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CHARACTERIZATION OF THE CLARKE REGULARITY OF SUBANALYTIC SETS

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Abstract. In this note, we will show that for a closed subanalytic subset $A \subset \mathbb{R}^n$, the Clarke tangential regularity of A at $x_0 \in A$ is equivalent to the coincidence of the Clarke tangent cone to A at x_0 with the set

$$\mathcal{L}(A, x_0) := \left\{ \dot{c}_+(0) \in \mathbb{R}^n : c : [0,1] \longrightarrow A \ \textit{ is Lipschitz}, \ c(0) = x_0 \right\},$$

where $\dot{c}_+(0)$ denotes the right-strict derivative of c at 0. The results obtained are used to show that the Clarke regularity of the epigraph of a function may be characterized by a new formula of the Clarke subdifferential of that function.

1. Introduction

Let $A \subset E$ be a closed subset of some real normed vector space E with $x_0 \in A$. The *Clarke tangent cone* of A at x_0 is defined by

$$T_c(A; x_0) := \liminf_{\substack{x \xrightarrow{A} x_0 \\ t \to 0^+}} \frac{A - x}{t}.$$

Equivalently, according to the definition of the limit inferior of a set-valued mapping, a vector v is a Clarke tangent vector to A at x_0 if for all neighborhood V of v there exist a neighborhood U of x_0 and some $\lambda > 0$ such that for all $(t, x) \in]0, \lambda[\times U$ we have $V \cap t^{-1}(A - x) \neq \emptyset$, that is,

$$(x + tV) \cap A \neq \emptyset.$$

When $x_0 \notin A$ one writes by convention $T_c(A, x_0) = \emptyset$.

This can be expressed in terms of sequences as follows: a vector $v \in E$ is a *Clarke tangent* to A at x_0 iff for any sequence $\{x_k\}_k$ of A converging to x_0 and any sequence of positive reals $\{t_k\}_k$ converging to 0, there exists a sequence $\{h_k\}_k$ in E converging to v such that for all $k \in \mathbb{N}$, $x_k + t_k h_k \in A$.

It has been shown in [10] that each vector in the Clarke tangent cone can be represented via a Lipschitz mapping. More precisely, the author established the following result.

Theorem 1. *Let A be a closed subset of $\mathbb{R}^n$ with $x_0 \in A$. For any $v \in T_c(A, x_0)$ and any real number $l > \|v\|$, there exists a Lipschitz continuous mapping $c : [0,1] \to \mathbb{R}^n$*

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with Lipschitz constant l such that

$$c(0) = x_0, \quad c([0,1]) \subset A,$$

and c is right strictly differentiable at 0 *with* $\dot{c}_+(0) = v$.

This result asserts that the following inclusion holds true for any closed set $A \subset \mathbb{R}^n$ and any $x_0 \in A$:

$$T_c(A, x_0) \subset \mathcal{L}(A, x_0), \tag{1.1}$$

where

$$\mathcal{L}(A, x_0) := \left\{ \dot{c}_+(0) \in \mathbb{R}^n : \, c : [0,1] \longrightarrow A \;\; \textit{is Lipschitz}, \, c(0) = x_0 \right\}$$

and $\dot{c}_+(0)$ denotes the right-strict derivative of c at 0, that is,

$$\lim_{t \neq s \to 0^+} \frac{c(t) - c(s) - (t-s)\dot{c}_+(0)}{t - s} = 0.$$

In general the inclusion (1.1) is strict, as is shown by the following example.

Example 1. Let $A = \{(x,y) \in \mathbb{R}^2 : \, y \geq -|x|\}$ and $x_0 = (0,0)$. Then the arc $c : [0,1] \longrightarrow A$ defined by $c(t) = t(1,0)$ for all $t \in [0,1]$ is Lipschitz and right strictly differentiable at 0. Moreover $\dot{c}_+(0) \notin T_c(A, x_0)$.

So the reverse inclusion in (1.1) is not always true except for sets satisfying the tangential equality

$$T_c(A, x_0) = K(A, x_0) \tag{1.2}$$

together with an additional geometrical property of A. Here $K(A, x_0)$ denotes the *contingent cone*, also called the Bouligand tangent cone, of the set A at $x_0 \in A$, and it is defined as the following limit superior of the set-differential quotient:

$$K(A, x_0) := \operatorname*{Lim\,sup}_{t \downarrow 0} \frac{1}{t}(A - x_0).$$

Otherwise stated, a vector $v \in K(A, x_0)$ if and only if for any neighborhood V of v and for any real $\varepsilon > 0$ one has

$$(x_0 +]0, \varepsilon[V) \cap A \neq \emptyset.$$

When $x_0 \notin A$ then one sets $K(A, x_0) = \emptyset$. It may already be said that convex sets as well as smooth manifolds are Clarke regular.

It is clear from the definition that

$$T_c(A, x_0) \subset K(A, x_0).$$

However for some examples of sets A, $T_c(A, x_0)$ is properly contained in $K(A, x_0)$. For instance when $A = \{(x,y) \in \mathbb{R}^2 : \;\; y \geq -|x|\}$ and $x_0 = (0,0)$, $T_c(A, x_0) = \{(x,y) \in \mathbb{R}^2 : \, y \geq |x|\}$ while $K(A, x_0) = A$. More examples and comments can be found in [4], and several tangential characterizations of $\mathcal{C}^1$ manifolds are given in the papers [2] and [3].

Nevertheless the tangential equality (1.2) of the Clarke and Bouligand tangent cones has many consequences of geometrical type; see [5]. It intervenes namely in the characterization of the Clarke tangent cone at an intersection of sets; that is, given two closed sets S_1 and S_2 with $x_0 \in S_1 \cap S_2$, how do we compute $T_c(S_1 \cap S_2)$? Unlike the contingent cone, the Clarke tangent cone does not satisfy the inclusion

$T_c(S_1 \cap S_2, x_0) \subset T_c(S_1, x_0) \cap T_c(S_2, x_0)$ (take $S_1 = \{(x, y) \in \mathbb{R}^2 : [y = x] \text{ or } [x \geq 0,\ x^2 + (y+2)^2 = 4]\}$, $S_2 = \{(x, y) \in \mathbb{R}^2 : [y = x] \text{ or } [x \geq 0,\ x^2 + (y+1)^2 = 1]\}$ and $x_0 = (0, 0)$, $T_c(S_1 \cap S_2, x_0) = \{(x, x) : x \in \mathbb{R}\}$ while $T_c(S_1, x_0) = T_c(S_2, x_0) = \{0\}$). If (1.2) is satisfied for S_1 and S_2 at x_0 and the transversality condition

$$T_c(S_1, x_0) - T_c(S_2, x_0) = \mathbb{R}^n$$

holds, then we have the characterization

$$T_c(S_1 \cap S_2, x_0) = T_c(S_1, x_0) \cap T_c(S_2, x_0).$$

Motivated by various consequences of the equality (1.2) of the Clarke and Bouligand tangent cones of the set A at x_0, Clarke baptizes this property in [5] as *Clarke tangential regularity*. Many inclusions for tangent and normal cones become equalities under Clarke tangential regularity. For example, given the constraint set $S = C \cap F^{-1}(D) := \{x \in C : F(x) \in D\}$ where $C \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^m$ are closed sets and $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a strictly differentiable mapping at $x_0 \in S$, with strict derivative $DF(x_0)$, we only have in general the following inclusions:

$$K(S, x_0) \subset K(C, x_0) \cap DF(x_0)^{-1} K(D, F(x_0)).$$

As above the tangential regularity of the sets C and D at x_0 and $F(x_0)$ respectively, together with the transversality assumption $DF(x_0)T_c(C, x_0) - T_c(D, F(x_0)) = \mathbb{R}^m$, ensures that

$$T_c(S, x_0) = T_c(C, x_0) \cap DF(x_0)^{-1} T_c(D, F(x_0)).$$

Here for a set $W \subset \mathbb{R}^m$, $DF(x_0)^{-1}(W) := \{h \in \mathbb{R}^n : DF(x_0)h \in W\}$.

Our aim in this paper is to characterize the Clarke tangent cone in terms of strictly differentiable mappings. This characterization occurs under the tangential regularity and subanalyticity of the set considered. In fact, we will establish (see Theorem 2) that for any subanalytic set $A \subset \mathbb{R}^n$, the tangential regularity of A at x_0 holds iff the inclusion (1.1) holds as an equality. An example is produced showing the necessity of the subanalyticity in this characterization. The results obtained are used to show that the Clarke regularity of the epigraph of a real-valued function f is characterized by a new formula of the Clarke subdifferential of f.

2. Subanalytic sets

The main result states that the tangential regularity can be characterized in terms of strictly differentiable mappings provided that the set involved is subanalytic.

In this section, we recall the definition and some properties of subanalytic sets. All the definitions used here concerning semianalytic and subanalytic sets are borrowed from [1].

Let M be a real analytic manifold. If U is an open set of M, let $\mathcal{A}(U)$ denote the ring of real analytic functions on U.

A subset A of M is *semianalytic* if each $a \in M$ has a neighborhood V such that

$$V \cap A = \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{x : f_{ij}(x) \sigma_{ij} 0\}$$

where $f_{ij} \in \mathcal{A}(V)$ and $\sigma_{ij} \in \{=, >\}$.

We have the following representation of semianalytic sets.

Proposition 1 ([1]). 1. *Every open semianalytic subset X of M is a finite union of semianalytic sets of the form*

$$\{x \in M : \quad f_i(x) > 0, \quad i = 1, \cdots, k\}$$

where $f_i \in \mathcal{A}(X)$.

2. *Every closed semianalytic subset X of M is a finite union of semianalytic sets of the form*

$$\{x \in M : \quad f_i(x) \geq 0, \quad i = 1, \cdots, k\}$$

where $f_i \in \mathcal{A}(X)$.

These sets are not stable under linear projection; that is, the linear projection of a semianalytic set need not be semianalytic (see [1]). This is the reason why we consider a larger class of subsets, called subanalytic, satisfying this property. A subset X of M is *subanalytic* if each point of X admits a neighborhood U such that $X \cap U$ is a projection of a relatively compact semianalytic set; i.e., there is a real analytic manifold N and a relatively compact semianalytic subset A of $M \times N$ such that $X \cap U = \pi(A)$, where $\pi : M \times N \mapsto M$ is the projection.

Some very interesting properties of these sets are listed in the following proposition. For the proofs the reader may consult [1].

Proposition 2 ([1]).

1. *The closure of a subanalytic set is a subanalytic set.*
2. *The complement of subanalytic set is a subanalytic set.*
3. *The distance function to a subanalytic set is subanalytic.*
4. *The projection of a relatively compact subanalytic set is subanalytic.*
5. *A finite union of subanalytic sets is subanalytic.*
6. *A finite intersection of subanalytic sets is subanalytic.*

Other characterizations of subanalytic sets can be found in [1].

The following version of the curve selection lemma for subanalytic sets [7], p. 328, will be used in the sequel.

Lemma 1. *Let E be a manifold and A be a subanalytic subset of E and let $x_0 \in \bar{A}$. Then there exists an analytic curve $c :]-1, 1] \mapsto E$, such that $c(0) = x_0$ and $c(t) \in A$ for all $t \in]0, 1]$.*

The following result gives an estimate of the distance to the Clarke tangent cone.

Proposition 3 ([6]). *Let $A \subset \mathbb{R}^n$ be a closed subanalytic set containing x_0. Then there exists $m > 1$ such that*

$$d(x_0 - x, T_c(A, x)) + d(x - x_0, T_c(A, x)) = o(\|x_0 - x\|^m) \text{ as } x \to x_0 \text{ in } A.$$

Remark 1. This proposition tells us that each differentiable mapping $c :]-1, 1] \mapsto E$ at 0, such that $c(0) = x_0$ and $c(t) \in A$ for all $t \in]0, 1]$, satisfies

$$\pm \dot{c}_+(0) \in \liminf_{t \to 0^+} T_c(A, c(t)).$$

But the later inclusion does not mean that $\dot{c}_+(0) \in T(A, x_0)$ (see Example 1).

3. Characterization of the Clarke regularity

The aim of this section is to give a characterization of the Clarke regularity in terms of strictly differentiable mappings and the classical tangent cone.

We recall that $v \in \mathbb{R}^n$ is a classical tangent vector to A at x_0 if for any sequence of positive reals $\{t_k\}$ such that $t_k \to 0$ there exists $x_k(\in A) \to x_0$ such that $t_k^{-1}(x_k - x_0) \to v$. The set of all the classical tangent vectors to A at x_0 is called the classical tangent cone; we will denote it by $T(A, x_0)$ so that

$$T(A, x_0) = \liminf_{t \to 0^+} \frac{A - x_0}{t}.$$

We always have

$$T_c(A, x_0) \subset T(A, x_0) \subset K(A, x_0).$$

In general $T(A, x_0) \subsetneq K(A, x_0)$. To see this take the set $A = \{(x, y) \in \mathbb{R}^2 : y = x \sin(\frac{1}{x})\} \cup \{(0, 0)\}$ and $x_0 = (0, 0)$. In this paper (see Proposition 5), we will show that the contingent and the classical cones coincide for closed subanalytic sets.

Now we state our main theorem.

Theorem 2. *Let A be a closed and subanalytic subset of $\mathbb{R}^n$ with $x_0 \in A$. Then the following assertions are equivalent:*

1. $T_c(A, x_0) = K(A, x_0)$;
2. $T_c(A, x_0) = T(A, x_0)$;
3. $T_c(A, x_0) = \left\{ \dot{c}_+(0) \in \mathbb{R}^n : c : [0, 1] \longrightarrow A \text{ is Lipschitz}, \ c(0) = x_0 \right\}$.

Remark 2. We have to mention that Theorem 2 remains valid for definable sets because the main tool used here is the curve selection lemma which holds for this class of sets (by replacing the term "analytic" by "$\mathcal{C}^1$") (see [8]).

Before establishing our main theorem, we give a counterexample showing the necessity of the subanalyticity of the set A.

Counterexample: Consider the following set in the real plan:

$$A = \{(x, y) \in \mathbb{R}^2 : \ y = x \sin(\frac{1}{x})\} \cup \{(0, 0)\}.$$

Suppose that there exists a Lipschitz mapping $v : [0, 1] \to \mathbb{R}^2$ such that $v(t) = (v_1(t), v_2(t)) \in A$, $v(0) = (0, 0)$ and v right differentiable at 0. We will prove that $v_1'(0) = v_2'(0) = 0$. Suppose that $v_1'(0) \neq 0$. Then $v_1(t) \neq 0$ for $t > 0$ near 0. Thus $v_2(t) = v_1(t) \sin(\frac{1}{v_1(t)})$ for $t > 0$ near 0. Dividing both sides by t and tending t to 0 we obtain

$$v_2'(0) = v_1'(0) \lim_{t \to 0^+} \sin(\frac{1}{v_1(t)}),$$

so that the limit $\lim_{t \to 0^+} \sin(\frac{1}{v_1(t)})$ exists, which is impossible (writing $v_1(t) = t v_1'(0) + o(t)$ and taking the sequences $t_k^1 = \frac{1}{v_1'(0)(\frac{\pi}{2} + 2k\pi)}$ and $t_k^2 = \frac{1}{v_1'(0)(\pi + 2k\pi)}$, we obtain two different values of this limit (1 and 0 respectively)). This contradiction asserts that $v_1'(0) = 0$ so that $v_2'(0) = 0$. Thus, by Proposition 4, $T_c(A, (0, 0)) = \{(0, 0)\}$, and by a simple calculation we obtain $T(A, (0, 0)) = \{(0, 0)\}$. This shows that items 2 and 3 of Theorem 2 hold. However item 1 of Theorem 2 does not hold since the contingent cone $K(A, (0, 0))$ contains the line $\mathbb{R}(1, 1)$.

Inspired by this counterexample, the following example shows the difference between the Clarke tangent cone, the contingent cone and the set on the right hand side of the equality in assertion 3 of Theorem 2.

Example 2. Let $A = \operatorname{epi} f$ be the epigraph of the function f defined by

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

and let $x_0 = (0,0)$. Then

- $T_c(A, x_0) = \{0\} \times \mathbb{R}_+$, $K(A, x_0) = \mathbb{R} \times \mathbb{R}_+$ and
- $\{\dot{c}_+(0) \in \mathbb{R}^2 : c : [0,1] \longrightarrow A$ *is Lipschitz*, $c(0) = x_0\} = \{(h,r) \in \mathbb{R}^2 : |h| \leq r\}$.

The proof of this theorem is a consequence of a series of propositions that will be established in the sequel. We begin with the following generalization of Theorem 1 for ball-compact sets in Banach spaces whose proof is similar to that of Theorem 3.1 in [10], and hence it is omitted. Let us recall that A is ball-compact if for all $r > 0$, the set $r\mathbb{B}_n \cap A$ is relatively compact. Here $\mathbb{B}_n$ denotes the closed unit ball of $\mathbb{R}^n$.

Proposition 4. *Let E be a Banach space and A be a ball-compact subset of E containing x_0. Then for any $v \in T_c(A, x_0)$ there exists a Lipschitz continuous mapping $z : [0,1] \longrightarrow E$ which is strictly differentiable at 0 such that*

$$z(0) = x_0, \quad z([0,1]) \subset A, \quad \dot{z}(0) = v.$$

In the sequel, we shall use the following notation:

(3.1)
$$\mathcal{A}(x_0) = \left\{ \frac{p'(0)}{\|p'(0)\|} : p : [0,1] \longrightarrow \mathbb{R}^n, \text{ analytic with } p([0,1]) \subset A, p(0) = x_0 \right\}.$$

Let's prove the following result.

Proposition 5. *Let A be a closed subanalytic subset of $\mathbb{R}^n$ with $x_0 \in A$. Then the following holds:*

$$\overline{\mathcal{A}(x_0)} = T(A, x_0) \cap \mathbb{S}_n = K(A, x_0) \cap \mathbb{S}_n.$$

Consequently,

$$T(A, x_0) = K(A, x_0).$$

Here $\mathbb{S}_n$ denotes the unit sphere of $\mathbb{R}^n$.

Proof. The proof of the proposition is similar to that given in [9]. Let $v \in \mathcal{A}(x_0)$. From (3.1) we have

$$v = \lim_{t \downarrow 0} \frac{p(t) - p(0)}{\|p(t) - p(0)\|}$$

where $p : [0,1] \longrightarrow \mathbb{R}^n$ is analytic with $p([0,1]) \subset A, p(0) = x_0$.

Let $\{t_k\}_k \subset (0,1)$ such that $t_k \to 0$ as $k \to +\infty$, and define $h_k = \frac{p(t_k) - p(0)}{\|p(t_k) - p(0)\|}$. We have $h_k \to v$, further

$$\frac{p(t_k) - p(0)}{t_k} = h_k \frac{\|p(t_k) - p(0)\|}{t_k} \longrightarrow \|p'(0)\| v.$$

This yields that $\|p'(0)\| v \in T(A, x_0)$. Since $T(A, x_0)$ is a cone, we have $v \in T(A, x_0)$. It follows that $\mathcal{A}(x_0) \subset T(A, x_0)$. Therefore since $T(A, x_0)$ is closed we have

$$\overline{\mathcal{A}(x_0)} \subset T(A, x_0) \cap \mathbb{S}_n.$$

Pick $v \in K(A, x_0)$ with $\|v\| = 1$. Fix $\varepsilon > 0$. Then there exist sequences $\{t_k\}_k \subset (0,1)$, with $\lim t_k = 0$, and $x_k (\in A) \to x_0$ such that $\lim \frac{x_k - x_0}{t_k} = v$:

$$\lim \frac{x_k - x_0}{\|x_k - x_0\|} = \lim \frac{x_k - x_0}{t_k} \cdot \lim \frac{t_k}{\|x_k - x_0\|} = v.$$

This implies that there exists $k_0 \in \mathbb{N}$ such that

$$\left\| \frac{x_k - x_0}{\|x_k - x_0\|} - v \right\| < \varepsilon, \ \forall\, k \geq k_0.$$

Now consider the set

$$A_\varepsilon = \left\{ x \in A \setminus \{x_0\} : \left\| v - \frac{x - x_0}{\|x - x_0\|} \right\| < \varepsilon \right\}.$$

Then $A_\varepsilon = A \cap B_\varepsilon$ where $B_\varepsilon = \{v \in \mathbb{R}^n : \|v\|x - x_0\| - x + x_0\| < \varepsilon \|x - x_0\|$. Since B_ε is a subanalytic (semialgebraic) set, Proposition 1 ensures that A_ε is subanalytic.

Therefore by considering the sequence $\{x_k\}_{k \geq k_0}$ and the subanalytic set A_ε we have $\{x_k\}_{k \geq k_0} \subset A_\varepsilon$ and $x_k \longrightarrow x_0$. It follows that $x_0 \in \bar{A_\varepsilon} \setminus A_\varepsilon \subset \partial A_\varepsilon$. Hence using the curve selection Lemma 1 for subanalytic sets, there exists an analytic curve $p_\varepsilon : [0,1] \longrightarrow \mathbb{R}^n$ such that $p_\varepsilon(t) \in A_\varepsilon \ \ \forall\, t \in (0,1]$ and $p_\varepsilon(0) = x_0$.

This implies that

$$\frac{p_\varepsilon(t) - p_\varepsilon(0)}{\|p_\varepsilon(t) - p_\varepsilon(0)\|} \in B(v, \varepsilon) \ \forall\ t \in (0,1],$$

where $B(v, \varepsilon) := \{x \in \mathbb{R}^n : \|x - v\| < \varepsilon\}$. Taking the limit when $t \to 0$ we get

$$\frac{p'_\varepsilon(0)}{\|p'_\varepsilon(0)\|} \in B(v, \varepsilon).$$

From this analysis, we deduce that for all $\varepsilon > 0$, $\mathcal{A}(x_0) \cap B(v, \varepsilon) \neq \emptyset$. This implies that $v \in \overline{\mathcal{A}(x_0)}$. Therefore

$$T(A, x_0) \cap \mathbb{S}_n \subset \overline{\mathcal{A}(x_0)}.$$

So we conclude that

$$T(A, x_0) \cap \mathbb{S}_n = \overline{\mathcal{A}(x_0)}.$$

This completes the proof. □

Proposition 6. *Let A be a closed subset of $\mathbb{R}^n$ and $x_0 \in A$. Assume that $T_c(A, x_0) = T(A, x_0)$; then*

$$\begin{aligned} T_c(A, x_0) &= \left\{ \dot{c}_+(0) : c : [0,1] \longrightarrow A, \ \ \textit{Lipschitz}, \ c(0) = x_0 \right\} \\ &= \left\{ c'_+(0) : c : [0,1] \longrightarrow A, \ \ \textit{Lipschitz}, \ c(0) = x_0 \right\}, \end{aligned}$$

where

$$c'_+(0) = \lim_{t \to 0^+} \frac{c(t) - c(0)}{t}.$$

Proof. The first inclusion comes from Theorem 1. Let's prove the reverse inclusion. Given $v = \dot{c}_+(0)$, with $c : [0,1] \longrightarrow A$ a Lipschitz mapping satisfying $c(0) = x_0$, and $\{t_k\}_k \subset (0,1)$ such that $\lim t_k = 0$. Put $x_k = c(t_k)$ for all integers k. Then we have $x_k \in A$ and $\lim \frac{x_k - x_0}{t_k} = v$. So $v \in T(A, x_0)$, that is, $v \in T_c(A, x_0)$. This proves the second inclusion. □

Let's prove the converse of the above result for closed subanalytic subsets.

Proposition 7. *Let A be a closed subanalytic subset of $\mathbb{R}^n$ and $x_0 \in A$. Assume that*

$$T_c(A, x_0) = \left\{ \dot{c}_+(0) : c : [0,1] \longrightarrow A, \ \ \textit{Lipschitz, } c(0) = x_0 \right\}.$$

Then $T_c(A, x_0) = K(A, x_0)$.

Proof. Proposition 5 asserts that

$$\overline{\mathcal{A}(x_0)} = K(A, x_0) \cap \mathbb{S}_n,$$

so it is enough to show that $\overline{\mathcal{A}(x_0)} \subset T_c(A, x_0)$. Indeed, as all analytic mappings are Lipschitz continuous on bounded sets, we deduce that for $p : [0,1] \to A$ analytic, $p(0) = x_0$, we have, because of our assumption, $p'(0) \in T_c(A, x_0)$. Since $T_c(A, x_0)$ is a cone $\frac{p'(0)}{\|p'(0)\|} \in T_c(A, x_0) \cap \mathbb{S}_n$. This implies that

$$\mathcal{A}(x_0) \subset T_c(A, x_0) \cap \mathbb{S}_n.$$

But $T_c(A, x_0) \cap \mathbb{S}_n$ is closed, so

$$\overline{\mathcal{A}(x_0)} \subset T_c(A, x_0) \cap \mathbb{S}_n.$$

From Proposition 5 we deduce that

$$K(A, x_0) \cap \mathbb{S}_n \subset T_c(A, x_0) \cap \mathbb{S}_n.$$

It ensues that $K(A, x_0) \subset T_c(A, x_0)$. The reverse inclusion being always true we have

$$K(A, x_0) = T_c(A, x_0).$$

This completes the proof. □

Let us now deduce the proof of Theorem 2.

Proof of Theorem 2. Assume that item 1 holds, that is, $T_c(A, x_0) = K(A, x_0)$. By definition we have

$$T_c(A, x_0) \subset T(A, x_0) \subset K(A, x_0).$$

Therefore $T_c(A, x_0) = T(A, x_0)$. So $1 \Rightarrow 2$. Suppose that $T_c(A, x_0) = T(A, x_0)$. By Proposition 6 we have

$$T_c(A, x_0) = \left\{ c'_+(0) : c : [0,1] \longrightarrow A, \ \ \text{Lipschitz, } c(0) = x_0 \right\}.$$

Therefore $2 \Rightarrow 3$. Suppose now that item 3 holds. Proposition 7 implies that $T_c(A, x_0) = K(A, x_0)$. So $3 \Rightarrow 1$.

In conclusion $1 \Leftrightarrow 2 \Leftrightarrow 3$ and the proof is completed. □

4. Characterization of the Clarke subdifferential

The negative polar cone $N_c(A, x_0)$ of the Clarke tangent cone $T_c(A, x_0)$ is the Clarke normal cone to A at $x_0 \in A$, that is,

$$N_c(A, x_0) = \big(T_c(A, x_0)\big)^0 := \{v \in X \colon \langle v, h\rangle \le 0 \quad \forall h \in T_C(A, x_0)\}.$$

As usual, $N_c(A, x_0) = \emptyset$ if $x_0 \notin A$. Through that normal cone, the Clarke subdifferential of the function $f \colon X \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\partial f(x_0) := \{v \in X \colon (v, -1) \in N_c\left(\operatorname{epi} f, (x_0, f(x_0))\right)\}, \tag{4.1}$$

where $\operatorname{epi} f := \{(y,r) \in \mathbb{R}^n \times \mathbb{R} \colon f(y) \le r\}$ is the epigraph of f. When the function is finite and locally Lipschitzian around x_0, the Clarke subdifferential is characterized (see [5]) in the following simple and amenable way:

$$\partial f(x_0) = \{v \in X \colon \langle v, h\rangle \le f^\circ(x_0; h) \quad \forall h \in \mathbb{R}^n\},$$

where

$$f^\circ(x_0; h) := \limsup_{(t,x)\to(0^+,x_0)} t^{-1}\left[f(x+th) - f(x)\right]$$

is the generalized directional derivative of the locally Lipschitzian function f at x_0 in the direction $h \in \mathbb{R}^n$. The function $f^\circ(x_0; \cdot)$ is in fact the support function of $\partial f(x_0)$.

Using our main theorem, we will establish that the Clarke regularity of the epigraph of f may be characterized by a new formula of the Clarke subdifferential of f.

Theorem 3. *Let $f \colon X \to \mathbb{R} \cup \{+\infty\}$ be a locally Lipschitzian function around x_0 whose epigraph is a subanalytic set. Then the following assertions are equivalent:*

1. $\operatorname{epi} f$ *is Clarke regular at* $(x_0, f(x_0))$,
2. $\partial f(x_0) = \{x^* \in \mathbb{R}^n : \langle x^*, c_1'(0)\rangle \le c_2'(0)\, \forall (c_1, c_2) \in \mathcal{L}_f(x_0)\}$, *where*

$$\mathcal{L}_f(x_0) := \{(c_1, c_2) : [0,1] \mapsto \operatorname{epi} f \ \textit{Lipschitz function which is right strictly differentiable at } 0 \textit{ with } c_1(0) = x_0,\ c_2(0) = f(x_0)\}.$$

Proof. The proof uses the following fact : Since f is locally Lipschitzian around x_0,

$$(x^*, \beta) \in N_c(\operatorname{epi} f, (x_0, f(x_0)) \implies \beta \le 0 \text{ and } (\beta = 0 \Rightarrow x^* = 0). \tag{4.2}$$

Indeed using the relation $T_c(\operatorname{epi} f, (x_0, f(x_0))) = \operatorname{epi} f^\circ(x_0; \cdot)$ (see [5]), we have $\langle (x^*, \beta), (h, f^\circ(x_0; h) + \varepsilon)\rangle \le 0$ for all $h \in X$ and $\varepsilon > 0$. That is, $\langle x^*, h\rangle + \beta f^\circ(x_0; h) + \beta\varepsilon \le 0$. Taking $h = 0$ we get $\beta\varepsilon \le 0$, so $\beta \le 0$. Now if $\beta = 0$ we have $\langle x^*, h\rangle \le 0$ for all $h \in X$. This implies that $x^* = 0$.

$1 \Rightarrow 2$: It is a consequence of the geometric characterization (4.1) of the Clarke subdifferential.

$2 \Rightarrow 1$: Suppose that there exist $(\bar{h}, \bar{\alpha}) \in K(\operatorname{epi}, (x_0, f(x_0)))$ and $(\bar{h}, \bar{\alpha}) \notin T_c(\operatorname{epi}, (x_0, f(x_0)))$. We may assume that $\|(\bar{h}, \bar{\alpha})\| = 1$. By the separation theorem there exist $v \in \mathbb{R}^n$, $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$, with $(v, \beta) \ne 0$, such that

$$\langle v, h\rangle + \beta\alpha \le \gamma < \langle v, \bar{h}\rangle + \beta\bar{\alpha} \quad \forall (h, \alpha) \in T_c(\operatorname{epi}, (x_0, f(x_0)))$$

or equivalently

$$(v, \beta) \in N_c(\operatorname{epi} f, (x_0, f(x_0)) \text{ and } 0 \le \gamma < \langle v, \bar{h}\rangle + \beta\bar{\alpha}. \tag{4.3}$$

Using the observation (4.2) and the fact that $(v, \beta) \ne 0$, we may assume that $\beta = -1$. This yields that $v \in \partial f(x_0)$. By 2), we have

$$\langle v, c_1'(0)\rangle \le c_2'(0)\, \forall (c_1, c_2) \in \mathcal{L}_f(x_0). \tag{4.4}$$

By Proposition 5, there exists a sequence $(p_{1k}, p_{2k})_k$ of analytic mappings $(p_{1k}, p_{2k}) : [0,1] \mapsto \operatorname{epi} f$ with $p_{1k}(0) = x_0$, $p_{2k}(0) = f(x_0)$ and such that

$$\lim_{k\to+\infty} \frac{(p_{1k}'(0), p_{2k}'(0))}{\|(p_{1k}'(0), p_{2k}'(0))\|} = (\bar{h}, \bar{\alpha}).$$

Since $(p_{1k}, p_{2k}) \in \mathcal{L}_f(x_0)$, relation (4.4) ensures that

$$\langle (v,-1), \frac{(p'_{1k}(0), p'_{2k}(0))}{\|(p'_{1k}(0), p'_{2k}(0))\|}\rangle \leq 0,$$

and passing to the limit, we get

$$\langle (v,-1), (\bar{h}, \bar{\alpha})\rangle \leq 0.$$

This inequality contradicts the last part of relation (4.3), and the proof is completed. □

5. Concluding remarks

In this paper, we have showed the fundamental role that the curve selection lemma plays in the characterization of the Clarke regularity. This lemma is related to the geometry of the set considered, and this geometry is due to the subanalyticity or, more generally, to the o-minimality properties. This regularity is also preserved for any strictly differentiable sets. The following example shows that Fréchet differentiability is not enough to guarantee this property.

Example 3. Consider the function $f : \mathbb{R} \mapsto \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then f is Fréchet differentiable but not strictly differentiable at 0. Let $A = \operatorname{epi} f$ and $x_0 = (0,0)$. Then $T_c(A, x_0) = \{(h,\alpha) \in \mathbb{R}^2 : |h| \leq \alpha\}$ while $K(A, x_0) = \mathbb{R}\times\mathbb{R}_+$.

Remark 3. This example can be used to show that both members of the assertion 2 of Theorem 3 are different. Indeed, $\partial_c f(0) = [-1,1]$ while the right hand side is exactly $\{0\}$. Even in the semialgebraic situation, without Clarke regularity, the equality in that assertion does not hold (take $f(x) = -|x|$).

Remark 4. The implication $1 \Rightarrow 2$ in Theorem 3 holds true for any lower semicontinuity and without the subanalyticity property. This is due to the fact that for any closed set $A \subset \mathbb{R}^n$ and any $x_0 \in A$, the inclusions hold (see Proposition 4):

$$T_c(A, x_0) \subset \left\{ \dot{c}_+(0) \in \mathbb{R}^n : c : [0,1] \longrightarrow A \ \textit{ is Lipschitz}, \ c(0) = x_0 \right\} \subset K(A, x_0).$$

Remark 5. For any locally Lipschitzian function $f : \mathbb{R}^n \mapsto \mathbb{R}$ around x_0 (whose $\operatorname{epi} f$ is not necessarily subanalytic), assertion 2 of Theorem 3 is equivalent to the following one:

$$T_c(\operatorname{epi} f, (x_0, f(x_0))) = \big\{ \dot{c}_+(0) : c := (c_1, c_2) : [0,1] \mapsto \operatorname{epi} f \text{ Lipschitz function with } c(0) = (x_0, f(x_0)) \big\}. \tag{5.1}$$

This equivalence is due to Proposition 4, the polarity property and the geometric characterization (4.1) of the Clarke subdifferential. Indeed, it suffices to establish the implication $2 \Rightarrow$ (5.1). We will show that $N_c(\operatorname{epi} f, (x_0, f(x_0))) \subset \mathcal{A}^0$, where $\mathcal{A}$ is the set on the right hand side of (5.1). Let $(x^*, \beta) \in N_c(\operatorname{epi} f, (x_0, f(x_0)))$. Using the observation (4.2), we may assume that $\beta < 0$. Then $(-\frac{x^*}{\beta}, -1) \in N_c(\operatorname{epi} f, (x_0, f(x_0)))$ or equivalently $-\frac{x^*}{\beta} \in \partial f(x_0)$. Using 2, we obtain

$$\langle x^*, h\rangle + \beta\alpha \leq 0 \, \forall (h,\alpha) \in \mathcal{A},$$

which is equivalent to saying that $(x^*, \beta) \in \mathcal{A}^0$. Thus

$$N_c(\operatorname{epi} f, (x_0, f(x_0))) = \big(T_c(\operatorname{epi} f, (x_0, f(x_0)))\big)^0 \subset \mathcal{A}^0$$

or equivalently $\mathcal{A} \subset T_c(\operatorname{epi} f, (x_0, f(x_0)))$. To conclude, it suffices to apply Proposition 4.

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