonup x$ as $n \to +\infty$;
(v) if $B \subset H$ is relatively compact then $\sup_{x\in B} \|x - P_n(x)\| \to 0$ as $n \to +\infty$.

Proof. It is enough to prove (iv): Let $j \in \mathbb{N}$. Then, for $n \geq j$:

$$\langle x_n - P_n(x_n), e_j\rangle = \sum_{k=n+1}^{+\infty} \langle x_n, e_k\rangle \langle e_k, e_j\rangle = 0.$$

Thus, by linearity,

$$\lim_{n\to+\infty} \langle x_n - P_n(x_n), v\rangle = 0 \quad \forall v \in \mathrm{span}\left(\{e_j\}_{j\in\mathbb{N}}\right).$$

Let $v \in H$. Then, there is $v_m \to v$ with $v_m \in \mathrm{span}(\{e_j\}_{j\in\mathbb{N}})$. Hence,

$$\begin{aligned} |\langle x_n - P_n(x_n), v\rangle| &\leq |\langle x_n - P_n(x_n), v - v_m\rangle| + |\langle x_n - P_n(x_n), v_m\rangle| \\ &\leq \|x_n - P_n(x_n)\| \cdot \|v_m - v\| + |\langle x_n - P_n(x_n), v_m\rangle| \\ &\leq 2\sup_{n\in\mathbb{N}} \|x_n\| \cdot \|v_m - v\| + |\langle x_n - P_n(x_n), v_m\rangle|. \end{aligned}$$

Therefore, taking the limit $n \to +\infty$ and then the limit $m \to +\infty$ we get the result.□

Let A be a bounded subset of H. We define the Kuratowski measure of non-compactness of A, $\alpha(A)$, as

$$\alpha(A) = \inf\{d > 0\colon A \text{ admits a finite cover by sets of diameter } \leq d\},$$

and the Hausdorff measure of non-compactness of A, $\beta(A)$, as

$$\beta(A) = \inf\{r > 0\colon A \text{ can be covered by finitely many ball of radius } r\}.$$

The following proposition gather the main properties of Kuratowski and Hausdorff measures of non-compactness (see [26, Section 9.2]).

PROPOSITION 3. *Let H be an infinite dimensional Hilbert space and B, B_1, B_2 be bounded subsets of H. Let γ be either the Kuratowski or the Hausdorff measures of non-compactness. Then,*

(i) $\gamma(B) = 0$ *if and only if* $\overline{B}$ *is compact;*
(ii) $\gamma(\lambda B) = |\lambda|\gamma(B)$ *for every* $\lambda \in \mathbb{R}$*;*
(iii) $\gamma(B_1 + B_2) \leq \gamma(B_1) + \gamma(B_2)$*;*
(iv) $B_1 \subset B_2$ *implies* $\gamma(B_1) \leq \gamma(B_2)$*;*
(v) $\gamma(\operatorname{conv} B) = \gamma(B)$*;*
(vi) $\gamma(\bar{B}) = \gamma(B)$*.*

The following lemma (see [26, Proposition 9.3]) is a useful rule for interchange of γ and integration.

LEMMA 4. *Let (v_n) be a sequence of measurable functions $v_n\colon [T_0, T] \to H$ such that $\sup_n \|v_n(t)\| \leq \psi(t)$ a.e. $t \in [T_0, T]$, where $\psi \in L^1(T_0, T)$. Then*

$$\gamma\left(\left\{\int_t^{t+h} v_n(s)ds\colon n \in \mathbb{N}\right\}\right) \leq \int_t^{t+h} \gamma\left(\{v_n(s)\colon n \in \mathbb{N}\}\right) ds,$$

for $T_0 \leq t < t + h \leq T$.

We recall the definition of the class of positively α-far sets, introduced in [33] and widely studied in [38].

DEFINITION 5. *Let $\alpha \in]0, 1]$ and $\rho \in]0, +\infty]$. Let S be a nonempty closed subset of X with $S \neq X$. We say that the Clarke subdifferential of the distance function $d(\cdot, S)$ keeps the origin α-far-off on the open ρ-tube around S, $U_\rho(S) := \{x \in H\colon 0 < d(x, S) < \rho\}$, provided*

$$0 < \alpha \leq \inf_{x \in U_\rho(S)} d(0, \partial d(\cdot, S)(x)). \tag{15}$$

Moreover, if E is a given nonempty set, we say that the family $(S(t))_{t\in E}$ is positively α-far if every $S(t)$ satisfies (15) with the same $\alpha \in]0, 1]$ and the same $\rho > 0$.

This notion includes strictly the notion of uniformly subsmooth sets (see Proposition 7) and the notion of uniformly prox-regular sets (see [38]).

DEFINITION 6. *Let S be a closed subset of H. We say that S is* uniformly subsmooth*, if for every $\varepsilon > 0$ there exists $\delta > 0$, such that*

$$\langle x_1^* - x_2^*, x_1 - x_2\rangle \geq -\varepsilon\|x_1 - x_2\|, \tag{16}$$

holds for all $x_1, x_2 \in S$ satisfying $\|x_1 - x_2\| < \delta$ and all $x_i^* \in N(S; x_i) \cap \mathbb{B}$ for $i = 1, 2$. Also, if E is a given nonempty set, we say that the family $(S(t))_{t\in E}$ is equi-uniformly subsmooth*, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that (16) holds for each $t \in E$ and all $x_1, x_2 \in S(t)$ satisfying $\|x_1 - x_2\| < \delta$ and all $x_i^* \in N(S(t); x_i) \cap \mathbb{B}$ for $i = 1, 2$.*

PROPOSITION 7 ([38]). *Assume that S is uniformly subsmooth. Then, for all $\varepsilon \in]0, 1[$ there exists $\rho \in]0, +\infty[$ such that the origin is kept positively $\sqrt{1-\varepsilon}$-far* *from the Clarke subdifferential of the distance function* $d(\cdot, S)$ *on the open* ρ*-tube* $U_\rho(S) = \{y \in H \colon 0 < d(y,S) < \rho\}$*, i.e.*

$$\sqrt{1-\varepsilon} \leq \inf_{y \in U_\rho(S)} d(0, \partial d(y,S)).$$

3. Technical assumptions. For the sake of readability, in this section we collect the hypotheses used along the paper.

Hypotheses on the set-valued map $C \colon [T_0,T] \times U \times V \rightrightarrows V$*.* C is a set-valued map with nonempty and closed values. Also, we will consider the following conditions:

($\mathcal{H}_1$) There exist $\zeta \in \mathrm{AC}([T_0,T];\mathbb{R})$, $L_1 \geq 0$ and $L_2 \in [0,1[$ such that for all $s,t \in [0,T]$ and all $x,y \in U$ and $u,v,w \in V$

$$|d(w, C(t,x,u)) - d(w, C(s,y,v))| \leq |\zeta(t) - \zeta(s)| + L_1\|x-y\| + L_2\|u-v\|.$$

($\mathcal{H}_2$) There exist two constants $\alpha_0 \in]0,1]$ and $\rho \in]0,+\infty]$ such that for every $(u,v) \in U \times V$

$$0 < \alpha_0 \leq \inf_{x \in U_\rho(C(t,u,v))} d\left(0, \partial d(\cdot, C(t,u,v))(x)\right) \quad \text{a.e. } t \in [T_0,T],$$

where $U_\rho\left(C(t,u,v)\right) = \{x \in V \colon 0 < d(x, C(t,u,v)) < \rho\}$.

($\mathcal{H}_3$) The family $\{C(t,u,v) \colon (t,u,v) \in [T_0,T] \times U \times V\}$ is equi-uniformly subsmooth.

($\mathcal{H}_4$) For every $t \in [T_0,T]$, every $r > 0$ and every pair of bounded sets $A \subset U$ and $B \subset V$, the set $C(t,A,B) \cap r\mathbb{B}$ is relatively compact.

Hypotheses on the set-valued map $C \colon [T_0,T] \times H \rightrightarrows H$*.* C is a set-valued map with nonempty and closed values. Also, we will consider the following conditions:

($\mathcal{H}_5$) There exist $\zeta \in \mathrm{AC}([T_0,T];\mathbb{R})$ and $L_2 \in [0,1[$ such that for all $s,t \in [0,T]$ and all $x,y,z \in H$

$$|d(z, C(t,x)) - d(z, C(s,y))| \leq |\zeta(t) - \zeta(s)| + L_2\|x-y\|.$$

($\mathcal{H}_6$) The family $\{C(t,v) \colon (t,v) \in [T_0,T] \times H\}$ is equi-uniformly subsmooth.

($\mathcal{H}_7$) For every $t \in [T_0,T]$, every $r > 0$ and every bounded set $A \subset H$ the set $C(t,A) \cap r\mathbb{B}$ is relatively compact.

Hypotheses on the set-valued map $C \colon [T_0,T] \rightrightarrows H$*.* C is a set-valued map with nonempty and closed values. Also, we will consider the following conditions:

($\mathcal{H}_8$) There exists $\zeta \in \mathrm{AC}([T_0,T];\mathbb{R})$ such that for all $s,t \in [0,T]$ and all $x \in H$

$$|d(x, C(t)) - d(x, C(s))| \leq |\zeta(t) - \zeta(s)|.$$

($\mathcal{H}_9$) There exist two constants $\alpha_0 \in]0,1]$ and $\rho \in]0,+\infty]$ such that

$$0 < \alpha_0 \leq \inf_{x \in U_\rho(C(t))} d\left(0, \partial d(x, C(t))\right) \quad \text{a.e. } t \in [T_0,T],$$

where $U_\rho\left(C(t)\right) = \{x \in H \colon 0 < d(x, C(t)) < \rho\}$ for all $t \in [T_0,T]$.

($\mathcal{H}_{10}$) For all $t \in [T_0,T]$ the set $C(t)$ is ball-compact, that is, for every $r > 0$ the set $C(t) \cap r\mathbb{B}$ is compact in H.

Remark 8.
1. Under $(\mathcal{H}_4)$, the set $\operatorname{Proj}_{C(t,u,v)}(v) \neq \emptyset$ for all $(t,u,v) \in [T_0,T]\times U\times V$. Indeed, let $(z_n)_n \subset C(t,u,v)$ such that $\|v-z_n\| \to d_{C(t,u,v)}(v)$ as $n \to +\infty$. Then, $(z_n)_n \subset r\mathbb{B}\cap C(t,\{u\},\{v\})$ for some $r>0$, which implies, by virtue of $(\mathcal{H}_4)$, that $(z_n)_n$ is relatively compact. Thus, a subsequence of (z_n) converges to an element of $\operatorname{Proj}_{C(t,u,v)}(v)$.
2. Let $L_2 \in [0,1[$. Under $(\mathcal{H}_3)$ for every $\alpha_0 \in]\sqrt{L_2},1]$ there exists $\rho>0$ such that $(\mathcal{H}_2)$ holds. This is a consequence of Proposition 7.

Hypotheses on the set-valued map $F\colon [T_0,T]\times U\times V \rightrightarrows V$. F is a set-valued map with nonempty, closed and convex values. Also, we will consider the following conditions:

$(\mathcal{H}_1^F)$ For each $(u,v)\in U\times V$, $F(\cdot,u,v)$ is measurable.
$(\mathcal{H}_2^F)$ For a.e. $t\in[T_0,T]$, $F(t,\cdot,\cdot)$ is upper semicontinuous from $U\times V$ into V_w.
$(\mathcal{H}_3^F)$ There exist $c,d\in L^1(T_0,T)$ such that

$$d\left(0,F(t,u,v)\right) := \inf\{\|w\|\colon w\in F(t,u,v)\} \le c(t)\|(u,v)\| + d(t),$$

for all $(u,v)\in U\times V$ and a.e. $t\in[T_0,T]$.

Hypotheses on the set-valued map $F\colon [T_0,T]\times H \rightrightarrows H$. F is a set-valued map with nonempty, closed and convex values. Also, we will consider the following conditions:

$(\mathcal{H}_4^F)$ For each $v\in H$, $F(\cdot,v)$ is measurable.
$(\mathcal{H}_5^F)$ For a.e. $t\in[T_0,T]$, $F(t,\cdot)$ is upper semicontinuous from H into H_w.
$(\mathcal{H}_6^F)$ There exist $c,d\in L^1(T_0,T)$ such that

$$d\left(0,F(t,v)\right) := \inf\{\|w\|\colon w\in F(t,v)\} \le c(t)\|v\| + d(t),$$

for all $v\in H$ and a.e. $t\in[T_0,T]$.

4. Preparatory lemmas. In this section we give some preliminary lemmas that will be used in the following sections. They are related to properties of the distance function and set-valued maps.

Lemma 9 ([37]). *Let $S\subset H$ be a ball-compact set. Then, for all $x\notin S$ we have*

$$\partial d_S(x) = \frac{x-\overline{\operatorname{co}}\operatorname{Proj}_S(x)}{d_S(x)}.$$

Lemma 10 ([37]). *Assume that $(\mathcal{H}_1)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$ hold. Then, for all $t\in[T_0,T]$ the set-valued map $(u,v)\rightrightarrows \partial d(\cdot,C(t,u,v))(v)$ is upper semicontinuous from $U\times V$ into V_w.*

The following lemma will be used in the proof of Proposition 13

Lemma 11. *Assume that $(\mathcal{H}_4^F)$, $(\mathcal{H}_5^F)$ and $(\mathcal{H}_6^F)$ hold and let $r\colon[T_0,T]\to\mathbb{R}_+$ be a continuous function. Then, the set-valued map $G\colon[T_0,T]\times H\rightrightarrows H$ defined by*

$$G(t,x) := F(t,p_{r(t)}(x)) \cap \left(c(t)\|p_{r(t)}(x)\| + d(t)\right)\mathbb{B} \quad (t,x)\in[T_0,T]\times H,$$

where $p_{r(t)}(x) = \begin{cases} x & \text{if } \|x\|\le r(t);\\ r(t)\frac{x}{\|x\|} & \text{if } \|x\|>r(t),\end{cases}$, *satisfies:*

(i) $G(t,x)$ *is nonempty, closed and convex for all* $(t,x) \in [T_0,T] \times H$;
(ii) *for each* $x \in H$, $G(\cdot,x)$ *is measurable;*
(iii) *for a.e.* $t \in [T_0,T]$, $G(t,\cdot)$ *is upper semicontinuous from* H *into* H_w;
(iv) *for all* $x \in H$ *and a.e.* $t \in [T_0,T]$

$$\|G(t,x)\| := \sup\{\|w\| \colon w \in G(t,x)\} \leq c(t)r(t) + d(t).$$

Proof. (i) is direct. (iii) follows from $(\mathcal{H}_5^F)$ and [4, Theorems 17.23 and 17.25]. Also, due to $(\mathcal{H}_6^F)$, we have

$$\begin{aligned} \|G(t,x)\| &= \sup\{\|w\| \colon w \in G(t,x)\} \\ &\leq c(t)\|p_{r(t)}(x)\| + d(t) \\ &\leq c(t)r(t) + d(t) \end{aligned}$$

which proves (iv). Thus, by virtue of (i) and (iv), G takes weakly compact and convex values. Therefore, (ii) follows from $(\mathcal{H}_4^F)$ and [36, Proposition 2.2.37]. ∎

The following result may be proved in much the same way as [37, Lemma 4.4] (see also [38, Lemma 5.7]).

LEMMA 12. *Let* $x, z \colon [T_0,T] \to V$ *and* $y \colon [T_0,T] \to U$ *be three absolutely continuous functions and let* $C \colon [T_0,T] \times U \times V \rightrightarrows V$ *be a set-valued map with nonempty closed values satisfying* $(\mathcal{H}_1)$. *Then*

(i) *The function* $t \to d(x(t); C(t,y(t),z(t)))$ *is absolutely continuous over* $[T_0,T]$.
(ii) *For all* $t \in]T_0,T[$, *where* $\dot{\zeta}(t)$, $\dot{y}(t)$ *and* $\dot{z}(t)$ *exist,*

$$\begin{aligned} &\limsup_{s\downarrow 0} \frac{1}{s}\left[d_{C(t+s,y(t+s),z(t+s))}(x(t+s)) - d_{C(t,y(t),z(t))}(x(t))\right] \\ &\leq |\dot{\zeta}(t)| + L_1\|\dot{y}(t)\| + L_2\|\dot{z}(t)\| \\ &\qquad\qquad + \limsup_{s\downarrow 0} \frac{1}{s}\left[d_{C(t,y(t),z(t))}(x(t+s)) - d_{C(t,y(t),z(t))}(x(t))\right]. \end{aligned}$$

(iii) *For all* $t \in]T_0,T[$, *where* $\dot{x}(t)$ *exists,*

$$\begin{aligned} &\limsup_{s\downarrow 0} \frac{1}{s}\left[d_{C(t,y(t),z(t))}(x(t+s)) - d_{C(t,y(t),z(t))}(x(t))\right] \\ &\qquad\qquad\qquad \leq \max_{y^* \in \partial d_{C(t,y(t),z(t))}(x(t))} \langle y^*, \dot{x}(t)\rangle. \end{aligned}$$

(iv) *For all* $t \in \{s \in [T_0,T] \colon x(s) \notin C(s,y(s),z(s))\}$, *where* $\dot{x}(t)$ *exists,*

$$\begin{aligned} &\lim_{s\downarrow 0} \frac{1}{s}\left[d_{C(t,y(t),z(t))}(x(t+s)) - d_{C(t,y(t),z(t))}(x(t))\right] \\ &\qquad\qquad\qquad = \min_{y^* \in \partial d(x(t),C(t,y(t),z(t)))} \langle y^*, \dot{x}(t)\rangle. \end{aligned}$$

(v) *For every* $x \in V$ *the set-valued map* $t \rightrightarrows \partial d(x, C(t,y(t),z(t)))$ *is measurable.*

5. Galerkin-like method. In this section we study existence of solutions to the following differential inclusion:

$$\left\{ \begin{aligned} \dot{x}(t) &\in F(t,x(t)) \quad \text{a.e. } t \in [T_0,T]; \\ x(T_0) &= x_0, \end{aligned} \right. \tag{17}$$

where $F\colon [T_0, T] \times H \rightrightarrows H$ is a set-valued map with nonempty closed and convex values. For every $n \in \mathbb{N}$ let us consider the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in F(t, P_n(x(t))) & \text{a.e. } t \in [T_0, T]; \\ x(T_0) = P_n(x_0), \end{cases} \tag{18}$$

where $P_n\colon H \to \operatorname{span}\{e_1, \ldots, e_n\}$ is the linear operator defined in Lemma 2. The next proposition asserts the existence of solutions for the approximate problem (18).

Proposition 13. *Assume that $(\mathcal{H}_4^F)$, $(\mathcal{H}_5^F)$ and $(\mathcal{H}_6^F)$ hold. Then, for each $n \in \mathbb{N}$ there exists at least one solution $x_n \in \mathrm{AC}([T_0, T]; H)$ of (18). Moreover,*

$$\|x_n(t)\| \le r(t) := \left(\|x_0\| + \int_{T_0}^t d(s)ds\right) \exp\left(\int_{T_0}^t c(s)ds\right) \quad \textit{for all } t \in [T_0, T], \tag{19}$$

and

$$\|\dot{x}_n(t)\| \le \psi(t) := c(t)r(t) + d(t) \quad \textit{a.e. } t \in [T_0, T]. \tag{20}$$

Proof. Let us consider $G(t, x) := F(t, p_{r(t)}(x)) \cap \big(c(t)\|p_{r(t)}(x)\| + d(t)\big)$, where $p_{r(t)}\colon H \to H$ is given by

$$p_{r(t)}(x) = \begin{cases} x & \text{if } \|x\| \le r(t); \\ r(t)\frac{x}{\|x\|} & \text{if } \|x\| > r(t), \end{cases}.$$

Then, due to Lemma 11, G satisfies $(\mathcal{H}_4^F)$, $(\mathcal{H}_5^F)$ and

$$\|G(t, x)\| := \sup\{\|w\|\colon w \in G(t, x)\} \le c(t)r(t) + d(t), \tag{21}$$

for all $x \in H$ and a.e. $t \in [T_0, T]$.

Consider the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in G(t, P_n(x(t))) & \text{a.e. } t \in [T_0, T]; \\ x(T_0) = P_n(x_0). \end{cases} \tag{22}$$

Let $K \subset L^1([T_0, T]; H)$ be defined by

$$K := \left\{f \in L^1([T_0, T]; H) : \|f(t)\| \le \psi(t) \text{ a.e. } t \in [T_0, T]\right\},$$

where ψ is defined by (20). This set is nonempty, closed and convex. In addition, since $\psi \in L^1(T_0, T)$, K is bounded and uniformly integrable, hence, it is compact in $L^1_w([T_0, T]; H)$ (see [30, Theorem 2.3.24]). Since $L^1([T_0, T]; H)$ is separable, we also note that K, endowed with the relative $L^1_w([T_0, T]; H)$ topology is a metric space (see [27, Theorem V.6.3]). Define the map $\mathcal{F}_n\colon K \rightrightarrows L^1([T_0, T]; H)$ by

$$\mathcal{F}_n(f) := \left\{v \in L^1([T_0, T]; H) : v(t) \in G(t, P_n(x_0 + \int_{T_0}^t f(s)ds)) \text{ a.e. } t \in [T_0, T]\right\},$$

for $f \in K$. By $(\mathcal{H}_4^F)$, $(\mathcal{H}_5^F)$, (21) and [2, Lemma 6], we conclude that $\mathcal{F}_n(f)$ has nonempty, closed and convex values. Moreover, $\mathcal{F}_n(K) \subset K$. Indeed, let $f \in K$ and

$v \in \mathcal{F}_n(f)$. Then,

$$\begin{aligned}\|v(t)\| &\leq \sup\{\|w\| \colon w \in G(t, P_n(x_0 + \int_{T_0}^t f(s)ds))\} \\ &\leq c(t)r(t) + d(t) \\ &= \psi(t).\end{aligned}$$

We denote K_w the set K seen as a compact convex subset of $L^1_w([T_0, T]; H)$.
Claim 1: $\mathcal{F}_n$ is upper semicontinuous from K_w into K_w.
Proof of Claim 1: By virtue of [36, Proposition 1.2.23] it is sufficient to prove that its graph graph$(\mathcal{F}_n)$ is sequentially closed in $K_w \times K_w$.

Let $(f_m, v_m) \in \text{graph}(\mathcal{F}_n)$ with $f_m \to f$ and $v_m \to v$ in $L^1_w([T_0, T]; H)$ as $m \to +\infty$. We have to show that $(f, v) \in \text{graph}(\mathcal{F}_n)$. To do that, let us define

$$u_m(t) := P_n(x_0) + \int_{T_0}^t f_m(s)ds \quad \text{for every } t \in [T_0, T].$$

Thus,

$$v_m(t) \in G(t, P_n(u_m(t))) \text{ for a.e. } t \in [T_0, T]. \tag{23}$$

Also, since $f_m \in K$, we have that

$$\|\dot{u}_m(t)\| \leq \psi(t) \quad \text{a.e. } t \in [T_0, T].$$

Hence, due to Lemma 1, there exists a subsequence of $(u_m)_m$ (without relabeling) and an absolutely continuous function $u \colon [T_0, T] \to H$ such that

$$\begin{aligned} u_m(t) &\to u(t) \text{ weakly for all } t \in [T_0, T]; \\ \dot{u}_m &\to \dot{u} \text{ in } L^1_w([T_0, T]; H), \end{aligned}$$

which implies that $\dot{u} = f$. Moreover, since $(u_m(t))_m$ is bounded for every $t \in [T_0, T]$, $P_n(u_m(t)) \to P_n(u(t))$ for every $t \in [T_0, T]$. Consequently, by virtue of [30, Proposition 2.3.1], (23) and the upper semicontinuity of G from H into H_w, for a.e. $t \in [T_0, T]$

$$\begin{aligned} v(t) &\in \overline{\text{conv}}\, w\text{-}\limsup_{m \to +\infty}\{v_m(t)\} \\ &\subset \overline{\text{conv}}\, G(t, P_n(u(t))) \\ &= G(t, P_n(u(t))), \end{aligned}$$

which shows that $(f, v) \in \text{graph}(\mathcal{F}_n)$, as claimed. $\square$

Now, we can invoke the Kakutani-Fan-Glicksberg fixed point theorem (see [4, Corollary 17.55]) to the set-valued map $\mathcal{F}_n \colon K_w \rightrightarrows K_w$ to deduce the existence of $\widehat{f}_n \in K$ such that $\widehat{f}_n \in \mathcal{F}_n(\widehat{f}_n)$. Then, the function $x_n \in \text{AC}([T_0, T]; H)$ defined for every $t \in [T_0, T]$ as:

$$x_n(t) = P_n(x_0) + \int_{T_0}^t \widehat{f}_n(s)ds,$$

is a solution of (22). Moreover, $x_n \in \text{AC}([T_0, T]; H)$ is a solution of (18). Indeed, for a.e. $t \in [T_0, T]$,

$$\begin{aligned} \|\dot{x}_n(t)\| &\leq c(t)\|p_{r(t)}(P_n(x_n(t)))\| + d(t) \\ &\leq c(t)\|x_n(t)\| + d(t), \end{aligned}$$

which, by Gronwall's inequality and the fact that $\|P_n(x_0)\| \leq \|x_0\|$, implies (19). Finally, $\|P_n(x_n(t))\| \leq r(t)$ and $p_{r(t)}(P_n(x_n(t))) = P_n(x_n(t))$ for all $t \in [T_0, T]$, which finishes the proof. $\square$

The following theorem asserts the existence of solution of (17) under a compactness condition on the sequence $(P_n(x_n(t)))_n$ for every $t \in [T_0, T]$.

THEOREM 14. *Let assumptions* $(\mathcal{H}_4^F)$, $(\mathcal{H}_5^F)$ *and* $(\mathcal{H}_6^F)$ *hold. Assume that the sequence* $(P_n(x_n(t)))_n$ *is relatively compact for all* $t \in [T_0, T]$. *Then, there exists a subsequence* $(x_{n_k})_k$ *of* $(x_n)_n$ *converging strongly pointwisely to a solution* $x \in \mathrm{AC}([T_0, T]; H)$ *of* (17). *Moreover,*

$$\|x(t)\| \leq r(t) := \left(\|x_0\| + \int_{T_0}^t d(s)ds\right) \exp\left(\int_{T_0}^t c(s)ds\right) \quad \textit{for all } t \in [T_0, T],$$

and

$$\|\dot{x}(t)\| \leq \psi(t) := c(t)r(t) + d(t) \quad \textit{a.e. } t \in [T_0, T].$$

Proof. We will show the existence of the subsequence via Lemma 1.
Claim 1: There exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ and an absolutely continuous function x such that (i), (ii), (iii) and (iv) from Lemma 1 hold with ψ defined as in the statement of the theorem.
Proof of Claim 1: According to Proposition 13, $\|\dot{x}_n(t)\| \leq \psi(t) = c(t)r(t) + d(t)$ for a.e. $t \in [T_0, T]$, which shows that (14) holds with the function ψ defined as above. Also, $P_n(x_0) \to x_0$ as $n \to +\infty$. Therefore, the claim follows from Lemma 1. $\square$

By simplicity we denote $x_k := x_{n_k}$ for $k \in \mathbb{N}$.
Claim 2: $P_k(x_k(t)) \rightharpoonup x(t)$ as $k \to +\infty$ for all $t \in [T_0, T]$.
Proof of Claim 2: Since $x_k(t) \rightharpoonup x(t)$ as $k \to +\infty$ for all $t \in [T_0, T]$, the result follows from (iv) of Lemma 2. $\square$
Claim 3: $P_k(x_k(t)) \to x(t)$ as $k \to +\infty$ for all $t \in [T_0, T]$.
Proof of Claim 3: The result follows from Claim 2 and the relative compactness of the sequence $(P_n(x_n(t)))_n$ for a.e. $t \in [T_0, T]$. $\square$

Summarizing, we have

(i) For each $x \in H$, $F(\cdot, x)$ is measurable;
(ii) for a.e. $t \in [T_0, T]$, $F(t, \cdot)$ is upper semicontinuous from H into H_w.
(iii) $\dot{x}_k \rightharpoonup \dot{x}$ in $L^1([T_0, T]; H)$;
(iv) $P_k(x_k(t)) \to x(t)$ as $k \to +\infty$ for a.e. $t \in [T_0, T]$;
(v) for all $k \in \mathbb{N}$, $\dot{x}_k(t) \in F(t, P_k(x_k(t)))$ for a.e. $t \in [T_0, T]$.

These conditions and the Convergence Theorem (see [2, Proposition 5] for more details) implies that $x \in \mathrm{AC}([T_0, T]; H)$ is a solution of (17), which finishes the proof. $\square$

6. A generalized perturbed sweeping process with nonregular sets. In this section we study the generalized perturbed sweeping process:

$$(24) \quad \begin{cases} -\dot{u}(t) = Bv(t) & \text{a.e. } t \in [T_0, T]; \\ -\dot{v}(t) \in N\left(C(t, u(t), v(t)); v(t)\right) + F(t, u(t), v(t)) + Au(t) & \text{a.e. } t \in [T_0, T]; \\ u(T_0) = u_0, v(T_0) = v_0 \in C(T_0, u_0, v_0), \end{cases}$$

where $A\colon U \to V$ and $B\colon V \to U$ are two *bounded linear operators*, $C\colon [T_0, T] \times U \times V \rightrightarrows V$ is a set-valued mapping with *nonempty closed values* and $F\colon [T_0, T] \times U \times V \rightrightarrows V$ is a set-valued mapping with *nonempty closed and convex values.*

The following theorem, which is the main result of this section, gives an existence result for (24).

Theorem 15. *Assume that the set-valued mapping C satisfies $(\mathcal{H}_1)$, $(\mathcal{H}_3)$, $(\mathcal{H}_4)$ and the set-valued mapping F satisfies $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_3^F)$. Then, for all $\alpha_0 \in]\sqrt{L_2}, 1]$ there exists at least one solution $(u,v) \in \mathrm{W}^{2,1}([T_0,T];U) \times \mathrm{AC}([T_0,T];V)$ of (24) satisfying*

$$\|(u(t),v(t))\| \leq \mu(t) := \left(\|(u_0,v_0)\| + \int_{T_0}^{t} \widetilde{d}(s)ds\right)\exp\left(\int_{T_0}^{t} \widetilde{c}(s)ds\right),$$

for all $t \in [T_0,T]$, where

$$\begin{aligned}\widetilde{c}(t) &:= \frac{\alpha_0^2+1}{\alpha_0^2-L_2}\left(c(t)+\|A\|\right) + \left(1+\frac{L_1}{\alpha_0^2-L_2}\right)\|B\|;\\ \widetilde{d}(t) &:= \frac{\alpha_0^2+1}{\alpha_0^2-L_2}d(t) + \frac{1}{\alpha_0^2-L_2}|\dot{\zeta}(t)|,\end{aligned}$$

for all $t \in [T_0,T]$.

Proof. The proof will be divided into two steps.

<u>*Step 1*</u>: We first prove the theorem under the additional assumption:

(25)
$$\frac{\alpha_0^2+1}{\alpha_0^2-L_2}\int_{T_0}^{T}\Big(|\dot{\zeta}(s)| + L_1\|B\|\mu(s) + (1+L_2)(c(s)\mu(s)+d(s)+\|A\|\mu(s))\Big)ds < \rho,$$

where $\rho > 0$ is defined by Remark 8.

Let $m\colon [T_0,T]\times U\times V \to \mathbb{R}$ be defined by

(26)
$$m(t,u,v) := \frac{1}{\alpha_0^2-L_2}\Big(|\dot{\zeta}(t)| + L_1\|B\|\|v\| + (1+L_2)(c(t)\|(u,v)\| + d(t) + \|A\|\|u\|)\Big),$$

for all $(t,u,v)\in[T_0,T]\times U\times V$.

Define the set-valued map $G\colon [T_0,T]\times U\times V \rightrightarrows U\times V$ as

$$G(t,u,v) = (-Bv, -m(t,u,v)\partial d_{C(t,u,v)}(v) - F(t,u,v) - Au),$$

for all $(t,u,v)\in[T_0,T]\times U\times V$. We will show, by using Theorem 14, that the following differential inclusion has at least one solution:

$$\begin{cases} (\dot{u}(t),\dot{v}(t)) \in G(t,u(t),v(t)) & \text{a.e. } t\in[T_0,T];\\ (u(T_0),v(T_0)) = (u_0,v_0). \end{cases} \tag{27}$$

Claim 1:

(i) For each $(u,v)\in U\times V$, $G(\cdot,u,v)$ is measurable.
(ii) for a.e. $t\in[T_0,T]$, $G(t,\cdot,\cdot)$ is upper semicontinuous from $U\times V$ into $U_w\times V_w$;
(iii) for all $(u,v)\in U\times V$ and a.e. $t\in[T_0,T]$

$$d(0,G(t,u,v)) \leq \widetilde{c}(t)\|(u,v)\| + \widetilde{d}(t),$$

where $\widetilde{c}$ and $\widetilde{d}$ are defined as in the statement of the theorem.

Proof of Claim 1: (i) follows from Lemma 12 and $(\mathcal{H}_1^F)$. Also, (ii) follows from Lemma 10 and $(\mathcal{H}_2^F)$. To prove (iii) let $(u,v) \in U \times V$ and $t \in [T_0, T]$. Then, by virtue of $(\mathcal{H}_3^F)$,

$$\begin{aligned} d(0, G(t,u,v)) &= \inf\{\|w\| \colon w \in G(t,u,v)\} \\ &\leq \|B\|\|v\| + m(t,u,v) + \inf\{\|w\| \colon w \in F(t,u,v)\} + \|A\|\|u\| \\ &\leq \|B\|\|v\| + m(t,u,v) + c(t)\|(u,v)\| + d(t) + \|A\|\|u\| \\ &\leq \widetilde{c}(t)\|(u,v)\| + \widetilde{d}(t), \end{aligned}$$

which finishes the proof of Claim 1. □

For each $n \in \mathbb{N}$, let us consider the following differential inclusion:

$$\tag{28} \begin{cases} (\dot{u}(t), \dot{v}(t)) \in G(t, P_n(u(t)), Q_n(v(t))) & \text{a.e. } t \in [T_0, T]; \\ (u(T_0), v(T_0)) = (P_n(u_0), Q_n(v_0)), \end{cases}$$

where $(P_n)_n$ and $(Q_n)_n$ are, respectively, orthonormal basis of U and V. By virtue of Proposition 13, the differential inclusion (28) has at least one solution $(u_n, v_n) \in \mathrm{AC}([T_0,T];U) \times \mathrm{AC}([T_0,T];V)$. Moreover,

$$\tag{29} \|(u_n(t), v_n(t))\| \leq \mu(t) \quad \text{for all } t \in [T_0, T],$$

and

$$\tag{30} \|(\dot{u}_n(t), \dot{v}_n(t))\| \leq \widetilde{c}(t)\mu(t) + \widetilde{d}(t) \quad \text{a.e. } t \in [T_0, T],$$

where μ, $\widetilde{c}$ and $\widetilde{d}$ are defined as in the statement of the theorem.
To simplify the notation, we write

$$\begin{aligned} m_n(t) &:= m(t, P_n(u_n(t)), Q_n(v_n(t))) \\ \Gamma_n(t) &:= \partial d_{C(t, P_n(u_n(t)), Q_n(v_n(t)))}(Q_n(v_n(t))), \end{aligned}$$

and we note that (see (29) and (30))

$$\tag{31} \begin{aligned} m_n(t) \leq \delta(t) :=& \frac{1}{\alpha_0^2 - L_2}\left(|\dot{\zeta}(t)| + L_1\|B\|\mu(t)\right) \\ &+ \frac{1}{\alpha_0^2 - L_2}\left((1 + L_2)(c(t)\mu(t) + d(t) + \|A\|\mu(t))\right), \end{aligned}$$

for a.e. $t \in [T_0, T]$. Moreover, there exist $f_n(t) \in F(t, P_n(u_n(t)), Q_n(v_n(t)))$ and $d_n(t) \in \Gamma_n(t)$ such that

$$\begin{cases} -\dot{u}_n(t) = B(Q_n(v_n(t))) & \text{a.e. } t \in [T_0, T]; \\ -\dot{v}_n(t) = m_n(t)d_n(t) + f_n(t) + A(Q_n(v_n(t))) & \text{a.e. } t \in [T_0, T], \end{cases}$$

Define $\varphi_n(t) = d_{C(t, P_n(u_n(t)), Q_n(v_n(t)))}(Q_n(v_n(t)))$ for $t \in [T_0, T]$.
Claim 2: For all $t \in [T_0, T]$

$$\varphi_n^3(t) \leq 3\int_{T_0}^{t} \delta(s) \sup_{x \in D(s)} \|x - Q_n(x)\|^2 ds,$$

where by $(\mathcal{H}_4)$ the set $D(t) := \overline{\mathrm{co}}\,(C(t, \mu(t)\mathbb{B}, \mu(t)\mathbb{B}) \cap (\rho + \mu(t))\mathbb{B})$ is relatively compact for every $t \in [T_0, T]$.

Proof of Claim 2: The idea of the proof is to use $(\mathcal{H}_2)$ (see Remark 8). To do that, we proceed to show first that $\varphi_n(t) < \rho$ for all $t \in [T_0, T]$. Indeed, let $t \in [T_0, T]$ where $\dot{u}_n(t)$ and $\dot{v}_n(t)$ exist. Then, due to Lemma 12, (31) and (26),

$$\begin{aligned}
\dot{\varphi}_n(t) &\leq |\dot{\zeta}(t)| + L_1\|P_n(\dot{u}_n(t))\| + L_2\|Q_n(\dot{v}_n(t))\| + \max_{y^*\in\Gamma_n(t)} \langle y^*, Q_n(\dot{v}_n(t))\rangle \\
&\leq |\dot{\zeta}(t)| + L_1\|B\|\|Q_n(v_n(t))\| + (1+L_2)\|Q_n(\dot{v}_n(t))\| \\
&\leq |\dot{\zeta}(t)| + L_1\|B\|\|Q_n(v_n(t))\| + (1+L_2)\left[m_n(t) + c(t)\|(P_n(u_n(t)), Q_n(v_n(t)))\|\right. \\
&\quad \left. + d(t) + \|A\|\|P_n(u_n(t))\|\right] \\
&= (\alpha_0^2+1)m_n(t) \\
&\leq (\alpha_0^2+1)\delta(t),
\end{aligned}$$

Therefore, according to (25), $\varphi_n(t) < \rho$ for all $t \in [T_0, T]$.

Now, let $t \in \Omega_n := \{t \in [T_0, T] \colon Q_n(v_n(t)) \notin C(t, P_n(u_n(t)), Q_n(v_n(t)))\}$ where $\dot{u}_n(t)$ and $\dot{v}_n(t)$ exist. Then, due to Lemma 12,

$$\begin{aligned}
\dot{\varphi}_n(t) &\leq |\dot{\zeta}(t)| + L_1\|P_n(\dot{u}_n(t))\| + L_2\|Q_n(\dot{v}_n(t))\| + \min_{y^*\in\Gamma_n(t)} \langle y^*, Q_n(\dot{v}_n(t))\rangle \\
&= |\dot{\zeta}(t)| + L_1\|B\|\|Q_n(v_n(t))\| + L_2(m_n(t) + c(t)\|(P_n(u_n(t)), Q_n(v_n(t)))\| \\
&\quad + d(t) + \|A\|\|P_n(u_n(t))\|) + \min_{y^*\in\Gamma_n(t)} \langle y^*, Q_n(\dot{v}_n(t))\rangle
\end{aligned}$$

Also, since $d_n(t) \in \Gamma_n(t)$,

$$\begin{aligned}
\min_{y^*\in\Gamma_n(t)} \langle y^*, Q_n(\dot{v}_n(t))\rangle &\leq \langle d_n(t), Q_n(\dot{v}_n(t))\rangle \\
&= \langle d_n(t), Q_n\left(-m_n(t)d_n(t) - f_n(t) - A(P_n(u_n(t)))\right)\rangle \\
&\leq \|f_n(t)\| + \|A\|\|P_n(u_n(t))\| - m_n(t)\langle d_n(t), Q_n(d_n(t))\rangle \\
&\leq c(t)\|(P_n(u_n(t)), Q_n(v_n(t)))\| + d(t) + \|A\|\|P_n(u_n(t))\| \\
&\quad - m_n(t)\langle d_n(t), Q_n(d_n(t))\rangle .
\end{aligned}$$

Hence, by using the last two estimations and (26), we obtain

$$\dot{\varphi}_n(t) \leq m_n(t)\left(\alpha_0^2 - \langle d_n(t), Q_n(d_n(t))\rangle\right).$$

Moreover, due to $(\mathcal{H}_2)$,

$$\begin{aligned}
\langle d_n(t), -Q_n(d_n(t))\rangle &= \langle d_n(t), d_n(t) - Q_n(d_n(t))\rangle + \langle d_n(t), -d_n(t)\rangle \\
&\leq \langle d_n(t), d_n(t) - Q_n(d_n(t))\rangle - \alpha_0^2 \\
&= \|d_n(t) - Q_n(d_n(t))\|^2 - \alpha_0^2.
\end{aligned}$$

Then,

$$\begin{aligned}
\dot{\varphi}_n(t) &\leq m_n(t)\left(\alpha_0^2 - \langle d_n(t), -Q_n(d_n(t))\rangle\right) \\
&\leq m_n(t)\|d_n(t) - Q_n(d_n(t))\|^2 \\
&\leq \delta(t)\|d_n(t) - Q_n(d_n(t))\|^2.
\end{aligned}$$

Furthermore, for $t \in \Omega_n$, since $d_n(t) \in \Gamma_n(t)$, Lemma 9 ensures the existence of $g_n(t) \in \overline{\operatorname{co}}\,\operatorname{Proj}_{C(t,P_n(u_n(t)),Q_n(v_n(t)))}(Q_n(v_n(t)))$ such that

$$d_n(t) = \frac{1}{\varphi_n(t)}\left(Q_n(v_n(t)) - g_n(t)\right). \tag{32}$$

Then,

$$\begin{aligned}\|g_n(t)\| &\leq \varphi_n(t) + \|Q_n(v_n(t))\| \\ &\leq \rho + \mu(t),\end{aligned}$$

which entails that $g_n(t) \in D(t)$ for all $t \in \Omega_n$. Thus, for every $t \in \Omega_n$ (see 32)

$$\begin{aligned}\varphi_n(t)^2\|d_n(t) - Q_n(d_n(t))\|^2 &= \|g_n(t) - Q_n(g_n(t))\|^2 \\ &\leq \sup_{x\in D(t)} \|x - Q_n(x)\|^2.\end{aligned}$$

Let $t \in [T_0, T]$. Then,

$$\begin{aligned}\varphi_n^3(t) &= \varphi_n^3(T_0) + 3\int_{T_0}^{t} \varphi_n^2(s)\dot{\varphi}_n(s)ds \\ &\leq 3\int_{T_0}^{t} \delta(s) \sup_{x\in D(s)} \|x - Q_n(x)\|^2 ds,\end{aligned}$$

as claimed. $\square$

Claim 3: $\lim_{n\to+\infty} \varphi_n(t) = 0$ for all $t \in [T_0, T]$.

Proof of Claim 3: Fix $t \in [T_0, T]$. Then, since $D(t)$ is relatively compact and (v) from Lemma 2,

$$\lim_{n\to+\infty} \sup_{x\in D(t)} \|x - Q_n(x)\| = 0.$$

Hence, by Fatou's lemma and Claim 2,

$$\begin{aligned}\limsup_{n\to+\infty} \varphi_n^3(t) &\leq 3\limsup_{n\to+\infty} \int_{T_0}^{t} \delta(s) \sup_{x\in D(s)} \|x - Q_n(x)\|^2 ds \\ &\leq 3\int_{T_0}^{t} \delta(s) \limsup_{n\to+\infty} \sup_{x\in D(s)} \|x - Q_n(x)\|^2 ds \\ &= 0,\end{aligned}$$

as required. $\square$

Claim 4: $(P_n(u_n(t)))_n$ and $(Q_n(v_n(t)))_n$ are relatively compact for all $t \in [T_0, T]$.

Proof of Claim 4: Let $\gamma = \alpha$ or $\gamma = \beta$ be either the Kuratowski or the Hausdorff measure of non-compactness. On the one hand, let

$$s_n(t) \in \operatorname{Proj}_{C(t,P_n(u_n(t)),Q_n(v_n(t)))}\left(Q_n(v_n(t))\right).$$

Then, $s_n(t) \in (\rho + \mu(t))\mathbb{B}$ and, due to Claim 3,

$$\begin{aligned}\gamma\left(\{Q_n(v_n(t))\colon n \in \mathbb{N}\}\right) &= \gamma\left(\{s_n(t)\colon n \in \mathbb{N}\}\right) \\ &\leq \gamma\left(C\left(t, \mu(t)\mathbb{B}, \mu(t)\mathbb{B}\right) \cap (\rho + \mu(t))\mathbb{B}\right) \\ &= 0,\end{aligned}$$

which shows that $(Q_n(v_n(t)))_n$ is relatively compact. On the other hand, by using

Lemma 4 and the relative compactness of $(Q_n(v_n(t)))_n$ for all $t \in [T_0, T]$, we obtain

$$\begin{aligned}\gamma(\{u_n(t)\colon n \in \mathbb{N}\}) &= \gamma(\{P_n(u_0) + \int_{T_0}^{t} \dot{u}_n(s)ds\colon n \in \mathbb{N}\})\\ &\leq \gamma(\{P_n(u_0)\colon n \in \mathbb{N}\}) + \gamma\left(\left\{\int_{T_0}^{t} \dot{u}_n(s)ds\colon n \in \mathbb{N}\right\}\right)\\ &= \gamma\left(\left\{-\int_{T_0}^{t} B(Q_n(v_n(s)))ds\colon n \in \mathbb{N}\right\}\right)\\ &= 0,\end{aligned}$$

which shows that $(u_n(t))_n$ is relatively compact for all $t \in [T_0, T]$. Therefore, the sequence $(P_n(u_n(t)))_n$ is relatively compact for all $t \in [T_0, T]$, as claimed. $\square$

Hence, we have verified all the hypotheses of Theorem 14. Therefore, there exists at least one solution $(u, v) \in \mathrm{AC}([T_0, T]; U) \times \mathrm{AC}([T_0, T]; V)$ of (27). Now it remains to show that (u, v) is a solution of (24).

Claim 5: For all $t \in [T_0, T]$, $v(t) \in C(t, u(t), v(t))$.

Proof of Claim 5: Fix $t \in [T_0, T]$. Then, as in the proof of Theorem 14, $P_k(u_k(t)) \to u(t)$ and $Q_k(v_k(t)) \to v(t)$, where $(u_k, v_k)_k$ is a subsequence of $(u_n, v_n)_n$. Thus, due to Claim 3,

$$\begin{aligned}&d_{C(t,u(t),v(t))}(v(t))\\ &= \limsup_{k\to+\infty} \left(d_{C(t,u(t),v(t))}(v(t)) - \varphi_k(t) + \varphi_k(t)\right)\\ &\leq \limsup_{k\to+\infty} ((1 + L_2)\|v(t) - Q_k(v_k(t))\| + L_1\|u(t) - P_k(u_k(t))\| + \varphi_k(t))\\ &= 0,\end{aligned}$$

as claimed. $\square$

Finally, by virtue of (13) and Claim 5, (u, v) is also a solution of (24).

Step 2. In the general case, without any restriction on the length of T, let us consider $\{T_0, T_1 \ldots, T_N\}$ be a partition of $[T_0, T]$ such that for every $k \in \{1, \ldots, N\}$

$$\frac{\alpha_0^2 + 1}{\alpha_0^2 - L_2} \int_{T_{k-1}}^{T_k} \left(|\dot{\zeta}(s)| + L_1\|B\|\mu(t) + (1 + L_2)(c(t)\mu(t) + d(t) + \|A\|\mu(t))\right) ds < \rho.$$

For $k = 1$, due to Step 1, let (u^1, v^1) be a solution of (24) over $[T_0, T_1]$. Then, $v^1(t) \in C(t, u^1(t), v^1(t))$ for all $t \in [T_0, T_1]$ and $u^1(T_0) = u_0$ and $v^1(T_0) = v_0 \in C(T_0, u_0, v_0)$. Inductively, for $k = 2, \ldots, N$, since $v^{k-1}(T_{k-1}) \in C(T_{k-1}, u^{k-1}(T_{k-1}), v^{k-1}(T_{k-1}))$, let (u^k, v^k) be a solution of (24) over $[T_{k-1}, T_k]$. Then, $v^k(t) \in C(t, u^k(t), v^k(t))$ for all $t \in [T_{k-1}, T_k]$ and $v^k(T_{k-1}) = v^{k-1}(T_{k-1})$.

Finally, we define $u(t) = u^k(t)$ and $v(t) = v^k(t)$ over $[T_{k-1}, T_k]$, for $k = 1, \ldots, N$. Then (u, v) is a solution of (24), which finishes the proof of the theorem. $\square$

According to the proof of Theorem 15, we observe that $(\mathcal{H}_3)$ was used only to obtain the upper semicontinuity of $\partial d_{C(t,\cdot,:)}(:)$ from $U \times V$ into V_w for all $t \in [T_0, T]$. Since, when $C(t, u, v) \equiv C(t)$ for all $(u, v) \in U \times V$ and $t \in [T_0, T]$ the subdifferential $\partial d_{C(t)}(\cdot)$ is always upper semicontinuous from V into V_w for all $t \in [T_0, T]$, we have the following existence result for (24) with positively α_0-far sets.

THEOREM 16. *Suppose that the set-valued mapping $C : [T_0, T] \times V \rightrightarrows V$ is nonempty and closed-valued and satisfies $(\mathcal{H}_8)$, $(\mathcal{H}_9)$, $(\mathcal{H}_{10})$ and that the set-valued* *mapping F satisfies $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_3^F)$. Then there exists at least one solution $(u,v) \in \mathrm{AC}([T_0,T];U) \times \mathrm{AC}([T_0,T];V)$ of*

$$\begin{cases} -\dot{u}(t) = Bv(t) & \text{a.e. } t \in [T_0,T]; \\ -\dot{v}(t) \in N\left(C(t);v(t)\right) + F(t,u(t),v(t)) + Au(t) & \text{a.e. } t \in [T_0,T]; \\ u(T_0) = u_0, v(T_0) = v_0 \in C(T_0), \end{cases}$$

satisfying

$$\|(u(t),v(t))\| \leq \mu(t) := \left(\|(u_0,v_0)\| + \int_{T_0}^t \widetilde{d}(s)ds\right) \exp\left(\int_{T_0}^t \widetilde{c}(s)ds\right),$$

for all $t \in [T_0,T]$, where

$$\begin{aligned} \widetilde{c}(t) &:= \frac{\alpha_0^2+1}{\alpha_0^2}\left(c(t) + \|A\|\right) + \|B\|; \\ \widetilde{d}(t) &:= \frac{\alpha_0^2+1}{\alpha_0^2} d(t) + \frac{1}{\alpha_0^2}|\dot{\zeta}(t)|, \end{aligned}$$

for all $t \in [T_0,T]$.

7. Perturbed state-dependent sweeping process. In this section we give and existence result for the perturbed state-dependent sweeping process:

$$\begin{cases} -\dot{v}(t) \in N\left(C(t,v(t));v(t)\right) + F(t,v(t)) & \text{a.e. } t \in [T_0,T]; \\ v(T_0) = v_0 \in C(T_0,v_0), \end{cases} \tag{33}$$

where $C : [T_0,T] \times V \rightrightarrows V$ is set-valued mapping with nonempty and closed values and $F : [T_0,T] \times V \rightrightarrows V$ is a set-valued mapping with *nonempty closed and convex values.*

The following result, consequence of Theorem 15, gives a very general existence result for the perturbed state-dependent sweeping process. The following theorem is related to [37, Theorem 6.1] and improves the results given in [46, 47].

THEOREM 17. *Suppose that the set-valued mapping C satisfies $(\mathcal{H}_5)$, $(\mathcal{H}_6)$ and $(\mathcal{H}_7)$ and that the set-valued mapping F satisfies $(\mathcal{H}_4^F)$, $(\mathcal{H}_5^F)$ and $(\mathcal{H}_6^F)$. Then, for all $\alpha_0 \in]\sqrt{L_2}, 1]$ there exists at least one solution $v \in \mathrm{AC}([T_0,T];H)$ of (33) satisfying:*

$$\|v(t)\| \leq \left(\|v_0\| + \int_{T_0}^t \widetilde{d}(s)ds\right) \exp\left(\int_{T_0}^t \widetilde{c}(s)ds\right) \quad \text{for all } t \in [T_0,T],$$

where for all $t \in [T_0,T]$

$$\begin{aligned} \widetilde{c}(t) &:= \frac{\alpha_0^2+1}{\alpha_0^2 - L_2} c(t); \\ \widetilde{d}(t) &:= \frac{\alpha_0^2+1}{\alpha_0^2 - L_2} d(t) + \frac{1}{\alpha_0^2 - L_2}|\dot{\zeta}(t)|. \end{aligned}$$

Remark 18. 1. The hypothesis $L_2 \in [0,1[$ in Theorem 17 cannot be improved. In fact, there are counterexamples to the existence of solutions to (33) when $L_2 \geq 1$ (see [39]).

2. It is well known that under the conditions of Theorem 17, uniqueness of solution to (33) (even for convex sets) does not necessarily hold (see [8, 39] for more details). However, Krejčí and Schnabel [17] have proved existence of solutions to (5) when the dependence of the Minkowski function and its gradient are Lipschitz functions.
3. Existence results for the state-dependent sweeping process with uniformly subsmooth sets have been proved in [46] under very strong conditions. In fact, in [46] it is assumed that for any bounded set A, the set $C([T_0,T],A)$ is relatively ball-compact, C has a Lipschitz variation in both variables and the perturbation term F is upper semicontinuous from $[T_0,T]\times H$ into H_w, with bounded perturbation term F.

As an application of Theorem 17 we get existence of solutions for the Bensoussan-Lions-Mosco problem (10). The following proposition improves [32, Proposition 17.5] where the authors assume that $\Gamma(\cdot)\subset K$ for some convex compact set K and $l\in W^{1,2}([T_0,T];H)\cap L^{\infty}([T_0,T];H)$.

Proposition 19. *Let $a(\cdot,\cdot)$ be a bilinear, symmetric, bounded and elliptic form and $l\in L^1([T_0,T];H)$. Assume that $\Gamma\colon H\rightrightarrows H$ is Lipschitz continuous with ratio $0<L<1$, takes closed convex values and for any bounded set A, the set $\Gamma(A)$ is relatively ball-compact. Then, for every $v_0\in\Gamma(v_0)$, there exists at least one solution of (10).*

8. Perturbed Moreau's sweeping process. In this section we give an existence result for the perturbed sweeping process:

$$\begin{cases} -\dot v(t)\in N\left(C(t);v(t)\right)+F(t,v(t)) & \text{a.e. } t\in[T_0,T];\\ v(T_0)=v_0\in C(T_0,v_0), \end{cases} \tag{34}$$

when the set-valued map takes positively α_0-far values. The following result, consequence of Theorem 16, was established in [38] by using a completely different approach.

Theorem 20. *Assume that $(\mathcal{H}_8)$, $(\mathcal{H}_9)$ and $(\mathcal{H}_{10})$ hold. Let $F\colon[T_0,T]\times H\rightrightarrows H$ be a set-valued mapping with nonempty closed and convex values satisfying $(\mathcal{H}_4^F)$, $(\mathcal{H}_5^F)$ and $(\mathcal{H}_6^F)$. Then, there exists at least one solution $v\in\mathrm{AC}([T_0,T];H)$ of (34) satisfying*

$$\|v(t)\|\le\left(\|v_0\|+\int_{T_0}^t\widetilde{d}(s)ds\right)\exp\left(\int_{T_0}^t\widetilde{c}(s)ds\right)\quad\textit{for all } t\in[T_0,T],$$

where for all $t\in[T_0,T]$

$$\begin{aligned}\widetilde{c}(t)&:=\frac{\alpha_0^2+1}{\alpha_0^2}c(t);\\ \widetilde{d}(t)&:=\frac{\alpha_0^2+1}{\alpha_0^2}d(t)+\frac{1}{\alpha_0^2}|\dot\zeta(t)|.\end{aligned}$$

Related to uniqueness for (34) with positively α_0-far sets, we have the following negative example.

Example 21. Let us consider the set-valued map $C\colon[0,1]\rightrightarrows\mathbb{R}^2$ defined by $C(t)=S-(t,0)$ for $t\in[0,1]$, where $S=\{(x,y)\in\mathbb{R}^2\colon|y|\ge x\}\cap\mathbb{B}$ (see Figure 1). Then, $C(t)$ is $\sqrt{2}/2$-far. Also, $v_1(t)=(-t/2,t/2)$ and $v_2(t)=(-t/2,-t/2)$ defined for $t\in[0,1]$

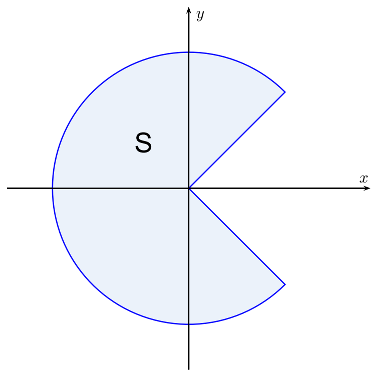

FIG. 1. *Set S in (38).*

are solutions of (34) with $F \equiv 0$. Thus, in general, there is no uniqueness of solutions to (34) with positively α_0-far sets. This is not the case when the sets C are convex or r-uniformly prox-regular (see for instance [11]).

9. Perturbed second-order sweeping process. In this section we give an existence result for the perturbed second-order sweeping process:

$$\text{(35)}\qquad \begin{cases} -\ddot{u}(t) \in N\left(C(t,u(t),\dot{u}(t));\dot{u}(t)\right) + F(t,u(t),\dot{u}(t)) & \text{a.e. } t \in [T_0,T], \\ u(T_0) = u_0, \dot{u}(T_0) = v_0 \in C(T_0,u_0,v_0), \end{cases}$$

The following result, consequence of Theorem 15, extends several works present in the literature [19, 10, 13, 6, 12, 5, 3] where the authors assume that the set-valued map takes convex or uniformly prox-regular values.

THEOREM 22. *Assume that $(\mathcal{H}_1)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$ hold. Let $F\colon [T_0,T]\times H \times H \rightrightarrows H$ be a set-valued mapping with nonempty closed and convex values satisfying $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_3^F)$. Then, for all $\alpha_0 \in]\sqrt{L_2},1]$ there exists at least one solution $u \in \mathrm{W}^{2,1}([T_0,T];H)$ of (35) satisfying*

$$\|(u(t),\dot{u}(t))\| \leq \left(\|(u_0,v_0)\| + \int_{T_0}^t \widetilde{d}(s)ds\right)\exp\left(\int_{T_0}^t \widetilde{c}(s)ds\right) \quad \textit{for all } t \in [T_0,T],$$

where

$$\widetilde{c}(t) := \frac{\alpha_0^2+1}{\alpha_0^2-L_2}c(t) \quad \textit{and} \quad \widetilde{d}(t) := \frac{\alpha_0^2+1}{\alpha_0^2-L_2}d(t) + \frac{1}{\alpha_0^2-L_2}|\dot{\zeta}(t)|,$$

for all $t \in [T_0,T]$.

By using Theorem 15, we can get existence of solutions for a variant of the second-order sweeping process with perturbation considered by Bounkhel and Haddad [15]. The next proposition greatly extends [15, Theorem 3.1], where the authors assume that $C(\cdot)$ is uniformly prox-regular, $C(\cdot) \subset K$ for some convex compact set K and $F\colon [T_0,T]\times H \rightrightarrows H$ is an upper semicontinuous set-valued mapping from $[T_0,T]\times H$ into H_w with nonempty closed convex values satisfying the stronger linear growth condition: There exists $L>0$ such that $F(t,x) \subset L\left(1+\|x\|\right)$ for all $(t,x) \in [T_0,T]\times H$.

Proposition 23. *Let $C\colon [T_0,T] \rightrightarrows H$ be a set-valued map satisfying $(\mathcal{H}_9)$, $(\mathcal{H}_8)$ and $(\mathcal{H}_{10})$, $A\colon H \to H$ be a linear bounded operator and let $F\colon [T_0,T]\times H \rightrightarrows H$ be a set-valued map with nonempty closed and convex values satisfying $(\mathcal{H}_4^F)$, $(\mathcal{H}_5^F)$ and $(\mathcal{H}_6^F)$. Then, there exists at least one solution $u \in \mathrm{W}^{2,1}([T_0,T];H)$ of the problem*

$$\begin{cases} -\ddot{u}(t) \in N\left(C(t);\dot{u}(t)\right) + F(t,\dot{u}(t)) + Au(t) & a.e.\ t\in[T_0,T],\\ u(T_0)=u_0, \dot{u}(T_0)=v_0\in C(T_0). \end{cases}$$

As a consequence of Proposition 23, we obtain the existence of solutions of the following dynamic analogue to the Signorini problem : Find $u\colon [T_0,T]\to H$, $u(T_0)=u_0$, $\dot{u}(T_0)=v_0\in C(T_0)$ such that

$$-\ddot{u}(t) \in \partial J\left(t,\dot{u}(t)\right) + N\left(C(t);\dot{u}(t)\right) + Au(t) - l(t) \quad \text{a.e. } t\in[T_0,T]. \tag{36}$$

Here $C\colon [T_0,T]\rightrightarrows H$ is a set-valued map with *closed values*, $a(\cdot,\cdot) := \langle A(\cdot),\cdot\rangle$ is a *real bilinear, symmetric, bounded and elliptic form* on $H\times H$, $l\in L^1([T_0,T];H)$ and $J\colon [T_0,T]\times H\to\mathbb{R}$ is a *locally Lipschitz continuous* function.

The following corollary extends [15, Corollary 1], where the authors assume that $C(\cdot)\subset K$ for some convex compact set K, l is uniformly bounded and J is time-independent and uniformly Lipschitz continuous.

Corollary 24. *Suppose that the set-valued mapping C satisfies $(\mathcal{H}_8)$, $(\mathcal{H}_9)$ and $(\mathcal{H}_{10})$, $l\in L^1([T_0,T];H)$ and that the function J is such that the set-valued mapping $F:=\partial J$ satisfies $(\mathcal{H}_4^F)$, $(\mathcal{H}_5^F)$ and $(\mathcal{H}_6^F)$. Then, for every $u_0\in H$ and any $v_0\in C(T_0)$, there exists at least one solution of (36).*

10. The necessity of the compactness assumptions. The existence of a solution of unperturbed sweeping process has been established in the literature (see the introduction) in the case where the sets $(C(t))$ are uniformly prox-regular. But the situation becomes more complicate in presence of the perturbation. As shown by the following counter-example, the compactness assumptions $(\mathcal{H}_4)$, $(\mathcal{H}_7)$ and $(\mathcal{H}_{10})$ in the previous theorems cannot be removed. It shows that a sweeping process governed by a single-valued continuous perturbation mapping and a normal cone to a closed bounded convex and autonomous set may have no solution. This example is based on the reference [35] where the authors have shown that in every separable Banach space X there is a continuous function $f\colon X\to X$ such that the autonomous differential equation

$$\dot{x}(t) = f(x(t)) \tag{37}$$

has no solutions in any interval of the real line (see [35, Theorem 8]). Since f is continuous at 0, we may assume that f is bounded on $r\mathbb{B}$, for some $r>0$. By considering this function we define

$$g(x) = \begin{cases} f(x) & \text{if } \|x\|\le r,\\ f\left(r\frac{x}{\|x\|}\right) & \text{if } \|x\|>r, \end{cases}$$

which is continuous in X and uniformly bounded. Now, consider the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in -N\left((M+1)\mathbb{B};x(t)\right) + g(x(t)) & \text{a.e. } t\in[0,T]\\ x(0)=0, \end{cases} \tag{38}$$

where $M := \sup_{x\in X} \|g(x)\|$. Assume that (38) has a solution $x\colon [0,T] \to X$ for some $T > 0$. Then,

$$\langle -\dot{x}(t) + g(x(t)), y - x(t)\rangle \leq 0 \quad \forall y \in (M+1)\mathbb{B}.$$

Since $x(t) \in (M+1)\mathbb{B}$ for all $t \in [0,T]$, we have for every $t \in [0,T]$ where $\dot{x}(t)$ exists

$$\langle -\dot{x}(t) + g(x(t)), \dot{x}(t)\rangle = 0.$$

Thus $\|\dot{x}(t)\| \leq M$ for a.e. $t \in [0,T]$ and hence $\|x(t)\| \leq MT$ for all $t \in [0,T]$. Thus, if $T \leq \min(\frac{r}{M}, 1)$, $x(t) \in \operatorname{int}(M+1)\mathbb{B}$ for all $t \in [0,T]$. Therefore,

$$\dot{x}(t) = g(x(t)) \quad \text{ a.e. } t \in [0,T],$$

which, since g and x are continuous and $\|x\|_\infty \leq r$, implies that x is a solution of (37). Therefore, the system (38) has no solutions.

Remark 25. The function $g\colon X \to X$ is continuous, thus is usc strongly-weakly from X into X.

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