The validity of the "lim inf" formula and a characterization of Asplund spaces

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Abstract

Our basic object in this paper is to show that for a given bornology β on a Banach space X the following "lim inf" formula holds true

$$\liminf_{x' \xrightarrow{C} x} T_\beta(C; x') \subset T_c(C; x)$$

for every closed set $C \subset X$, and $x \in C$, provided that the space $X \times X$ is ∂_β-trusted. Here $T_\beta(C; x)$ and $T_c(C; x)$ denote the β-tangent cone and the Clarke tangent cone to C at x. The trustworthiness includes spaces with an equivalent β-differentiable norm or more generally with a Lipschitz β-differentiable bump function. As a consequence, we show that for the Fréchet bornology, this "lim inf" formula characterizes in fact the Asplund property of X. We use our results to obtain new characterizations of T_β-pseudoconvexity of X.

Key Words. Tangent cones, subdifferentials, bornology, Asplund space, Gâteaux (Fréchet) differentiability, psodoconvexity, trustworthiness.

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1 Introduction

Let X be a real Banach space and X^* be its topological dual with pairing $\langle \cdot, \cdot \rangle$. A *bornology* β on X is a family of bounded and centrally symmetric subsets of

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X whose union is X, which is closed under multiplication by positive scalars and is directed upwards (i.e., the union of any two members of β is contained in some member of β). The most important bornologies are Gâteaux bornology consisting of all finite subset of X, Hadamard bornology consisting of all norm compact sets, weak Hadamard bornology consisting of all weakly compact sets and Fréchet bornology consisting of all bounded sets.
Each bornology β generates a β-subdifferential which in turn gives rise to the β-normal cone, and hence by making polars to the β-tangent cone.
In this paper, we are concerned with sufficient conditions on a Banach space X satisfying the following "lim inf" formula

$$\liminf_{x' \xrightarrow{C} x} T_\beta(C; x') \subset T_c(C; x) \tag{1.1}$$

for each closed set $C \subset X$, and for each $x \in C$. Here $T_\beta(C; x)$ and $T_c(C; x)$ denote the β-tangent cone and the Clarke tangent cone to C at x.
This kind of formulas has been studied by many authors in special situations. They started with the work by Cornet [6] who found a topological connection between the Clarke tangent cone and the contingent cone $K(C; x)$ to C at x. He has shown that if $C \subset \mathbb{R}^m$, then

$$T_c(C; x) = \liminf_{x' \xrightarrow{C} x} K(C; x').$$

Using his new characterization of Clarke tangent cone, Treiman [20]-[21](see also [8] for an independent proof) showed that the inclusion

$$\liminf_{x' \xrightarrow{C} x} K(C; x') \subset T_c(C; x)$$

is true in any Banach space and equality holds whenever C is epi-Lipschitzian at x in the sense of Rockafellar [19]. But this result does not include the finite dimensional case where this formula holds true for any closed set. In [4],[5], Borwein and Strojwas introduced the concept of compactly epi-Lipschitz sets to show that the previous equality holds for C in this class unifying the finite and infinite dimensional situations. In the case when the space is reflexive, these authors obtained the following equality

$$T_c(C, x) = \liminf_{x' \xrightarrow{C} x} WK(C, x')$$

where $WK(C, x)$ denotes the weak-contingent cone to C at x. They generalize the results of Penot [16] for finite dimensional and reflexive Banach spaces and of Cornet [6] for finite dimensional spaces. Aubin-Frankowska [2] obtained the following formula

$$T_c(C; x) = \liminf_{x' \xrightarrow{C} x} WK(C; x') = \liminf_{x' \xrightarrow{C} x} co(WK(C; x'))$$

in the case when the space X is uniformly smooth and the norm of X^* is Fréchet differentiable off the origin.

In their paper [5], Borwein and Strojwas gave a characterization of reflexive Banach spaces. They showed that the following assertions are equivalent :

(i) $T_c(C;x) \subset \liminf_{x' \xrightarrow{C} x} coWK(C;x')$, *for all closed sets* $C \subset X$, *and all* $x \in C$;
(ii) X *is reflexive.*

The validity of the "lim inf" formula (1.1) has been accomplished in Borwein and Ioffe [3] in the case when the space X admits a β-differentiable equivalent norm.

Our aim in this paper is to show that if the space $X \times X$ is ∂_β-trusted or equivalently basic fuzzy principle is satisfied on $X \times X$ (this includes spaces with equivalent β-smooth norm or more generaly spaces with Lipschitz β-smooth bump function) then the "lim inf" formula (1.1) holds. As a consequence, we show that for the Fréchet bornology, the formula (1.1) characterizes in fact the Asplund property of X. We then use our results to obtain new characterizations of β-pseudoconvexity.
The plan of the present paper is as follows: After recalling some tools of nonsmooth analysis in the second section, we establish in the third one a connection between Gâteaux (Fréchet) differentiability of the norm and the regularity of the set $D = \overline{\mathbb{B}^c} = \{x \in X : \|x\| \geq 1\}$. For $\bar{x} \in D$, with $\|\bar{x}\| = 1$, Borwein and Strojwas [5] showed that Gâteaux differentiability of the norm at $\bar{x}$ is equivalent to $\bar{co}K(C;\bar{x}) \neq X$. We prove that Gâteaux differentiability of the norm at $\bar{x}$ is equivalent to $K(D;\bar{x})$ equal to a half space which in turn is equivalent to the Clarke tangential regularity of D at $\bar{x}$. Similar results are obtained for Fréchet differentiability by means of the Fréchet normal cone to D. In the fourth section, we prove our main theorem and some of its consequences. In the fifth section, we give some corollaries, namely a new characterization of Asplund spaces: A Banach space is Asplund space if and only if the "lim inf" formula holds true with the Fréchet bornology for any closed set $C \subset X$. The last section concerns characterizations of T_β-pseudoconvex sets.

2 Notation and Preliminaries

Let X be a Banach space with a given norm $\|\cdot\|$, X^* be its topological dual space and $\langle \cdot, \cdot \rangle$ be the duality pairing between X and X^*. The sphere of X and the open ball in X centered at x and of radius δ are defined by $S_X = \{h \in X : \|h\| = 1\}$ and $B(x,\delta) = \{h \in X : \|h - x\| < \delta\}$.

Let C be a closed subset of X. The *contingent cone* $K(C;x)$ (resp. weak-*contingent cone* $WK(C;x)$) to C at x is the set of all $h \in X$ for which there are a sequence (h_n) in X converging strongly (resp. weakly) to h and a sequence of positive numbers (t_n) converging to zero such that

$$x + t_n h_n \in C,$$

for all $n \in \mathbb{N}$. A vector $h \in X$ belongs to the *Clarke tangent* cone $T_c(C;x)$ of C

at x provided that for any real $\varepsilon > 0$ there exists a real $\delta > 0$ such that

$$\big(u + tB(h, \varepsilon)\big) \cap C \neq \emptyset,$$

for all $u \in C \cap B(x, \delta)$ and $t \in]0, \delta[$. It is known that $h \in T_c(C; x)$ if and only if for any sequences $(x_n) \in C$ converging to x and (t_n) of positive numbers converging to zero there is a sequence (h_n) in X converging to h such that

$$x_n + t_n h_n \in C,$$

for all $n \in \mathbb{N}$. It is obvious that $T_c(C; x) \subset K(C; x)$. The *Clarke normal cone* is defined as the negative polar cone of the Clarke tangent cone, that is,

$$N_c(C; x) := \{x^* \in X^* : \ \langle x^*, h\rangle \leq 0 \text{ for all } h \in T_c(C; x)\}.$$

Let us recall that the (negative) polar cone of a convex cone K is given by

$$K^0 = \{x^* \in X^* : \langle x^*, h\rangle \leq 0 \ \ \forall h \in K\}.$$

Definition 2.1 *Let $f : X \to \mathbb{R} \cup \{\pm\infty\}$ be a function finite at x and β be a bornology on X.*
(a) f is said to be β-differentiable (or β-smooth) at x if it is Gâteaux differentiable at x uniformly on members of β, that is, there is $x^* \in X^*$ such that for each set $S \in \beta$

$$\lim_{t\to\infty} t^{-1} \sup_{h\in S} |f(x + th) - f(x) - \langle x^*, th\rangle\, | = 0,$$

(b) $x^* \in X^*$ is a β-subgradient of f at x, if for each $\epsilon > 0$ and each set $S \in \beta$ there is $\delta > 0$ such that for all $0 < t < \delta$ and all $h \in S$

$$t^{-1}\big(f(x + th) - f(x)\big) - \langle x^*, h\rangle \geq -\epsilon.$$

We denote by $\partial_\beta f(x)$ the set of all β-subgradient of f at x.

It follows from this definition that if $\beta_1 \subset \beta_2$, then $\partial_{\beta_2} f(x) \subset \partial_{\beta_1} f(x)$.

Applying Definition 2.1(a) to the bounded bornology, Gâteaux bornology and Hadamard bornology, we obtain the following classical definitions of:

- Fréchet differentiability: There is $x^* \in X^*$ such that

$$\lim_{h\to 0} \|h\|^{-1}(f(x + h) - f(x) - \langle x^*, h\rangle) = 0.$$

- Gâteaux differentiability: There is $x^* \in X^*$ such that

$$\forall h \in X, \quad \lim_{t\to 0^+} t^{-1}(f(x + th) - f(x)) = \langle x^*, h\rangle.$$

- Hadamard differentiability: There is $x^* \in X^*$ such that

$$\forall h \in X, \quad \lim_{\substack{t \to 0^+, \\ u \to h}} t^{-1}(f(x+tu) - f(x)) = \langle x^*, h \rangle .$$

While Definition 2.1(b) leads ([14]) in the case of the bounded bornology, Gâteaux bornology and Hadamard bornology, to the following classical definitions of :

- Fréchet-subdifferential of f at x:

$$\partial_\beta f(x) = \partial_F f(x) = \left\{ x^* \in X^* : \liminf_{y \to x} \frac{f(y) - f(x) - \langle x^*, y-x \rangle}{\|y-x\|} \geq 0 \right\} .$$

- Gâteaux-subdifferential of f at x:

$$\partial_\beta f(x) = \partial_G f(x) = \left\{ x^* \in X^* : \liminf_{t \to 0^+} \frac{f(x+th) - f(x)}{t} \geq \langle x^*, h \rangle , \quad \forall h \in X \right\} .$$

- Hadamard-subdifferential (or Dini-Hadamard-subdifferential) of f at x:

$$\partial_\beta f(x) = \partial_H f(x) = \left\{ x^* \in X^* : \liminf_{\substack{t \to 0^+ \\ u \to h}} \frac{f(x+tu) - f(x)}{t} \geq \langle x^*, h \rangle , \quad \forall h \in X \right\} .$$

We denote by ∂ the Fenchel (or Moreau-Rockafeller) subdifferential that is

$$\partial f(x) = \{x^* \in X^* : f(x+h) - f(x) \geq \langle x, h \rangle , \forall h \in X\}.$$

It is important to note that in case of lower semicontinuous convex function f, we have

$$\partial_\beta f(x) = \partial f(x).$$

We will denote by $N_\beta(C; x)$ the β-normal cone of C at x which is defined by

$$N_\beta(C; x) = \partial_\beta \psi_C(x)$$

where ψ_C is the indicator fonction of C, that is,

$$\psi_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{if } x \notin C \end{cases}$$

and by $T_\beta(C; x)$ the β-tangent cone which is defined as the negative polar cone of the β-normal cone intersected with X, that is

$$T_\beta(C, x) = (N_\beta(C, x))^0 \cap X.$$

For any bornology β the following inclusions hold:

$$N_F(C; x) \subset N_\beta(C; x) \subset N_G(C; x), \quad T_G(C; x) \subset T_\beta(C; x) \subset T_F(C; x).$$

When β is the Hadamard (resp. Fréchet) bornology, then ([1],[17]) we obtain that

$$T_H(C; x) = \overline{\text{co}} K(C; x), \text{ (resp. } N_F(C; \bar{x}) = \left\{ x^* \in X^* : \limsup_{u \xrightarrow{C} x} \frac{\langle x^*, u-x \rangle}{\|u-x\|} \leq 0 \right\}).$$

Definition 2.2 *Let X be a Banach space and let β be a bornology on it, we say that X is ∂_β trusted, if the following fuzzy minimization rule holds: let f be a lower semicontinuous function on X finite at $\bar{x} \in X$, and let g be a Lipschitz continuous function on X. Assume that $f+g$ attains a local minimum at x. Then for any $\epsilon > 0$ there are $x, u \in X$ and $x^* \in \partial_\beta f(x)$, $u^* \in \partial_\beta g(u)$ such that*

$$\|x - \bar{x}\| < \epsilon, \quad \|u - \bar{x}\| < \epsilon, \quad |f(x) - f(\bar{x})| < \epsilon, \quad \textit{and} \quad \|x^* + y^*\| < \epsilon.$$

We recall that a bump function on X is a real-valued function ϕ which has bounded nonempty support $supp(\phi) = \{x \in X : \phi(x) \neq 0\}$.

Proposition 2.3 *[14] If there is on X a β-differentiable Lipschitz bump function, then X is $\partial_\beta-$trusted,*

Proposition 2.4 *[9] A Banach space is trusted for the Fréchet subdifferential if and only if it is Asplund.*

3 Characterizations of Gâteaux and Fréchet differentiability of the norm

In this section, we study the connection between differentiability of the norm $\|\cdot\|$ on X and some property of the subset $D := \overline{\mathbb{B}^c} = \{x \in X : \|x\| \geq 1\}$. In [5] Borwein and Srojwas showed several properties of D in various Banach spaces. In particular they showed that if $\|\bar{x}\| = 1$ then Gâteaux differentiability of the norm at $\bar{x}$ is equivalent to the P-properness of D et $\bar{x}$, i.e., $\bar{co}K(D; x) \neq X$. In this section we will show furthur properties for various norms. We denote by $P_C(x)$ the set of projections of x on the subset C of X, i.e.,

$$P_C(x) = \{y \in C : \|x - y\| = d_C(x)\}.$$

Proposition 3.1 *Assume that X is a Banach space with a given norm $\|\cdot\|$. Let $\bar{x} \in D$ with $\|\bar{x}\| = 1$. Then*

(a) $K(D; \bar{x})$ *contains at least a one half space,*

(b) $\bar{x} + K(D; \bar{x}) \subset D$,

(c) $K(D; \bar{x}) \neq X$,

(d) $\forall \lambda \in]0, 1[$, $D \cap B(\bar{x}, 1 - \lambda) + tB(\bar{x}, \lambda) \subset D$, *for all* $t > 0$,

(e) $B(\bar{x}, 1) \subset T_c(D; \bar{x})$,

(f) $\dfrac{x}{\|x\|} \in P_D(x)$ *and* $d_D(x) = 1 - \|x\|$ *for all* $x \in \mathbb{B} \backslash \{0\}$.

Proof. (a) Since $\mathbb{B}$ is convex and $\|\bar{x}\| = 1$, Hahn-Banach Theorem ensures the existence of $x^* \in X^*$, with $\|x^*\| = 1$, such that

$$\mathbb{B} \subset \{h \in X : \langle x^*, h - \bar{x}\rangle \leq 0\}.$$

Thus

$$\{h \in X : \langle x^*, h - \bar{x}\rangle \geq 0\} \subset D,$$
$$\bar{x} + \{h \in X : \langle x^*, h\rangle \geq 0\} \subset D,$$
$$\{h \in X : \langle x^*, h\rangle \geq 0\} \subset D - \bar{x}.$$

Therefore we receive that

$$\{h \in X : \langle x^*, h\rangle \geq 0\} \subset K(D; \bar{x}).$$

(b) Suppose that there is $h \in K(D; \bar{x})$ such that $\bar{x} + h \in int\mathbb{B}$. Then there is $\delta > 0$ such that $\bar{x} + B(h, \delta) \subset \mathbb{B}$. Since $\mathbb{B}$ is convex for any $t \in]0, 1]$

$$\bar{x} + tB(h, \delta) \subset \mathbb{B}.$$

Therefore for any sequences (h_n) converging to h and any $t_n \to 0$ there is $n_0 \in \mathbb{N}$ such that

$$\bar{x} + t_n h_n \in int\mathbb{B}, \quad \forall n \geq n_0.$$

This is in contradiction with $h \in K(D, \bar{x})$, therefore $\bar{x} + h \in D$.

(c) It is a direct consequence of (b).

(d) For any $x \in D$, $z \in X$ and $t > 0$

$$\|z - (1+t)x\| \leq t \implies (1+t)\|x\| - \|z\| \leq t \implies 1 \leq \|z\|.$$

Therefore $B\big((1+t)x, t\big) \subset D$ or equivalently $x + tB(x, 1) \subset D$. Let $\lambda \in]0, 1[$ and pick $x \in B(\bar{x}, 1-\lambda) \cap D$, then $B(\bar{x}, \lambda) \subset B(x, 1)$ and hence $x + tB(\bar{x}, \lambda) \subset D$. Finally we receive that

$$D \cap B(\bar{x}, 1-\lambda) + tB(\bar{x}, \lambda) \subset D.$$

(e) Let $(x_n)_n$ be a sequence in D converging to $\bar{x}$, $(t_n)_n$ be a sequence of positive numbers converging to 0 and $h \in B(\bar{x}, 1)$. Then there exist $\lambda > 0$ and $n_0 > 0$ such that for all $n \geq n_0$, $x_n \in D \cap \mathbb{B}(\bar{x}, 1-\lambda)$. The property (d) ensures that $x_n + t_n h \in D$ for all $n > n_0$. By the definition of the Clarke tangent cone, we get $h \in T_c(D; \bar{x})$.

(f) Suppose that $x \in \mathbb{B}_X$ and $z \in D$, then

$$\|x - z\| \geq \|z\| - \|x\| \geq 1 - \|x\| = \left\| x - \frac{x}{\|x\|} \right\|$$

therefore $\dfrac{x}{\|x\|} \in P_D(x)$.□

The following proposition contains several characterizations of the Gâteaux differentiability of the norm.

Proposition 3.2 *Let X be a Banach space with a given norm $\| \cdot \|$. Assume that $\|\bar{x}\| = 1$. Then the following assertions are equivalent:*

(a) $\| \ \|$ is Gâteaux differentiable at $\bar{x}$,

(b) there is $x^* \in X^*$, $\|x^*\| = 1$ such that $K(D;\bar{x}) = \{h \in X : \langle x^*, h\rangle \geq 0\}$,

(c) $T_c(D;\bar{x}) = K(D;\bar{x})$.

Proof. (a) $\Rightarrow$ (b). Suppose that $\| \cdot \|$ is Gâteaux differentiable at $\bar{x}$ with derivative x^*. Then for any $h \in X$

$$\lim_{t\to 0} \frac{\|\bar{x}+th\| - \|\bar{x}\|}{t} - \langle x^*, h\rangle = 0$$

By (a) of Proposition 3.1 the cone $K(D;\bar{x})$ contains at least one half space. If we show that $K(D;\bar{x}) \subset \{h \in X : \langle x^*, h\rangle \geq 0\}$ then this inclusion will become equality. Take $h \in K(D;\bar{x})$ and find $(h_n)_n \subset X$ converging strongly to h and a sequence $(t_n)_n$ of positive numbers converging to zero such that for all $n \in \mathbb{N}$ large enough

$$\bar{x} + t_n h_n \in D.$$

Thus, as $\|\bar{x} + t_n h_n\| \geq 1$,

$$\begin{aligned} \frac{\|\bar{x}+t_n h\| - \|\bar{x}\|}{t_n} - \langle x^*, h\rangle &\geq \frac{\|\bar{x}+t_n h_n\| - \|\bar{x}\|}{t_n} - \langle x^*, h\rangle - \|h - h_n\| \\ &\geq -\langle x^*, h\rangle - \|h - h_n\|. \end{aligned}$$

Therefore

$$\lim_{n\to\infty} \frac{\|\bar{x}+t_n h\| - \|\bar{x}\|}{t_n} - \langle x^*, h\rangle \geq -\langle x^*, h\rangle,$$

$$0 \geq -\langle x^*, h\rangle,$$

$$\langle x^*, h\rangle \geq 0.$$

(b)$\Rightarrow$(a) Assume that $K(D;\bar{x}) = \{h : \langle x^*, h\rangle \geq 0\}$ for some $x^* \in X^*$. Let $z^* \in \partial\| \ \|(\bar{x})$. Then $\|z^*\| = 1$ and

$$\lim_{t\to 0} \frac{\|\bar{x}+th\| - \|\bar{x}\|}{t} - \langle z^*, h\rangle \geq 0.$$

Suppose that $\langle z^*, h\rangle > 0$. Then $\lim_{t\to 0} \frac{\|\bar{x}+th\| - \|\bar{x}\|}{t} > 0$ and thus there is $t_0 > 0$ such that

$$\frac{\|\bar{x}+th\| - \|\bar{x}\|}{t} > 0, \quad \forall \ 0 < t < t_0,$$

and hence for all $t \in]0, t_0[$, $\bar{x} + th \in D$, which asserts that $h \in K(D;\bar{x})$. We receive, because of the closedness of $K(D;x)$, that

$$\{h : \langle z^*, h\rangle > 0\} \subset K(D;\bar{x}) \subset \{h : \langle x^*, h\rangle \geq 0\}.$$

By Farkas Lemma ([11]), we conclude that $z^* = \lambda x^*$ with $\lambda > 0$. Thus

$$\lambda = \frac{\|z^*\|}{\|x^*\|} = 1 \quad \text{and} \quad z^* = x^*.$$

(a) $\Rightarrow$ (c) Suppose that the norm $\|\cdot\|$ is Gâteaux differentiable at $\bar{x}$. It suffices to show that there exists a unique $x^* \in X^*$, with $\|x^*\| = 1$ such that

$$T_c(D;\bar{x}) = \{h \in X : \langle x^*, h\rangle \geq 0\}.$$

Assertions (c) and (d) of Proposition 3.1 ensure that 0 is a boundary point of $T_c(D;\bar{x})$ and $\mathrm{int}T_c(D,\bar{x}) \neq \emptyset$. So the separation theorem produces $x^* \in X^*$, with $\|x^*\| = 1$ such that

$$T_c(D;\bar{x}) \subset \{h \in X : \langle x^*, h\rangle \geq 0\}$$

and as $B(\bar{x}, 1) \subset T_c(D;\bar{x})$ (by (d) of Proposition 3.1), the assumption (a) implies that x^* is exactly the Gâteaux derivative of the norm $\|\cdot\|$ at $\bar{x}$. It remains to establish the reverse inclusion

$$T_c(D;\bar{x}) \supset \{h \in X : \langle x^*, h\rangle \geq 0\}.$$

Suppose that there exists $v \in X$ satisfying $\langle x^*, v\rangle \geq 0$ and $v \notin T_c(D;\bar{x})$. Once again, the separation theorem yields $u^* \in X^*$, with $\|u^*\| = 1$, such that

$$T_c(D;\bar{x}) \subset \{h \in X : \langle u^*, h\rangle \geq 0\} \quad \text{and} \quad \langle u^*, v\rangle < 0.$$

As before we show that u^* is also a Gâteaux derivative of the norm $\|\cdot\|$ at $\bar{x}$, and by (a), $x^* = u^*$ and this contradicts the relations

$$\langle x^*, v\rangle \geq 0 \quad \text{and} \quad \langle u^*, v\rangle < 0.$$

(c) $\Rightarrow$ (b) Suppose that $T_c(D;\bar{x}) = K(D;\bar{x})$. Then $T_c(D;\bar{x})$ contains at least one half space. By Proposition 3.1, $T_c(D,\bar{x}) \neq X$ and by the separation Theorem (recall that the Clarke cone is convex and closed) there is $x^* \in X^*$, $\|x^*\| = 1$ such that

$$T_c(D;x) \subset \{h \in X : \langle x^*, h\rangle \geq 0\}.$$

By the Farkas lemma we have

$$T_c(D;x) = \{h \in X : \langle x^*, h\rangle \geq 0\}.$$

$\square$

The following corollary on the density of points of Gâteaux differentiability of the norm is a consequence of Propositions 3.1 and 3.2.

Corollary 3.3 *Let $(X, \|\cdot\|)$ be a Banach space and put $D = \{u \in X : \|u\| \geq 1\}$. The following assertions are equivalent:*

(1) For each $x \in S_X$, $\liminf_{x' \xrightarrow{D} x} \bar{co}K(D;x') \neq X$.

(2) The norm $\|\cdot\|$ is Gâteaux differentiable on a dense subset of X.

Proof. First, we remark that

$$\liminf_{x' \xrightarrow{D} x} \bar{co}K(D;x') \neq X \iff \liminf_{x' \xrightarrow{S_X} x} \bar{co}K(D;x') \neq X$$

(1) $\Rightarrow$ (2): It suffices to show that $\|\cdot\|$ is Gâteaux differentiable on dense subset of S_X. Let $x \in S_X$. Then

$$\liminf_{x' \xrightarrow{D} x} \bar{co}K(D;x') \neq X.$$

Therefore for any $\epsilon > 0$ there is $z \in B(x,\epsilon) \cap D$ such that

$$\bar{co}K(D;z) \neq X.$$

That is the convex cone $\bar{co}K(D;z)$ belongs to a half space, thus $K(D;z)$ also belongs to a half space. Since by (a) of Proposition 3.1 we know that $K(D;z)$ contains at least one half space, then by Farkas Lemma we deduce that $K(D;z)$ is equal to the half space and $\|\cdot\|$ is Gâteaux differentiable at z according to the Proposition 3.2.

(2) $\Rightarrow$ (1): Let $x \in S_X$ and $x_n \xrightarrow{S_X} x$ such that the norm $\|\cdot\|$ is Gâteaux differentiable. Proposition 3.2 asserts that there exists $x_n^* \in X^*$, $\|x_n^*\| = 1$, such that $K(D,x_n) = \{h \in X : \langle x_n^*, h \leq 0\}$, and hence $\bar{co}K(D;x_n) = K(D;x_n)$. Applying Proposition 3.1 (b), we get $\bar{co}K(D;x_n) \subset D - x_n$. Thus $\liminf_{x' \xrightarrow{D} x} \bar{co}K(D;x') \subset D - x$, and the proof is completed.$\Box$

Proposition 3.4 *Let X be a Banach space with a given norm $\|\cdot\|$. Assume that $\|\bar{x}\| = 1$. Then the following assertions are equivalent:*

(a) $\|\ \|$ is Fréchet differentiable at $\bar{x}$,

(b) $N_F(D;\bar{x}) \neq \{0\}$,

Proof. (a) $\Rightarrow$ (b) If (a) holds then there is some $x^* \in X^*$, $\|x^*\| = 1$ which is the Fréchet derivative of $\|\cdot\|$ at $\bar{x}$, that is, for any $\epsilon > 0$ there is $\delta > 0$ such that

$$-\epsilon \leq \frac{\|y\| - \|\bar{x}\| - \langle x^*, y - \bar{x}\rangle}{\|y - \bar{x}\|} \leq \epsilon,$$

for all $y \in B(\bar{x},\delta)$. If $y \in D \cap B(\bar{x},\delta)$ then $\|y\| \geq 1 = \|\bar{x}\|$ and so

$$\frac{\langle -x^*, y - \bar{x}\rangle}{\|y - \bar{x}\|} \leq \epsilon.$$

This implies that $-x^* \in N_F(D;\bar{x})$.
(b) $\Rightarrow$ (a) Suppose that $x^* \in N_F(D;\bar{x})$ with $\|x^*\| = 1$. Since $N_F(D;\bar{x}) \subset \big(K(D;x)\big)^\circ$ then, by polarity, we get

$$K(D;\bar{x}) \subset \{h \in X : \langle x^*, h\rangle \leq 0\}.$$

As $K(D;\bar{x})^0$ contains at least one half space, we deduce by Farkas Lemma that $K(D;\bar{x})$ is a half space and therefore Proposition 3.2 asserts that $-x^*$ is a Gâteaux derivative of $\|\cdot\|$ at $\bar{x}$ and $\langle -x^*, \bar{x}\rangle = 1$. By the definition of $N_F(D;\bar{x})$, for any $\epsilon > 0$ there is $\delta > 0$ (with $\delta \leq 1$) such that

$$\langle x^*, x - \bar{x}\rangle \leq \epsilon\|x - \bar{x}\| \tag{3.1}$$

for all $x \in D \cap B(\bar{x}, \delta)$. We note that

$$\begin{aligned} \left\|\frac{x}{\|x\|} - \bar{x}\right\| = \frac{1}{\|x\|}\Big\|x - \|x\|\bar{x}\Big\| &\leq \frac{1}{\|x\|}\left[\Big\|x - \|x\|x\Big\| + \Big\|\|x\|x - \|x\|\bar{x}\Big\|\right] \\ &= \Big|\|x\| - 1\Big| + \|x - \bar{x}\| \\ &\leq 2\|x - \bar{x}\|. \end{aligned}$$

Thus if $x \in B(\bar{x}, \frac{\delta}{2})$, then $\frac{x}{\|x\|} \in B(\bar{x}, \delta) \cap D$ and therefore by inequality (3.1)

$$\left\langle x^*, \frac{x}{\|x\|} - \bar{x}\right\rangle \leq \epsilon\left\|\frac{x}{\|x\|} - \bar{x}\right\|,$$

$$1 + \left\langle x^*, \frac{x}{\|x\|}\right\rangle \leq 2\epsilon\big\|x - \bar{x}\big\|,$$

$$\|x\| + \langle x^*, x\rangle \leq 2\epsilon\|x\|\|x - \bar{x}\|,$$

$$\|x\| - 1 + \langle x^*, x - \bar{x}\rangle \leq 4\epsilon\|x - \bar{x}\|,$$

$$\|x\| - \|\bar{x}\| + \langle x^*, x - \bar{x}\rangle \leq 4\epsilon\|x - \bar{x}\|.$$

As $-x^*$ is the Gâteaux derivative of $\|\cdot\|$ at $\bar{x}$ we receive finally that

$$0 \leq \|x\| - \|\bar{x}\| + \langle x^*, x - \bar{x}\rangle \leq 4\epsilon\|x - \bar{x}\|,$$

for all $x \in B(\bar{x}, \frac{\delta}{2})$. Therefore $\|\cdot\|$ is Fréchet differentiable at $\bar{x}$.$\Box$

The following corollary on the density of points of Fréchet differentiability of the norm is a consequence of Propositions 3.1 and 3.4. Its proof is similar to that of Corollary 3.3.

Corollary 3.5 *Let $(X, \|\cdot\|)$ be a Banach space and put $D = \{u \in X : \|u\| \geq 1\}$. The following assertions are equivalent:*

(1) For each $x \in S_X$, $\liminf_{x' \xrightarrow{D} x} T_F(D; x') \neq X$.

(2) The norm $\|\cdot\|$ is Fréchet differentiable on a dense subset of X.

4 The validity of the "lim inf" formula

Theorem 4.1 *Let $(X, \|\cdot\|)$ be a Banach space and β a bornology on X such that $X \times X$ is ∂_β-trusted. Then for any closed subset C of X and $\bar{x} \in C$*

$$\liminf_{x \xrightarrow{C} \bar{x}} T_\beta(C; x) \subset T_c(C; \bar{x}).$$

Proof. Pick $w \in \liminf_{x \xrightarrow{C} \bar{x}} T_\beta(C, x)$. We want to show that $w \in T_c(C; x)$. Suppose that $w \notin T_c(C, x)$. Then by Lemma 1.2.1 in [20] there are a sequence x_n in C converging to x, a sequence (λ_n) in $(0, \frac{1}{2})$ of real positive numbers converging to zero, $\epsilon > 0$ and $n_0 \in \mathbb{N}$ such that

$$\big(x_n +]0, \lambda_n] B(w, \epsilon)\big) \cap C = \emptyset, \quad \forall n \geq n_0.$$

Let us fix an integer $n \geq n_0$ and put $D := x_n + \left[0, \frac{\lambda_n}{2}\right] B(w, \epsilon)$. Then $\big(D + \lambda_n^4 w\big) \cap C = \emptyset$. Define the function f by

$$f(x, y) = \|x - y - \lambda_n^4 w\|, \quad \forall (x, y) \in X \times X.$$

Thus $f(x_n, x_n) = \lambda_n^4$ and

$$\lambda_n^4 + \inf_{(x,y) \in C \times D} f(x, y) \geq f(x_n, x_n).$$

The well-known Ekeland's variational principle assures the existence of $(u_n, v_n) \in C \times D$ satisfying

$$\|u_n - x_n\| + \|v_n - x_n\| < \lambda_n^2,$$

and

$$\forall u \in C, \forall v \in D, \quad f(u_n, v_n) \leq f(u, v) + \lambda_n^2(\|u - u_n\| + \|v - v_n\|).$$

Thus

$$f(u_n, v_n) \leq f(u, v) + \lambda_n^2(\|u - u_n\| + \|v - v_n\|) + \psi_C(u) + \psi_D(v), \tag{4.1}$$

for all $u, v \in X$. Since $\big(D + \lambda_n^4 w\big) \cap C = \emptyset$, we get

$$\|u_n - v_n - \lambda_n^4 w\| > 0$$

and so there is $\delta_n > 0$ such that

$$\|t - \tau - \lambda_n^4 w\| > 0,$$

for all $t \in B(u_n, \delta_n)$ and $\tau \in B(v_n, \delta_n)$.

Since $X \times X$ is ∂_β-trusted, 4.1 provides there are $u_n^1, u_n^2, v_n^1, v_n^2 \in X$ and $u_n^{*1}, u_n^{*2}, v_n^{*1}, v_n^{*2} \in X^*$ such that

$$\|u_n^1 - u_n\| + \|u_n^2 - u_n\| + \|v_n^1 - v_n\| + \|v_n^2 - v_n\| < \alpha_n = \min\{\delta_n, \lambda_n^4\},$$

$$\|u_n^{*1}+u_n^{*2}\|+\|v_n^{*1}+v_n^{*2}\| \le \alpha_n = \min\{\delta_n, \lambda_n^4\} \tag{4.2}$$

and

$$(u_n^{*1}, v_n^{*1}) \in \partial_\beta\Big(f+\lambda_n^2\big(\|\cdot-u_n\|+\|\cdot-v_n\|\big)\Big)(u_n^1, v_n^1),$$
$$(u_n^{*2}, v_n^{*2}) \in \partial_\beta\big(\psi_C+\psi_D\big)(u_n^2, v_n^2).$$

By the convexity of separate summands

$$\begin{aligned} & \partial_\beta\Big(f+\lambda_n^2\big(\|\cdot-u_n\|+\|\cdot-v_n\|\big)\Big)(u_n^1, v_n^1) \\ = \quad & \partial\Big(f+\lambda_n^2\big(\|\cdot-u_n\|+\|\cdot-v_n\|\big)\Big)(u_n^1, v_n^1) \\ \subset \quad & \partial f(u_n^1, v_n^1)+\lambda_n^2(B_{X^*}\times B_{X^*}). \end{aligned}$$

Since $\|u_n^1-v_n^1-\lambda_n^4 w\| \ne 0$ we receive that $\partial f(u_n^1, v_n^1)$ is included in $\{(x^*, -x^*) : \|x^*\|=1\}$. That is there is $x_n^* \in X^*$ with $\|x_n^*\|=1$ such that

$$\|u_n^{*1}-x_n^*\| \le \lambda_n^2 \quad \text{and} \quad \|v_n^{*1}+x_n^*\| \le \lambda_n^2.$$

By the inequality (4.2) we receive that

$$\|x_n^*+u_n^{*2}\| \le \lambda_n^2+\lambda_n^4 \quad \text{and} \quad \|v_n^{*2}-x_n^*\| \le \lambda_n^2+\lambda_n^4 \tag{4.3}$$

and thus

$$\|u_n^{*2}+v_n^{*2}\| \le 2(\lambda_n^2+\lambda_n^4) \tag{4.4}$$

It is evident that

$$\partial_\beta(\psi_C(\cdot)+\psi_D(\cdot))(u_n^2, v_n^2) = \partial_\beta\psi_C(u_n^2)\times\partial_\beta\psi_D(v_n^2).$$

Thus

$$\left\langle v_n^{*2}, u-v_n^2\right\rangle \le 0 \qquad \forall u \in D,$$
$$\left\langle v_n^{*2}, x_n+\frac{\lambda_n}{2}(w+b)-v_n^2\right\rangle \le 0 \qquad \forall b \in B(0,\epsilon),$$
$$\epsilon\|v_n^{*2}\|\frac{\lambda_n}{2}+\left\langle v_n^{*2}, x_n-v_n^2+\frac{\lambda_n}{2}w\right\rangle \le 0,$$
$$\frac{\epsilon\lambda_n}{2}(1-\lambda_n^2-\lambda_n^4) \le \left\langle v_n^{*2}, v_n^2-x_n-\frac{\lambda_n}{2}w\right\rangle.$$

Using (4.3) and (4.4), we get

$$\begin{aligned} \frac{\epsilon\lambda_n}{2}(1-\lambda_n^2-\lambda_n^4) \quad & \le \quad \left\langle v_n^{*2}+u_n^{*2}, v_n^2-x_n-\frac{\lambda_n}{2}w\right\rangle+\left\langle -u_n^{*2}, v_n^2-x_n-\frac{\lambda_n}{2}w\right\rangle \\ & \le \quad 2(\lambda_n^2+\lambda_n^4)\big(\|v_n^2-x_n-\frac{\lambda_n}{2}w\|\big)+\left\langle -u_n^{*2}, v_n^2-x_n\right\rangle+\frac{\lambda_n}{2}\left\langle u_n^{*2}, w\right\rangle, \end{aligned}$$

$$\frac{\epsilon\lambda_n}{2}(1-\lambda_n^2-\lambda_n^4)+\left\langle u_n^{*2}, v_n^2-x_n\right\rangle \le 2(\lambda_n^2+\lambda_n^4)\big(\|v_n^2-x_n-\frac{\lambda}{2}w\|\big)+\frac{\lambda_n}{2}\left\langle u_n^{*2}, w\right\rangle,$$

$$\frac{\epsilon\lambda_n}{2}(1-\lambda_n^2-\lambda_n^4)-\|u_n^{*2}\|\|v_n^2-x_n\| \leq 2(\lambda_n^2+\lambda_n^4)\big(\|v_n^2-x_n-\frac{\lambda}{2}w\|\big)+\frac{\lambda_n}{2}\left\langle u_n^{*2},w\right\rangle,$$

$$\frac{\epsilon\lambda_n}{2}(1-\lambda_n^2-\lambda_n^4)-(1+\lambda_n^2+\lambda_n^4)(\lambda_n^2+\lambda_n^4) \leq 2(\lambda_n^2+\lambda_n^4)\big(\|v_n^2-x_n-\frac{\lambda_n}{2}w\|\big)+\frac{\lambda_n}{2}\left\langle u_n^{*2},w\right\rangle,$$

$$\epsilon(1-\lambda_n^2-\lambda_n^4)-2(1+\lambda_n^2+\lambda_n^4)(\lambda_n+\lambda_n^3) \leq 4(\lambda_n+\lambda_n^3)\big(\|v_n^2-x_n-\frac{\lambda_n}{2}w\|\big)+\left\langle u_n^{*2},w\right\rangle.$$

Now remember that $u_n^{*2} \in \partial_\beta\psi_C(u_n^2) = N^\beta(C,u_n^2)$, $\{u_n^2\}_n$ converges strongly to $\bar{x}$, $(\lambda_n)_n$ converges to zero and $w \in \liminf\limits_{x \xrightarrow{C} \bar{x}} T_\beta(C;x)$. Therefore there is $w_n \in T_\beta(C,u_n^2)$ converging strongly to w. Thus we receive that

$$\begin{aligned}\epsilon-(2\epsilon+8)\lambda_n &\leq 4(\lambda_n+\lambda_n^3)\big(\|v_n^2-x_n-\frac{\lambda_n}{2}w\|\big)+\left\langle u_n^{*2},w-w_n\right\rangle+\left\langle u_n^{*2},w_n\right\rangle\\ &\leq 4(\lambda_n+\lambda_n^3)\big(\|v_n^2-x_n-\frac{\lambda_n}{2}w\|\big)+\|u_n^{*2}\|\|w-w_n\|.\end{aligned}$$

as $u^{*2} \in N_\beta(C;u_n^2)$. Passing to the limit on n and taking into account that $\|u_n^{*2}\| \leq 1+\lambda_n^2+\lambda_n^4$ and $v_n^2-x_n-\frac{\lambda_n}{2}w$ converges to 0 we receive $\epsilon \leq 0$ which is contradiction.□

We know that if there is on X a β-differentiable Lipschitz bump function then there is also on $X \times X$ a β-differentiable Lipschitz bump function, therefore according to Proposition 2.3, $X \times X$ is ∂_β-trusted. So the following corollary is a direct consequence of Theorem 4.1.

Corollary 4.2 *Assume that there is on X a β-differentiable Lipschitz bump function. Then for any closed subset C of X containing x*

$$\liminf_{x' \xrightarrow{C} x} T_\beta(C;x') \subset T_c(C;x).$$

We recall that if X admits an equivalent β-differentiable norm (at all nonzero points), then there is on X a β-differentiable Lipschitz bump function [18]. Note that the reverse is not true. Haydon [12] constructed a nonseparable Banach space that has Fréchet differentiable Lipschitz bump function but doesn't admit an equivalent Gâteaux differentiable norm.

Corollary 4.3 *([3]) Let X be a Banach space with a norm which is β-differentiable away from the origin. Let C be a closed subset of X. Then for any $x \in C$ we have*

$$\liminf_{x' \xrightarrow{C} x} T_\beta(C;x') \subset T_c(C;x).$$

The following corollary is an extention of Theorem 3.4 in [4] from spaces with equivalent Fréchet differentiable norm away from the origin to Asplund spaces and without the weak compcatness assumption on the set C. We recall that $WK(C;x)$ denotes the weak-contingent cone to C at x.

Corollary 4.4 *([15]) Let X be Asplund space and C be a closed subset of X. Then for any $x \in C$ we have*

$$\liminf_{x' \xrightarrow{C} x} \bar{co}(WK(C;x')) \subset T_c(C;x).$$

Proof. Borwein and Strojwas [4] proved that for any closed subset C of X and $x \in C$

$$N_F(C;x) \subset (WK(C;x))^{\circ}.$$

Therefore

$$\bar{co}(WK(C;x)) \subset T_F(C;x).$$

On the other hand since X is Asplund thus $X \times X$ is also Asplund and therefore according to the Proposition 2.3 trusted for the Fréchet subdifferetnial. By Theorem 4.1 we receive that

$$\liminf_{x' \xrightarrow{C} x} T_F(C;x) \subset T_c(C;x),$$

and therefore

$$\liminf_{x' \xrightarrow{C} x} \bar{co}(WK(C;x)) \subset T_c(C;x).$$

The proof is completed.□

To end up this section, we give an extention of Theorem 5.4 in [5] where lower semicontinuity (LSC) of a multivalued mapping is involved. A multivalued mapping $F : C \rightrightarrows X$ is said to be lower semicontinuous at $x \in C$ if

$$F(x) \subset \liminf_{x' \xrightarrow{C} x} F(x')$$

and is LSC on C if it is LSC at each point x in C.

Theorem 4.5 *Let $(X, \|\cdot\|)$ be a Banach space, β be a bornology on X containing the Hadamard bornology such that $X \times X$ is ∂_β-trusted and C be a closed subset of X. Suppose that $F : C \rightrightarrows X$ is LSC on C. Then the following statements are equivalent:*

(*i*) *$F(x) \subset T_c(C;x)$, for all $x \in C$,*

(*ii*) *$F(x) \subset T_\beta(C;x)$, for all $x \in C$.*

Proof $(ii) \Rightarrow (i)$ Follows from the lower semicontinuity of F and Theorem 4.1. $(i) \Rightarrow (ii)$: Since $T_c(C;x) \subset T_H(C;x)$, our hypothesis on the bornology β ensures that $T_c(C;x) \subset \overline{\text{co}}K(C;x) = T_H(C;x) \subset T_\beta(C;x)$ and so (i) implies (ii). □

Remark 4.6

- *Statement (2) in Theorem 5.4 in [5] is extended from refelexive Banach spaces to Asplund spaces.*
- *The weak compactness assumptions and the smoothness of an equivalent norm (resp. the Fréchet differentiability of an equivalent norm) off zero assumed in the statement (4) (resp. (5)) of Theorem 5.4 in [5] are weakened by assuming that the space admits a Gâteaux differentiable lipschitz bump function (resp. the space is Asplund) and the set is closed.*

5 The "$\liminf$" formula as a characterization of Asplund spaces

We begin by recalling that X is an *Asplund* space if every continuous convex function on any open convex subset U of X is Fréchet differentiable at the points of a dense G_δ subset of U.
A well known theorem of Fabian and Mordukhovich [10] affirms that the space X is Asplund if and only if for every closed set $C \subset X$ and every $\bar{x} \in C$ one has the limiting representation

$$N(C;\bar{x}) = \limsup_{x \to \bar{x}} N^F(C;x)$$

where $N(\bar{x};C)$ denotes the limiting normal cone of C at $\bar{x}$. Here, we give a characterization of Asplund spaces by mean of the "liminf" formula.

Theorem 5.1 *A Banach space X is Asplund if and only if for every closed set C in it and every $x \in C$, the following inclusion holds*

$$\liminf_{x' \xrightarrow{C} x} T_F(C;x') \subset T_c(C;x).$$

Proof. (a) $\Rightarrow$ (b): We know that if X is Asplund space then $X \times X$ is also Asplund space. According to (c) of Proposition 2.4 $X \times X$ is trusted for Fréchet subdifferential. Theorem 4.1 asserts that

$$\liminf_{x' \xrightarrow{C} x} T_F(C;x') \subset T_c(C;x),$$

for any set $C \subset X$ and $x \in C$.

(b) $\Rightarrow$ (a): Suppose that X is not Asplund space then it is known [7] that there is an equivalent norm on X which is nowhere Fréchet differentiable. Therefore by Proposition 3.4 $N_F(C_1;x) = \{0\}$ for all $x \in C_1$, where $C_1 = \{z \in X : \|z\| \geq 1\}$. Thus $T_F(C_1;x) = X$ for all $x \in C_1$ and

$$X = \liminf_{x' \xrightarrow{C_1} x} T_F(C_1;x') \subset T_c(C_1;x).$$

This is in contradiction with $T_c(C_1;x) \subset K(C_1,x) \neq X$ (see Proposition 3.1 (c)).$\square$

6 Convexity of Pseudoconvex sets

Let C be a set in a Banach space X and let $x \in C$. Let $R(C;x)$ denotes one of the cones $T_c(C;x),\ T_\beta(C;x),\ K(C;x),\ \ldots$. We say that C is *R-pseudoconvex at x* if

$$C - x \subset R(C;x).$$

We say that C is *R-pseudoconvex* if the last inclusion holds for every $x \in C$.
Concerning this notion, Borwein and Strojwas [5] established the following result concerning the equivalence between convexity and R-pseudoconvexity. For the sake of completeness, we include its proof which is slightly different than that given in [5].

Theorem 6.1 *[5] For a closed set C in a Banach space X TFAE : (i) C is convex; (ii) C is T_c-pseudoconvex; (iii) C is K-pseudoconvex.*

Proof. $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are obvious.
$(iii) \Rightarrow (i)$: By Treiman Theorem ([20]-[21]) we receive that T_c-pseudoconvexity coincide with K-pseudoconvexity. Suppose that C is T_c-pseudoconvex, that is $C - x \subset T_c(C;x)$ for all $x \in C$. If C is not convex, then there exist distinct $u, v \in C$ such that $]u,v[\cap C = \emptyset$. Let $w \in]u,v[$ and consider the function $f(x) = \|x - w\|$. For every $n \in \mathbb{N}$ find $u_n \in C$ such that

$$\|u_n - w\| \le \inf_{x \in C} \|x - w\| + \frac{1}{n^2}. \tag{6.1}$$

By Ekeland's variational principle, there exists $x_n \in C$ such that

$$\|x_n - u_n\| \le \frac{1}{n} \tag{6.2}$$

and

$$f(x_n) \le f(x) + \frac{1}{n}\|x - x_n\| \quad \forall x \in C.$$

This later one ensures that x_n is a local minimum of the function

$$x \mapsto (1 + \frac{1}{n}) d_C(x) + \|x - w\| + \frac{1}{n}\|x - x_n\|$$

and hence

$$0 \in (1 + \frac{1}{n})\partial d_C(x_n) + \partial\| \cdot - w\|(x_n) + \frac{1}{n}\partial\| \cdot - x_n\|(x_n).$$

Since $x_n \neq w$, there exists $x_n^* \in \partial\| \cdot - w\|(x_n)$ and $b_n^* \in \frac{1}{n}\partial\| \cdot - x_n\|(x_n)$ such that

$$\|x_n^*\| = 1, \quad \langle x_n^*, x_n - w\rangle = \|x_n - w\|, \quad -\frac{x_n^* + b_n^*}{1 + \frac{1}{n}} \in \partial d_C(x_n) = N_c(C;x_n).$$

By T_c-pseudoconvexity, we get

$$\langle -\frac{x_n^* + b_n^*}{1 + \frac{1}{n}}, x - x_n\rangle \le 0 \quad \forall x \in C$$

or equivalently

$$\langle -\frac{x_n^* + b_n^*}{1+\frac{1}{n}}, w - x_n\rangle \leq \langle \frac{x_n^* + b_n^*}{1+\frac{1}{n}}, x - w\rangle \quad \forall x \in C. \tag{6.3}$$

Remark that

$$\begin{aligned}\langle -\frac{x_n^* + b_n^*}{1+\frac{1}{n}}, w - x_n\rangle &= \frac{1}{1+\frac{1}{n}}[\langle -x_n^*, w - x_n\rangle + \langle -b_n^*, w - x_n\rangle] \\ &= \frac{1}{1+\frac{1}{n}}\|x_n - w\| + \frac{1}{1+\frac{1}{n}}\langle -b_n^*, w - x_n\rangle \\ &\geq \frac{1}{1+\frac{1}{n}}d_C(w) + \frac{1}{1+\frac{1}{n}}\langle -b_n^*, w - x_n\rangle\end{aligned}$$

and, by (6.1) and (6.2), $\langle -b_n^*, w - x_n\rangle \to 0$. Thus extracting subnet, we may assume that $x_n^* \stackrel{w^*}{\to} x^*$, with $\|x^*\| \leq 1$, and, by relation (6.3), we obtain

$$d_C(w) \leq \langle x^*, x - w\rangle \quad \forall x \in C.$$

In particular this later one holds for $x = u$ and $x = v$, and hence on all the segment $[u, v]$ and particularly for $x = w$. Thus $d_C(w) \leq 0$ and the closeness of C ensures that $w \in C$ and this is in contradiction with $]u, w[\cap C = \emptyset$.□

Here we give another result in terms of the T_β-pseudoconvexity.

Theorem 6.2 *Let $(X, \| \|)$ be a Banach space and β be a bornology on X. If $X \times X$ is ∂_β-trusted then*

a closed set $C \subset X$ is T_β-pseudoconvex if and only if it is convex.

Proof. If C is T_β-pseudoconvex then

$$C - x \subset T_\beta(C; x), \quad \forall x \in C,$$

and hence

$$\liminf_{x' \stackrel{C}{\to} x} T_\beta(C; x') \subset T_c(C; x).$$

By Theorem 4.1 we obtain that

$$C - x = \lim_{x' \to x}(C - x') \subset \liminf_{x' \stackrel{C}{\to} x} T_\beta(C; x') \subset T_c(C; x),$$

and therefore by Theorem 6.1 C is convex.□

Using Proposition 2.3, we obtain the following corollaries.

Corollary 6.3 *Let $(X, \| \|)$ be a Banach space and β be a bornology on X. If there is on X a β-differentiable Lipschitz bump function then*

C is T_β-pseudoconvex if and only if C is convex.

Corollary 6.4 *Assume that X is an Asplund space and C is a closed subset of X. Then*

$$C \text{ is } \overline{co}WK\text{-pseudoconvex if and only if } C \text{ is convex.}$$

Proof. It is direct from Theorem 6.2 and Proposition 2.3.□

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