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# $C^{1,\omega(\cdot)}$-regularity and Lipschitz-like properties of subdifferential

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Abstract

It is known that the subdifferential of a lower semicontinuous convex function $f$ over a Banach space $X$ determines this function up to an additive constant in the sense that another function of the same type $g$ whose subdifferential coincides with that of $f$ at every point is equal to $f$ plus a constant, i.e., $g = f + c$ for some real constant $c$. Recently, Thibault and Zagrodny introduced a large class of directionally essentially smooth functions for which the subdifferential determination still holds. More generally, for extended real-valued functions in that class, they provided a detailed analysis of the enlarged inclusion

$$\partial g(x) \subset \partial f(x) + \gamma \mathbb{B} \quad \text{for all } x \in X,$$

where $\gamma$ is a nonnegative real number and $\mathbb{B}$ is the closed unit ball of the topological dual space. The aim of the present paper is to show how results concerning such an enlarged inclusion of subdifferentials allow us to establish the $\mathcal{C}^1$ or $\mathcal{C}^{1,\omega(\cdot)}$ property of an essentially directionally smooth function $f$ whose subdifferential set-valued mapping admits a continuous or Hölder continuous selection. The $\mathcal{C}^{1,\omega(\cdot)}$-property is also obtained under a natural Hölder-like behaviour of the set-valued mapping $\partial f$. Similar results are also proved for another class of functions that we call $\partial^{1,\varphi(\cdot)}$-subregular functions. When $X$ is a Hilbert space, the latter class contains prox-regular functions and hence our results extend old and recent results in the literature.

## 1. *Introduction*

Let $f, g : U \to \mathbb{R} \cup \{+\infty\}$ be two lower semicontinuous functions on a nonempty open convex set $U$ of a Banach space $X$. It is known that the subdifferential equality

$$\partial g(x) = \partial f(x) \quad \text{for all } x \in U \tag{1.1}$$

entails that the functions $f$ and $g$ are equal up to an additive constant (i.e. $f = g + c$ for some real constant $c$) provided the functions $f$ and $g$ are convex. This has been first established by Moreau [**35**] (see also [**36**]) when $X$ is a Hilbert (or reflexive) space and it has been extended to any Banach space by Rockafellar [**44**, **45**]. If both functions are nonconvex and Fréchet differentiable then (1.1) simply means that $D_F g(x) = D_F f(x)$ for every $x \in U$, where $D_F$ stands for the Fréchet derivative. There are several results in “classical” mathematical analysis allowing us to state that if derivatives of two functions are equal on $U$ then the functions are equal up to an additive constant. Thus having the equality $D_F g(x) = D_F f(x)$ for every $x \in U$, we know that the function $g$ inherits regularity properties of $f$, in other words information on regularity of $f$ are saved in properties of its derivative. In the case of nondifferentiable functions we also would like to know which regularity properties of the function are embedded in its subdifferential, although subdifferentials usually are not single valued. Frequently we are faced with the problem of evaluating subdifferential and it is often not possible to have knowledge on the whole subdifferential, so conditions like in (1.1) can be awkward to check. For this reason, we should look for new conditions more adapted to a subdifferential calculus. The first candidate to relax (1.1) is the inclusion instead of the equality. Thibault and Zagrodny [**53**]

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began the study of the enlarged inclusion

$$\partial g(x) \subset \partial f(x) + \gamma \mathbb{B}_{X^*} \quad \text{for all } x \in U, \tag{1.2}$$

where $\gamma$ is a nonnegative real number and $\mathbb{B}_{X^*}$ denotes the closed unit ball of $X^*$ centred at the origin. They showed that such an inclusion ensures that for all $x \in U$ and $y \in U \cap \operatorname{dom} f$,

$$f(x) - f(y) - \gamma\|x - y\| \leqslant g(x) - g(y) \leqslant f(x) - f(y) + \gamma\|x - y\| \tag{1.3}$$

whenever the lower semicontinuous function $f$ is convex. They proved later in [**54**] that conclusion (1.3) is still true when $f$ (instead of being convex) belong to the class of subdifferentially stable functions (see [**54**] for the definition), see also [**7**] where primal lower nice functions in Poliquin's sense ([**39**]) are involved to get (1.3). Recently, Thibault and Zagrodny continued the study of the enlarged inclusion above for the class of essentially directionally smooth functions; see Definition 3.1 for the definition of essentially directionally smooth functions, eds in short. The class of such functions includes convex functions, approximate convex functions, qualified convexly composite functions, directionally regular functions, essentially smooth functions, and so on; see Proposition 3.2 where the class is more specified. At this point it should be emphasized that one needs additional properties of $f$ in order to deduce (1.3) from (1.2). Indeed, it is known (see, e.g., [**12**]) that the Clarke subdifferential or Mordukhovich subdifferential alone cannot determine every Lipschitz continuous function up to an additive constant. In other words, (1.2) with $\gamma = 0$ does not imply (1.3) when additional properties on $f$ are not imposed.

Our first aim in the present paper is to study, through conclusion (1.3) for enlarged inclusions of subdifferentials of essentially directionally smooth functions, the behaviour of functions $f$ in the latter class whose subdifferential $\partial f$ admits a selection that is continuous, locally Hölder continuous with power $\alpha > 0$, or uniformly continuous with $\omega(\cdot)$ as the modulus of uniform continuity. We show that such a property of $f$ ensures that it is $C^1$, $C^{1,\alpha}$, or $C^{1,\omega(\cdot)}$. Doing so, we obtain a partial extension of a result of P. Kenderov [**31**] as well as a partial extension Asplund and Rockafellar [**1**, Theorem 3 and Corollary 2] to a large class of nonconvex functions. In fact, due to the state of the art on subdifferential calculus, we can use less restrictive conditions than that in (1.2) to get (1.3). So we are able to explore, employing (1.3), the integration of subdifferentials or differential properties of $f$ in more general cases. For example, even in the case of convex continuous functions a better characterization of the Fréchet differentiability at a given point than that in [**1**, Corollary 2] is established, see Corollary 4.2 for details. To make things more understandable, let us point out that dealing with nondifferentiable functions it is hard to expect that we can calculate the whole subdifferential, as said above. Sometimes we have a knowledge on some parts of it. Hence, it seems that it is better, as a second candidate for the relaxation of (1.1), to consider the condition

$$\partial g(x) \cap \partial f(x) \neq \emptyset \text{ for every } x \in Q,$$

instead of (1.1) whenever we integrate subdifferential, where $Q$ is a dense subset of $U$. In Section 4, we provide integration results where this condition is used and the function $g$ is either convex or DC; see Theorem 4.1 and Proposition 4.2. One of the important conclusions from this method of integration of subdifferential of eds functions is that, for any eds function and any selection of its Clarke subdifferential on $Q$, we have that the Clarke subdifferential is singleton at all points of continuity of the selection relative to $Q$, see Corollaries 4.2 and 4.5(c)–(e). Consequently, the function $f$ is strictly Fréchet differentiable at each point of a dense set $Q$ of $U$ whenever $\partial f$ admits a selection on $Q$ that is continuous relative to $Q$ (that is, with respect to the induced topology on $Q$); see Corollary 4.5. Thus, for a continuous essentially directionally smooth function and a dense subset of its domain a necessary and sufficient condition for strict Fréchet differentiability on this set is the existence of a continuous selection of the Clarke subdifferential on this set.

Our second objective is to show how conclusion (1.3) allows us to establish the $\mathcal{C}^{1,1}$-property of prox-regular functions (see Definition 5.1) whose subdifferentials enjoy the Aubin Lipschitz-like property. Such results for prox-regular functions started with Levy and Poliquin [**32**] through the study of single valuedness of hypomonotone set-valued mappings in finite-dimensional spaces. Their result for this kind of behaviour of prox-regular functions has been recently extended to Hilbert spaces by Bačák, Borwein, Eberhard and Mordukhovich [**3**]. The method in [**3**] is strongly based on infimal convolution techniques related to Moreau envelopes and proximal mappings of prox-regular functions in Hilbert spaces. Here, with the use of conclusion (1.3) for enlarged inclusions of subdifferentials, we establish under the same assumptions or other similar assumptions the $\mathcal{C}^{1,1}$ (or $C^{1,\omega(\cdot)}$) property, respectively, for the class of $\partial^{1,\varphi(\cdot)}$-subregular functions (see Definition 5.2) in any Banach space; see Theorem 5.1. The class of $\partial^{1,\varphi(\cdot)}$-subregular functions encompass convex functions, qualified convexly composite functions, prox-regular functions on Hilbert spaces, an so on. We thus extend the results mentioned above for prox-regular functions on finite-dimensional or Hilbert spaces to several classes of functions defined on general Banach spaces (instead of Hilbert spaces). As already emphasized in [**3**], such results have many important consequences in variational analysis (see, for example, [**32**, **34**, **46**, **47**]) and in the theory of Robinson generalized equations (see, for example, [**34**, **42**, **43**, **47**]).

The paper is organized as follows. In Section 2, we recall some notions and their properties. In Section 3, we state and recall the definition of essentially directionally smooth functions. The class of these functions is significantly large to encompass several important classes of functions. In this section, several examples of directionally essentially smooth functions, known in the literature, are provided. The main result of Section 4 establishes the $\mathcal{C}^1$ or $\mathcal{C}^{1,1}$-property of essentially directionally smooth function $f$ whose subdifferential $\partial f$ admits a continuous or locally Lipschitz continuous selection. More generally, the $\mathcal{C}^1$ or $\mathcal{C}^{1,1}$-property is established for such functions whose subdifferential set-valued mappings are Lipschitz-like continuous or Aubin continuous. The use of the Aubin continuity of subdifferentials allows us also to provide a characterization of the Fréchet differentiability of approximate convex functions in terms of the inner (lower) semicontinuity of the subdifferential. In Section 5, results of Section 4 are employed to get the $\mathcal{C}^{1,\omega(\cdot)}$ continuity for prox-regular functions on Hilbert spaces or $\partial^{1,\varphi(\cdot)}$-subregular functions on Banach spaces, under the Aubin or Lipschitz-like property of subdifferentials; see Theorem 5.1 and Corollary 5.3.

In Section 6, basic information on $\mathcal{C}^{1,\omega(\cdot)}$ continuity are pointed out. Additionally a characterization of this continuity is given in terms of paraconvexity, see [**27**, **48**–**51**] for several facts on the class of paraconvex functions; see also [**15**] for the parent class of semiconvex functions.

## 2. *Preliminaries*

Throughout, unless otherwise stated, $X$ is a *real Banach space*, $X^*$ its topological dual, and $B(x,\delta)$ the open ball of centre $x$ and radius $\delta$. By $\mathbb{B}_X$ and $\mathbb{B}_{X^*}$ we denote the closed unit balls of $X$ and $X^*$ respectively, centred at the origin. For an extended real-valued function $f : U \to \mathbb{R} \cup \{+\infty\}$ defined on a nonempty subset $U$ of $X$, its effective domain is defined by $\operatorname{dom} f := \{x \in U : f(x) < +\infty\}$. When $\operatorname{dom} f \neq \emptyset$, the function $f$ is said to be *proper*. In the same way the effective domain of a set-valued mapping $M : U \rightrightarrows Y$ (from $U$ into a nonempty set $Y$) is the set $\operatorname{Dom} M := \{x \in U : M(x) \neq \emptyset\}$. For such a set-valued mapping one also defines its graph as $\operatorname{gph} M := \{(u, y)) \in U \times Y : y \in M(u)\}$. A typical example of set-valued mappings with which we shall work in the next sections is the subdifferential in the sense of Convex Analysis. In such a context the subdifferential $\partial f(x)$ of a convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ at $x \in \operatorname{dom} f$ is the set

$$\partial f(x) := \{x^* \in X^* : \langle x^*, u - x\rangle + f(x) \leqslant f(u) \ \forall u \in X\}. \tag{2.1}$$

It is also known that for any $x \in \operatorname{dom} f$ its directional derivative

$$f'(x;h) := \lim_{t \downarrow 0} t^{-1}[f(x+th) - f(x)] \tag{2.2}$$

exists whenever the convexity of $f$ is assumed, and one also has $f'(x;h) = \inf_{t>0} t^{-1}[f(x+th) - f(x)]$. The latter equality ensures

$$\partial f(x) = \{x^* \in X^* : \langle x^*, h\rangle \leqslant f'(x;h)\ \forall h \in X\}, \tag{2.3}$$

and whenever $f$ is in addition continuous at $x \in \operatorname{int} \operatorname{dom} f$, the set $\partial f(x)$ is nonempty and $w^*$-compact, and $f'(x;\cdot)$ is the support function of the set $\partial f(x)$, that is,

$$f'(x;h) = \max\{\langle x^*, h\rangle : x^* \in \partial f(x)\} \quad \text{for all } h \in X. \tag{2.4}$$

Whenever nonconvex functions are considered, there are several ways to define their subdifferential. It is then natural to work with an abstract general concept of subdifferential allowing us to state many results in a general unified framework. Such a generality is achieved through the concept of presubdifferential. Following [**52**, **53**], a *presubdifferential* on $X$ is an operator $\partial$ that associates with any function $f : X \to \mathbb{R} \cup \{+\infty\}$ and any $x \in X$ a subset $\partial f(x)$ of $X^*$ and which satisfies the following properties:

(P1) $\partial f(x) \subset X^*$ and $\partial f(x) = \emptyset$ if $x \notin \operatorname{dom} f$;
(P2) $\partial f(x) = \partial g(x)$ whenever $f$ and $g$ coincide on a neighbourhood of $x$;
(P3) $\partial f(x)$ is equal to the subdifferential in the above sense (2.1) of Convex Analysis whenever $f$ is convex and lower semicontinuous;
(P4) if $f$ is lower semicontinuous near $x \in \operatorname{dom} f$, $g$ is (finite) convex continuous near $x$, and $x$ is a local minimum point of $f + g$, then one has

$$0 \in w^* - \operatorname*{Lim\,sup}_{u \to_f x} \partial f(u) + \partial g(x),$$

where $u \to_f x$ means $(u, f(u)) \to (x, f(x))$ and $w^* - \operatorname{Lim\,sup}_{u \to_f x} \partial f(u)$ denotes the weak* sequential outer (upper) limit of $\partial f(u)$ as $u \to_f x$, that is, the set of all $w^*$-limits $\lim_k u_k^*$ of sequences $(u_k^*)_k$ such that $u_k^* \in \partial f(u_k)$ and $u_k \to_f x$.

The graph of the presubdifferential set-valued mapping $\partial f$ is the set

$$\operatorname{gph} \partial f := \{(x, x^*) \in X \times X^* : x^* \in \partial f(x)\}.$$

When $f : U \to \mathbb{R} \cup \{+\infty\}$ is defined on a subset $U$ of $X$ its presubdifferential $\partial f(x)$ is defined as the presubdifferential of the extension of $f$ to $X$ with the value $+\infty$ outside of $U$.

We say that $\partial$ is a subdifferential with the exact inclusion sum rule when (P1)-(P3) hold and instead of (P4) one requires:

(P4') for any function $g$ finite and locally Lipschitz continuous near $x$

$$\partial(f+g)(x) \subset \partial f(x) + \partial g(x),$$

and $0 \in \partial f(x)$ whenever $x \in \operatorname{dom} f$ is a local minimum of $f$.

Obviously, any such subdifferential is a presubdifferential. A first example of useful presubdifferentials is the operator of *proximal subgradients* when $X$ is a Hilbert space (see, e.g., [**17**]). Another useful presubdifferential is the operator of Fréchet subgradients when $X$ is an Asplund space (see, for example, [**34**]). It will be used in many parts of the paper. We recall that, for an extended real-valued function $f$ defined on the Banach space $X$ and $x \in \operatorname{dom} f$, an element $x^* \in X^*$ is a *Fréchet subgradient* of $f$ at $x \in \operatorname{dom} f$, and we write $x^* \in \partial_F f(x)$, provided that, for each real $\varepsilon > 0$ there exists some neighbourhood $U$ of $x$ such that

$$\langle x^*, u - x\rangle \leqslant f(u) - f(x) + \varepsilon\|u - x\| \quad \text{for all } u \in U.$$

We also recall that the Banach space $X$ is Asplund when the topological dual of any separable subspace of $X$ is separable. So, any reflexive Banach space is Asplund.

A first important example of subdifferentials with the exact inclusion sum rule which we will deal with, is the *Mordukhovich limiting subdifferential* $\partial_L$ on Asplund spaces (see [**34**]) whose elements are $w^*$-limits of sequences of Fréchet subgradients; more precisely $\partial_L f(x)$ is the set of $x^* \in X^*$ for which there are sequences $x_k \to_f x$ and $x_k^* \overset{w^*}{\to} x^*$ with $x_k^* \in \partial_F f(x_k)$. It is worth emphasizing that $\partial_L f(x)$ is nonempty and bounded whenever $X$ is an Asplund space and $f$ is locally Lipschitz continuous near $x$. As two other important subdifferentials with exact inclusion sum rule, we have the Ioffe (geometric) subdifferential on any Banach space (see [**27**]), and the Clarke subdifferential on any normed vector space (see [**16**]). Besides the above properties (P1)–(P3) and (P4'), some other properties of the Clarke subdifferential will be involved in the development of the paper.

One of the best ways to introduce the Clarke subdifferential (called also the Clarke generalized gradient) is to define it first for locally Lipschitz continuous functions via the generalized directional derivative. Reacall that, for a locally Lipschitz continuous function $f : U \to \mathbb{R}$ on an open set $U$ of $X$, its *Clarke generalized directional derivative* (see [**16**]) is defined for $x \in U$ by

$$f^o(x; h) := \limsup_{u\to x; t\downarrow 0} t^{-1}[f(u + th) - f(u)]$$

and then its *Clarke subdifferential* at $x$ can be defined similarly to the convex setting (see (2.3))

$$\partial_C f(x) := \{x^* \in X^* : \langle x^*, h\rangle \leqslant f^o(x; h) \quad \forall h \in X\}.$$

It is not difficult to see that, for a locally Lipschitz continuous function $f$, we have

$$f^o(x; h) = \limsup_{(u,w)\to(x,h); t\downarrow 0} t^{-1}[f(u + tw) - f(u)],$$

and hence the function $f^o(\cdot;\cdot)$ is upper semicontinuous on $U \times X$, we refer the reader to [**16**] for details.

It is worth pointing out that, when $X$ is Asplund, the Clarke subdifferential $\partial_C f(x)$ is related to the Mordukhovich limiting subdifferential $\partial_L f(x)$ through the equality

$$\partial_C f(x) = \mathrm{cl}_{\,w*}\mathrm{co}\big(\partial_L f(x)\big), \tag{2.5}$$

where $\mathrm{cl}_{\,w*}\mathrm{co}$ denotes the $w^*$-closed convex hull, we refer the reader to [**34**] for details.

We recall that a mapping $G : U \to Y$ from the open set $U$ into a normed space $Y$ is strictly Fréchet differentiable at $a \in U$ provided that there exists some continuous linear mapping $A : X \to Y$ such that for each real $\varepsilon > 0$ there exists some neighbourhood $U' \subset U$ of $a$ such that

$$\|G(x) - G(y) - A(x - y)\| \leqslant \varepsilon\|x - y\| \quad \text{for all } x, y \in U';$$

in such a case $G$ is obviously Fréchet differentiable at $a$ and its Fréchet derivative $D_F G(a)$ coincides with $A$ (there are several notions of differentiability with a long history, for example, strict Fréchet differentiability and Fréchet differentiability were known in the nineteenth century; see [**20**] for some historical comments). When a function $f : U \to \mathbb{R}$ is strictly Fréchet differentiable at $a \in U$, it is easily seen that it is Lipschitz continuous near the point $a$ and

$$\partial_F f(a) = \partial_L f(a) = \partial_C f(a) = \{D_F f(a)\}. \tag{2.6}$$

Through the Clarke subdifferential of the distance function $d_S(u) = \inf_{y\in S}\|y - u\|$ to a subset $S \subset X$ the Clarke normal cone to $S$ at $x \in S$ is defined as the weak* closure of the cone generated by the Clarke subdifferential of the distance function, that is,

$$N_C(S;x) := \mathrm{cl}_{w*}([0,+\infty[\partial_C d_S(x))$$

and so the Clarke subdifferential may be extended to any function $f : X \to \mathbb{R} \cup \{+\infty\}$ at $x \in \mathrm{dom}\, f$ (see [**16**]) as

$$\partial_C f(x) = \{x^* \in X^* : (x^*, -1) \in N_C(\mathrm{epi}\, f; (x, f(x)))\},$$

where epi $f$ denotes the epigraph $\mathrm{epi}\, f = \{(u, r) \in X \times \mathbb{R} : f(u) \leqslant r\}$ of $f$.

A crucial property of the presubdifferential that will be considered later is that Dom $\partial f$ is graphically dense in dom $f$; see, for example, [**56**]. For the sake of completeness and convenience of the reader, we sketch a proof below. The proof is based on the Zagrodny mean value theorem (proved for the first time in [**58**] for the Clarke subdifferential, but as pointed out in [**52**] the same proof holds with a slight adaptation for any presubdifferential). We also refer the reader to [**22**, **59**, **60**] for other related results.

THEOREM 2.1. *Let $\partial$ be a presubdifferential on a Banach space $X$, let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, and let $a, b \in X$ with $a \neq b$ and $a \in \mathrm{dom}\, f$. Then for any real number $r \leqslant f(b)$, there exist sequences $x_k \to_f c \in \{a + t(b - a) : t \in [0, 1[\}$, $x_k^* \in \partial f(x_k)$ such that the following properties hold:*

(a) $r - f(a) \leqslant \lim_{k\to\infty} \langle x_k^*, b - a\rangle$;
(b) $(\|b - c\|/\|b - a\|)(r - f(a)) \leqslant \lim_{k\to\infty} \langle x_k^*, b - x_k\rangle$;
(c) $\|b - a\|(f(c) - f(a)) \leqslant \|c - a\|(r - f(a))$.

We point out that the proof provided below for the graphical density of Dom $\partial f$ in dom $f$ works for any operator for which the above mean value theorem holds.

PROPOSITION 2.1. *Let $\partial$ be a presubdifferential on $X$ and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Then* Dom $\partial f$ *is $f$-graphically dense in* dom $f$, *i.e., for any $a \in \mathrm{dom}\, f$ there exists a sequence $x_k \in \mathrm{Dom}\, \partial f$ such that $x_k \to_f a$.*

*Proof.* Fix any $a \in \mathrm{dom}\, f$ and $\varepsilon > 0$. Let a positive $\delta < \varepsilon/2$ be such that $f(a) - \varepsilon < f(x)$ for all $x \in B(a, \delta)$ and let any $b \in B(a, \delta)$ with $b \neq a$. Fix a real number $r \leqslant f(b)$ such that $r < f(a) + \varepsilon$. Theorem 2.1 gives a sequence $(x_k)_k$ with $\partial f(x_k) \neq \emptyset$ and $x_k \to_f c \in B(a, \delta)$, where the point $c$ satisfies the property (c) of the same Theorem 2.1. The latter and the inequality $r - f(a) < \varepsilon$ ensure that $\lim_k f(x_k) = f(c) < f(a) + \varepsilon$. We then deduce the existence of some integer $K$ such that $\|x_K - a\| < \varepsilon$ and $|f(x_K) - f(a)| < \varepsilon$, which completes the proof. □

Before passing to the next section, we recall that the lower Dini directional derivative of a function $f : U \to \mathbb{R} \cup \{+\infty\}$ at $u \in \mathrm{dom}\, f$, where $U$ is an open set of $X$, is given by

$$d^- f(u; h) := \liminf_{w\to h; t\downarrow 0} t^{-1}[f(u + tw) - f(u)]$$

and when $f$ is Lipschitz continuous near $u$ we obviously have for all $h \in X$

$$d^- f(u; h) = \liminf_{t\downarrow 0} t^{-1}[f(u + th) - f(u)].$$

When $f$ is Lipschitz continuous near $u$ and $f^o(u; \cdot) = d^- f(u; \cdot)$, the function $f$ is said to be *directionally subregular (or Clarke regular) at the point $u$*. It is not difficult to see that $f$ is directionally subregular at $u$ if and only if

$$f^o(u;h) = \lim_{t\downarrow 0} t^{-1}[f(u+th) - f(u)] \quad \text{for all } h \in X.$$

If the directional subregularity property of $f$ holds for all $u \in U$, one says that $f$ is *directionally subregular on* $U$.

## 3. *Essentially directionally smooth functions and enlarged inclusions of subdifferentials*

In this section, we introduce the class of essentially directionally smooth functions. We need first to recall some other concepts which motivated the study in [**55**] of essentially directionally smooth functions and which will be involved in the present paper.

Throughout the section and others, unless otherwise stated, $\partial$ is a *presubdifferential* on a Banach space $X$. Let $U$ be a nonempty *open convex subset* of $X$ and $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be two functions that are proper on $U$. Assume that

$$\partial g(x) = \partial f(x) \quad \text{for all } x \in U. \tag{3.1}$$

If $f$ and $g$ are lower semicontinuous (lsc, for short) and convex, then there exists a real constant $c$ such that

$$g(x) = f(x) + c \quad \text{for all } x \in U; \tag{3.2}$$

this means that proper lsc convex functions are *subdifferentially determined*. This result was obtained by Moreau when $X$ is a Hilbert space (with a proof that is still valid for any reflexive Banach space) (see [**35**, **36**]), and extended by Rockafellar to any Banach space; see [**44**, **45**]. As it is said in the introduction, more generally in place of the equality in (3.1), an enlarged inclusion has been considered in [**53**] with the subdifferential of the convex function $g$, say

$$\partial g(x) \subset \partial f(x) + \gamma \mathbb{B}_{X^*} \quad \text{for all } x \in U. \tag{3.3}$$

Correa and Jofre [**19**] also established that under condition (3.1) with the Clarke subdifferential the same conclusion (3.2) holds, provided that $f, g$ are over $U$ finite locally Lipschitz continuous functions that are *semismooth* and whose Clarke subdifferential is single-valued at any point of a dense subset $D$ of $U$ (see also [**56**] for the case of Lipschitz continuous directionally subregular functions). The locally Lipschitz continuous function $f$ is semismooth at $x \in U$, if for each $h \in X$ one has $\lim_{y\to h; t\downarrow 0} d^- f(x+ty; h) = d^- f(x; h)$. Later, Borwein and Moors [**9**, **11**] showed with the Clarke subdifferential that the result also holds whenever the locally Lipschitz continuous functions $f$ and $g$ are *essentially smooth* in the sense that they introduced (see also [**21**]). So proper lsc convex functions and locally Lipschitz continuous essentially smooth functions are subdifferentially determined. The locally Lipschitz continuous function $f$ is essentially smooth on the open set $U$ when, for each nonzero vector $h \in X$, the set $N := \{x \in U : f^o(x; -h) \neq -f^o(x; h)\}$ is Haar-null in $X$ in the sense that there exists a Radon probability measure $P$ on $X$ such that $P(N + y) = 0$ for all $y \in X$. The essential smoothness is not stable by composition, that is, there are locally Lipschitz continuous functions $f$ and mappings $F$ with values in $\mathbb{R}^n$ that are essentially smooth (that is, each component of $F$ is essentially smooth) such that $f \circ F$ fails to be essentially smooth (see [**10**]).

That nonclosedness property under composition led Borwein and Moors [**10**] to introduce for $X = \mathbb{R}^m$, as a large subclass, the concept of arc-wise essentially smooth functions which is in the line of Valadier's sound functions (”fonctions saines” in French; see [**57**]) and which is preserved under composition. Observing that the equality $f^o(x; -h) = -f^o(x; h)$ is equivalent to the existence of the limit $\lim_{u\to x; t\downarrow 0} t^{-1}[f(u+th) - f(u)]$, we may (as in [**13**, **14**]) extend the concept of essential smoothness to mappings $F$ from an open subset $V$ of a Banach space $Z$ into $U$ by requiring that, for each nonzero vector $v \in Z$, the set

$$\{z \in V : \lim_{y \to z; t \downarrow 0} t^{-1}[F(y+tv) - F(y)] \text{ does not exist}\}$$

is Haar-null in $Z$. So, the locally Lipschitz continuous function $f : U \to \mathbb{R}$ is *arc-wise essentially smooth* if, for each locally Lipschitz continuous essentially smooth mapping $x : ]0,1[ \to U$, the set

$$\{t \in ]0,1[ : \ f^o(x(t); -x'(t)) \neq -f^o(x(t); x'(t))\} \tag{3.4}$$

has null Lebesgue measure (or is equivalently Haar-null). When, for each $(u,v) \in U \times U$ and for $x(t) = u + t(v-u)$, the set in (3.4) has null Lebesgue measure, we will say that $f$ is *segment-wise essentially smooth*. The last class obviously contains that of arc-wise essentially smooth locally Lipschitz continuous functions.

The approaches in [**11**, **19**, **35**, **44**] are all distinct. Further, the finiteness over $U$ of the locally Lipschitz continuous arc-wise essentially smooth functions makes clear that neither the extended real-valued convex functions is included in that class nor the converse. The class of *essentially directionally smooth* functions introduced in [**55**] allowed their authors on the one hand, to unify several of the above results and, on the other hand, to identify many other interesting amenable functions that are subdifferentially determined. As we shall see below, the class even provides stronger results concerning enlarged inclusion (3.3). Before recalling the definition, we denote, according to (2.2), by $\varphi'(t;1)$ the right derivative of a function $\varphi$ at $t$ defined on an interval of $\mathbb{R}$, whenever it exists, that is,

$$\varphi'(t;1) := \lim_{\tau \downarrow 0} \frac{\varphi(t+\tau) - \varphi(t)}{\tau}.$$

DEFINITION 3.1 ([**55**]). Let $U$ be a nonempty open convex subset of the Banach space $X$ and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc function on $U$ with $U \cap \operatorname{dom} f \neq \emptyset$ and let $\mu > 0$ be fixed. Let $D$ be a subset of $X$ with $\operatorname{Dom} \partial f \subset D \subset \operatorname{dom} f$. We say that the function $f$ is essentially $\partial$, $\mu$-directionally smooth on $U$ relative to $D$ provided that, for each $u \in U \cap \operatorname{Dom} \partial f$, the following conditions are satisfied.

(i) For each $v \in U \cap \operatorname{dom} f$ the function $f_{u,v}(t) := f(u + t(v-u))$ is finite and continuous on $[0,1]$.

(ii) For each $v \in U \cap D$ there are real numbers $0 = t_0 < \cdots < t_p = 1$ such that the function $t \mapsto f_{u,v}(t)$ is absolutely continuous on each closed interval included in $[0,1] \setminus \{t_0, t_1, \cdots, t_p\}$.

(iii) For each $v \in U \cap D$ with $v \neq u$ there exists a subset $T \subset [0,1]$ of full Lebesgue measure (that is, of Lebesgue measure 1) such that for every $t \in T$ and every sequence $((x_k, x_k^*))_k \subset \operatorname{gph} \partial f$ with $x_k \to x(t) := u + t(v-u)$, there is some $w \in ]x(t), v]$ for which

$$\limsup_{k \to \infty} \langle x_k^*, w - x_k \rangle \leqslant \|w - x(t)\| (\|v-u\|^{-1} f'_{u,v}(t;1) + \mu).$$

Conditions (i) and (ii) are quite natural to have the subdifferential integration property. As regards (iii), we refer the reader to [**55**, Section 4] for a large discussion and several particular cases.

The function $f$ will be said to be *essentially $\partial$-directionally smooth on $U$ relative to $D$* when it is essentially $\partial, \mu$-directionally smooth on $U$ relative to $D$ for every $\mu > 0$. We use the abbreviation "$\partial$-eds on $U$ respectively to $D$" for this function.

The two most important cases correspond with $D = \operatorname{dom} f$ and $D = \operatorname{Dom} \partial f$. When $D = \operatorname{dom} f$ we shall omit writing "relative to dom $f$", that is, we shall only say that $f$ is *essentially $\partial, \mu$-directionally smooth* or *$\partial$-directionally smooth* ($\partial, \mu$-eds or $\partial$-eds, for short) on $U$.

Obviously for $\mathrm{Dom}\,\partial f \subset D \subset D' \subset \mathrm{dom}\, f$, the essential $\partial, \mu$-directional smoothness of $f$ relative to $D'$ entails the same property relative to $D$. Further, it is readily seen that $f$ is $\partial, \mu$-eds on $U$ if and only if it is $\partial, \mu$-eds on $U$ relative to $U \cap \mathrm{dom}\, f$.

The class of eds functions is stable under addition according to the following result of Thibault and Zagrodny [55].

Proposition 3.1 ([55, Proposition 4.3]). *Let $U$ be a nonempty open convex subset of $E$ and let $f_i : U \to \mathbb{R} \cup \{+\infty\}$ be a $\partial, \mu_i$-eds function on $U$ relative to $\mathrm{dom}\, f_i$ for every $i \in \{1, \cdots, n\}$, satisfying $U \cap \bigcap_{i=1}^n \mathrm{dom}\, f_i \neq \emptyset$. Assume that $\partial$ is any presubdifferential (fulfilling (P1)–(P4)) and that*

$$\partial(f_1 + \cdots + f_n)(x) \subset \partial f_1(x) + \cdots + \partial f_n(x) \quad \text{for all } x \in U.$$

*Then the function $f_1 + \cdots + f_n$ is $\partial, (\mu_1 + \cdots + \mu_n)$-eds on $U$ relative to $\mathrm{dom}\,(f_1 + \cdots + f_n)$.*

The proposition above and the next one show how large the class of eds functions is. Before giving the proposition, recall that a function $f : U \to \mathbb{R} \cup \{+\infty\}$ defined on an open convex set $U$ of $X$ is approximate convex (see [37]) at a point $a \in U$, provided, for each real $\rho > 0$, there exists some neighbourhood $U' \subset U$ of $a$ such that for all $x, y \in U'$ and $t \in ]0, 1[$

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) + \rho t(1-t)\|x - y\|. \tag{3.5}$$

When $f$ is approximate convex at each point of $U$, one says that $f$ is approximate convex on $U$.

Proposition 3.2. *Let $U$ be a nonempty open convex subset of $X$ and let $f : U \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc function. Then each one of the following conditions ensures that $f$ is $\partial$-eds on $U$:*

(a) *$f$ is convex on $U$;*
(b) *$f$ is locally Lipschitz continuous on $U$ and segment-wise essentially smooth on $U$, and the presubdifferential $\partial f$ is included in the Clarke subdifferential of $f$;*
(c) *$f$ is locally Lipschitz continuous on $U$ and directionally subregular on $U$, and the presubdifferential $\partial f$ is included in the Clarke subdifferential of $f$;*
(d) *$\partial$ is a subdifferential included in the Clarke subdifferential with the exact subdifferential sum rule (P4') and $f$ is DC on $U$, that is, for any $x \in U$ there exist an open convex neighbourhood $U' \subset U$ of $x$, an lsc convex function $f_1 : U' \to \mathbb{R} \cup \{+\infty\}$ and a convex continuous function $f_2 : U' \to \mathbb{R}$ such that $f(u) = f_1(u) - f_2(u)$ for all $u \in U'$.*

*Further, $f$ is $\partial$-eds on $U$ relative to $D := \mathrm{Dom}\,\partial f$ whenever $f$ is approximate convex on $U$ and $\partial$ is included in the Clarke subdifferential.*

*Proof.* The case of condition (a) corresponds to [55, Proposition 4.4] and the cases of conditions (b)-(d) are established in Corollaries 4.9, 4.8 and 4.13, respectively, of the same paper [55]. The final case of approximate convex function follows from, [54, Proposition 3.5] and [55, Proposition 4.15]. □

## 4. *Continuous-like and Lipschitz-like properties of subdifferentials of eds functions*

The aim of this section is to show that several generalized continuity or Lipschitz properties of the presubdifferential of an eds function ensure the $\mathcal{C}^1$- or $\mathcal{C}^{1,\omega(\cdot)}$-regularity of the function. More generally, we investigate the behaviour of an eds function $f$ under the existence of some specific selections of the presubdifferential set-valued mapping $\partial f$. Let us start with the following theorem from which we will deduce a new simple criterion for the Lipschitz property of eds functions. Let us first say that, for $\varepsilon > 0$, a set-valued mapping $M : S \rightrightarrows X^*$ defined on a subset $S$ of the Banach space $X$ is *$\varepsilon$-hypomonotone with power* 1 whenever

$$\langle x_1^* - x_2^*, x_1 - x_2\rangle \geqslant -\varepsilon\|x_1 - x_2\| \quad \text{for all } x_i \in S \text{ and } x_i^* \in M(x_i),\ i = 1, 2.$$

We also say that the set-valued mapping $M$ is *locally bounded* provided that, for each $x \in S$ there are $\delta > 0$ and $\beta \geqslant 0$ such that

$$\|y^*\| \leqslant \beta \quad \text{for all } y^* \in M(y) \text{ with } y \in S \cap B(x, \delta).$$

THEOREM 4.1. *Let $X$ be a Banach space, $U$ be a nonempty open convex subset of $X$, $\partial$ be a presubdifferential and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a $\partial, \mu$-eds function on $U$ relative to $\operatorname{Dom}\partial f$. Let $\varepsilon \geqslant 0$, $\gamma \geqslant 0$, $Q \subset U$ be given such that $U \cap \operatorname{cl}\operatorname{dom} f \subset \operatorname{cl} Q$, where $\operatorname{cl}$ stands for the closure with respect to the strong topology (the topology generated by the norm). Assume that there is a set-valued mapping $M_\varepsilon : U \cap \operatorname{cl} Q \rightrightarrows X^*$ with $M_\varepsilon(x) \neq \emptyset$ for all $x \in U \cap \operatorname{cl} Q$ that is $\varepsilon$-hypomonotone with power 1 and locally bounded on $U \cap \operatorname{cl} Q$ and such that*

$$(\partial f(q) + \gamma\mathbb{B}_{X^*}) \cap M_\varepsilon(q) \neq \emptyset \text{ for every } q \in Q. \tag{4.1}$$

*Then $f$ is finite on $U \cap \operatorname{cl} Q$ and, for all $v \in U$, $u \in U \cap \operatorname{cl} Q$, $u^* \in M_\varepsilon(u)$,*

$$f(v) - f(u) + (\mu + 2\varepsilon + \gamma)\|v - u\| \geqslant \langle u^*, v - u\rangle$$

*and*

$$f(v) - f(u) + (\mu + \varepsilon + \gamma)\|v - u\| \geqslant \langle u^*, v - u\rangle,$$

*whenever $\partial f(u) \neq \emptyset$.*

*Moreover, if $Q$ is a dense subset of $U$ and $g : U \to \mathbb{R}$ is a convex continuous function such that*

$$(\partial f(q) + \gamma\mathbb{B}_{X^*}) \cap \partial g(q) \neq \emptyset \text{ for every } q \in Q, \tag{4.2}$$

*then $f$ is finite on $U$ and*

$$f(v) - f(u) + (\mu + \gamma)\|v - u\| \geqslant g(v) - g(u) \geqslant f(v) - f(u) - (\mu + \gamma)\|v - u\| \text{ for all } u, v \in U.$$

*Proof.* Let us fix $u \in U \cap \operatorname{cl} Q$ and $u^* \in M_\varepsilon(u)$. In this part of the proof, let us also assume that $\partial f(u) \neq \emptyset$. Take any $v \in U \cap \operatorname{Dom}\partial f \setminus \{u\}$. Since the function $f$ is $\partial, \mu$-eds on $U$ relative to $\operatorname{Dom}\partial f$, there is a subset $T \subset [0, 1]$ of full Lebesgue measure such that condition (iii) of Definition 3.1 is satisfied. Fix any $t \in ]0, 1] \cap T$ and take a sequence $(q_k)_k \subset Q$ such that $\lim_{k\to\infty} q_k = u + t(v - u)$ (keep in mind that by (i) of Definition 3.1 we get $[u, v] \subset U \cap \operatorname{dom} f$ and $U \cap \operatorname{cl}\operatorname{dom} f \subset \operatorname{cl} Q$ by the assumptions). By (4.1) there are $q_k^* \in (\partial f(q_k) + \gamma\mathbb{B}_{X^*}) \cap M_\varepsilon(q_k)$. By (iii) of Definition 3.1, for some $w \in ]u + t(v - u), v]$ (with $w$ depending on $t$), we have

$$\limsup_{k\to\infty} \left(\langle q_k^*, w - q_k\rangle - \gamma\|w - q_k\|\right) \leqslant \|w - u - t(v - u)\| \left(\|v - u\|^{-1} f'_{u,v}(t; 1) + \mu\right).$$

Observe that by the $\varepsilon$-hypomonotonicity we get

$$\langle u^*, q_k - u\rangle - \varepsilon\|q_k - u\| \leqslant \langle q_k^*, q_k - u\rangle.$$

We know that there exists $t' \in ]t, 1]$ such that $w = u + t'(v - u)$ and hence (remembering that $(q_k)_k$ converges to $u + t(v - u) \in U \cap \operatorname{cl} Q$ and $M_\varepsilon : U \cap \operatorname{cl} Q \rightrightarrows X^*$ is locally bounded on $U \cap \mathrm{cl}\, Q$)

$$\limsup_{k\to\infty} \langle q_k^*, w - q_k\rangle = (t' - t) \limsup_{k\to\infty} \langle q_k^*, v - u\rangle.$$

We have also

$$\langle u^*, w - u - t(v-u)\rangle = (t'-t)\langle u^*, v-u\rangle = \frac{t'-t}{t} \lim_{k\to\infty} \langle u^*, q_k - u\rangle,$$

so

$$\begin{aligned}
\langle u^*, w - u - t(v-u)\rangle &= \frac{t'-t}{t} \lim_{k\to\infty} \langle u^*, q_k - u\rangle \\
&\leqslant \frac{t'-t}{t} \left( \limsup_{k\to\infty} \left( \langle q_k^*, q_k - u\rangle + \varepsilon \|q_k - u\| \right) \right) \\
&= \frac{t'-t}{t} \left( t \left( \limsup_{k\to\infty} \langle q_k^*, v - u\rangle + \varepsilon \|v-u\| \right) \right) \\
&= (t'-t) \left( \limsup_{k\to\infty} \langle q_k^*, v-u\rangle + \varepsilon \|v-u\| \right) \\
&= \limsup_{k\to\infty} \langle q_k^*, w - q_k\rangle + \varepsilon \|w - u - t(v-u)\|,
\end{aligned}$$

which implies

$$\begin{aligned}
&\langle u^*, w - u - t(v-u)\rangle - (\varepsilon+\gamma)\|w - u - t(v-u)\| \\
&\quad \leqslant \|w-u-t(v-u)\| \left( \|v-u\|^{-1} f'_{u,v}(t;1) + \mu \right).
\end{aligned}$$

By (ii) of Definition 3.1 take $0 = t_0 < \cdots < t_p = 1$ such that the function $f_{u,v}$ is absolutely continuous on each closed interval included in $[0,1] \setminus \{t_0, \cdots, t_p\}$. Fix any positive real $\eta < \frac{1}{2} \min_{1\leqslant i\leqslant p} (t_i - t_{i-1})$. Observing that

$$\frac{\|v-u\|}{\|w-u-t(v-u)\|}(w - u - t(v-u)) = v - u,$$

we see that

$$\begin{aligned}
f_{u,v}(t_i - \eta) - f_{u,v}(t_{i-1}+\eta) &= \int_{t_{i-1}+\eta}^{t_i-\eta} f'_{u,v}(t;1)\, dt \\
&\geqslant \int_{t_{i-1}+\eta}^{t_i-\eta} \big( \langle u^*, v-u\rangle - (\varepsilon+\gamma+\mu)\|v-u\| \big)\, dt \qquad (4.3)\\
&= \big( \langle u^*, v-u\rangle - (\varepsilon+\gamma+\mu)\|v-u\| \big)(t_i - t_{i-1} - 2\eta).
\end{aligned}$$

This yields

$$\sum_{i=1}^{p} \big( f_{u,v}(t_i-\eta) - f_{u,v}(t_{i-1}+\eta) \big) \geqslant \big( \langle u^*, v-u\rangle - (\varepsilon+\gamma+\mu)\|v-u\| \big)(t_p - t_0 - 2p\eta).$$

Using, by (i) of Definition 3.1, the continuity of $f_{u,v}$ on $[0,1]$ and taking the limit as $\eta \downarrow 0$ we obtain

$$f_{u,v}(1) - f_{u,v}(0) \geqslant \big( \langle u^*, v-u\rangle - (\varepsilon+\gamma+\mu)\|v-u\| \big)(t_p - t_0), \qquad (4.4)$$

that is,

$$f(v) - f(u) \geqslant \langle u^*, v-u\rangle - (\varepsilon+\gamma+\mu)\|v-u\|.$$

It then follows from Proposition 2.1 that

$$f(v) - f(u) \geqslant \langle u^*, v-u\rangle - (\mu+\varepsilon+\gamma)\|v-u\| \quad \text{for all } v \in U \cap \mathrm{dom}\, f.$$

Now let us take any $u \in U \cap \operatorname{cl} Q$, $v \neq u$, $v \in U \cap \operatorname{dom} f$, $u^* \in M_\varepsilon(u)$. Observe that the above reasonings ensure

$$[q, v] \subset U \cap \operatorname{dom} f \quad \text{for all } q \in Q,$$

so $[u, v] \subset U \cap \operatorname{cl} \operatorname{dom} f$. Put $\nu := \|v - u\|^{-1}(v - u)$ and fix a sequence of positive numbers $s_k \downarrow 0$ with $s_k \nu + u \in [u, v]$ for all $k$. By the inclusions

$$[u, v] \subset U \cap \operatorname{cl} \operatorname{dom} f \subset \operatorname{cl} Q$$

for each $k$ we can choose $u_k \in Q$ such that $\|s_k \nu + u - u_k\| < s_k^2$. Putting $\nu_k := s_k^{-1}(u_k - u)$, we see that $\nu_k \to \nu$ and hence by what precedes we have, for every $u_k^* \in (\partial f(u_k) + \gamma \mathbb{B}_{X^*}) \cap M_\varepsilon(u_k)$,

$$\langle u^*, u_k - u \rangle - \varepsilon \|u_k - u\| \leqslant \langle u_k^*, u_k - u \rangle$$

and

$$f(v) - f(u_k) \geqslant \langle u_k^*, v - u_k \rangle - (\mu + \varepsilon + \gamma)\|v - u_k\|.$$

Since $\|u_k - u\|^{-1}(u_k - u) \to \|v - u\|^{-1}(v - u)$ and $u_k \to u$, by the boundedness of the sequence $(u_k^*)_k$ and the lower semicontinuity of $f$, we get

$$\begin{aligned} f(v) &\geqslant \liminf_{k\to\infty} \big(f(u_k) + \langle u_k^*, v - u_k \rangle - (\mu + \varepsilon + \gamma)\|v - u_k\|\big) \\ &\geqslant \liminf_{k\to\infty} \big(f(u_k) + \|v - u\| \langle u_k^*, \|u_k - u\|^{-1}(u_k - u) \rangle - (\mu + \varepsilon + \gamma)\|v - u_k\|\big) \\ &\geqslant \liminf_{k\to\infty} \big(f(u_k) + \|v - u\| \langle u^*, \|u_k - u\|^{-1}(u_k - u) \rangle - (\mu + 2\varepsilon + \gamma)\|v - u_k\|\big) \\ &\geqslant f(u) + \langle u^*, v - u \rangle - (\mu + 2\varepsilon + \gamma)\|v - u\|. \end{aligned}$$

Therefore, since $u \in U \cap \operatorname{cl} Q$ and $v \in \operatorname{dom} f$, we get $U \cap \operatorname{cl} Q \subset U \cap \operatorname{dom} f$ and

$$f(v) - f(u) \geqslant \langle u^*, v - u \rangle - (\mu + 2\varepsilon + \gamma)\|v - u\| \text{ for all } v \in U, u \in U \cap \operatorname{cl} Q. \tag{4.5}$$

In order to finish the proof let us assume that $g : U \to \mathbb{R}$ is a convex continuous function such that

$$(\partial f(q) + \gamma \mathbb{B}_{X^*}) \cap \partial g(q) \neq \emptyset \text{ for every } q \in Q.$$

For every $u \in U$ let us put $M_0(u) := \partial g(u)$. It follows from (4.5) that

$$f(v) - f(u) \geqslant \langle u^*, v - u \rangle - (\mu + \gamma)\|v - u\| \text{ for all } v, u \in U, u^* \in M_0(u),$$

and so, at every point $u \in U$ and every direction $h$ such that $f'(u; h)$ exists, we get

$$f'(u; h) \geqslant g'(u; h) - (\mu + \gamma)\|h\|,$$

since $g'(u; h) = \max_{u^* \in \partial g(u)} \langle u^*, h \rangle$ according to the continuity and convexity of $g$ on $U$.

Fix any $u \in U \cap \operatorname{Dom} \partial f$ and $v \in U \cap \operatorname{Dom} \partial f \setminus \{u\}$ and by (ii) of Definition 3.1 take $0 = t_0 < \cdots < t_p = 1$ such that $f_{u,v}$ is absolutely continuous on each closed interval included in $[0, 1] \setminus \{t_0, \cdots, t_p\}$. Then, taking the local Lipschitz continuity of $g$ into account, the functions $f_{u,v}$, $g_{u,v}$, $(f - g)_{u,v}$ are absolutely continuous on each closed interval included in $[0, 1] \setminus \{t_0, \cdots, t_p\}$. Proceeding as for (4.3) and (4.4) with $f - g$ in place of $f$ and $\varepsilon = 0$, we obtain for all $u, v \in U \cap \operatorname{Dom} \partial f$

$$\begin{aligned} &(f - g)(v) - (f - g)(u) \geqslant -(\mu + \gamma)\|v - u\|, \text{ that is,} \\ &f(v) - f(u) + (\mu + \gamma)\|v - u\| \geqslant g(v) - g(u). \end{aligned}$$

Using Proposition 2.1 we see that the latter inequality holds for all $u, v \in U$, which completes the proof of the theorem. □

An immediate consequence of the theorem is that any $\partial$-eds function $f$ on $U$ relative to $\operatorname{Dom} \partial f$ with a locally bounded monotone selection has to be convex, whenever $\gamma = 0$.

COROLLARY 4.1. *Let $X$ be a Banach space, $U$ be a nonempty open convex subset of $X$, $\partial$ be a presubdifferential and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a $\partial$-eds function on $U$ relative to* $\operatorname{Dom} \partial f$. *Let $Q \subset U$ be given such that $\operatorname{cl} Q$ is a convex set and $U \cap \operatorname{cl} \operatorname{dom} f \subset \operatorname{cl} Q$. Assume that there is a set-valued mapping $M : U \cap \operatorname{cl} Q \rightrightarrows X^*$ that is monotone (i.e., it is $\varepsilon$-hypomonotone with $\varepsilon = 0$) and locally bounded on $U \cap \operatorname{cl} Q$ and such that*

$$\partial f(q) \cap M(q) \neq \emptyset \text{ for every } q \in Q.$$

*Then $f$ is finite and convex on $U \cap \operatorname{cl} Q$.*

*Proof.* It follows from the preceding theorem that $f$ is finite on $U \cap \operatorname{cl} Q$ and

$$f(v) - f(u) \geqslant \langle u^*, v - u \rangle \text{ for all } u, v \in U \cap \operatorname{cl} Q, u^* \in M(u).$$

Let us fix any $u, v \in U \cap \operatorname{cl} Q$, $t \in [0, 1]$ and $u_t^* \in M(u + t(v - u))$. Observe that we have

$$f(v) - f(u + t(v - u)) \geqslant \langle u_t^*, v - u - t(v - u) \rangle$$

and

$$f(u) - f(u + t(v - u)) \geqslant \langle u_t^*, u - u - t(v - u) \rangle,$$

which implies

$$tf(v) + (1 - t)f(u) \geqslant f(u + t(v - u)).$$ □

It appears that the main advantages of conditions (4.1) and (4.2), when compared with previous ones, are that it is enough to check whether the intersection is nonempty (we do not have to check the inclusions, which is much more difficult to verify). It is in fact enough to check the nonvacuity of this intersection on some dense subset of the interior of the domain, which is more useful in the case where we have some information on the presubdifferential on some generic set, for example, in Asplund spaces. In Proposition 4.2 (see also Remark 4.3) an example showing that checking (4.2) on some dense subset is easier than on $U$ is given in the case of integration of $DC$ functions. Moreover, in some cases we have that if (4.2) is satisfied for some dense subset, then it is also satisfied for $U$. In order to observe this, let us recall, following [**26**, D. page 150], the notion of bounded weak* topology, bw* in short. Namely, the topology is defined on the topological dual space of $X$ by defining bw* closed sets: *a subset $G$ of $X^*$ is bw* closed if and only if its intersection with every weak*-compact set is again weak*-compact* or equivalently every bounded weak* converging net in $G$ has its limit in $G$; see [**30**] for the definition and properties of nets. If the graph of $\partial f$ is $\|\cdot\| \times bw^*$-closed, then (4.2) holds for some dense subset $Q$ of $U$ if and only if it holds for $U$. Indeed, fix any $x \in U$ and choose a sequence $(q_k)_{k \in \mathbb{N}}$ of $Q$ converging to $x$, where $Q$ is a dense subset of $U$ such that (4.2) holds true. By (4.2) choose $q_k^* \in \partial f(q_k)$ and $b_k^* \in \gamma \mathbb{B}_{X^*}$ such that $q_k^* + b_k^* \in \partial g(q_k)$. Since $\partial g$ is locally bounded ($g$ is locally Lipschitz) the sequences $(q_k^*)_k$ and $(b_k^*)_k$ admit subnets $(q_{s(j)}^*)_{j \in J}$ and $(b_{s(j)}^*)_{j \in J}$ weak*-converging (and hence also $bw^*$-converging, because of the boundedness) to some $q^*$ and $b^*$, respectively. Put

$$p_j^* := q_{s(j)}^*, \quad d_j^* := b_{s(j)}^*, \quad \text{and} \quad p_j := q_{s(j)}.$$

By the assumption above on $\partial f$ we have $q^* \in \partial f(x)$. On the other hand the convexity of $g$ ensures that

$$(q^* + b^*) \in \partial g(x).$$

Consequently,

$$(q^* + b^*) \in (\partial f(x) + \gamma \mathbb{B}) \cap \partial g(x),$$

which entails the desired equivalence. Let us also observe that if $M$ is a maximal monotone set-valued mapping then a similar reasoning ensures that if (4.1) is satisfied for some dense subset then it is also satisfied for $U$.

REMARK 4.1. The last conclusion of Theorem 4.1 ensures in particular

$$f(tu+(1-t)v) \leqslant tf(u)+(1-t)f(v)+2(\gamma+\mu)t(1-t)\|v-u\|,$$

for all $u, v \in U$ and $t \in ]0,1[$, that is, $f$ is strongly 1-paraconvex on $U$; see the last section for the definition of strongly 1−paraconvex functions. Indeed, according to the conclusion of Theorem 4.1 and to the convexity of $g$, for $\rho := \gamma+\mu$ we have, for all $t \in ]0,1[$ and $x, y \in U$,

$$\begin{aligned}
t[f(u+(1-t)(v-u)) - f(u)] &+ (1-t)[f(u+(1-t)(v-u)) - f(v)] \\
&\leqslant \rho t(1-t)\|v-u\| + t[g(u+(1-t)(v-u)) - g(u)] + \rho t(1-t)\|v-u\| \\
&\quad + (1-t)[g(u+(1-t)(v-u)) - g(v)] \\
&= 2\rho t(1-t)\|v-u\| + [g(u+(1-t)(v-u)) - tg(u) - (1-t)g(v)] \\
&\leqslant 2\rho t(1-t)\|v-u\|,
\end{aligned}$$

and hence

$$f(tu+(1-t)v) \leqslant tf(u)+(1-t)f(v)+2(\gamma+\mu)t(1-t)\|v-u\|.$$

This proves the strong 1-paraconvexity of $f$ on $U$.

REMARK 4.2. Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be an lsc proper convex function on $U$ and $\varepsilon \geqslant 0$ be given (in the reasoning below, $\varepsilon$ under consideration is small). Put $S(g) := \operatorname{gph} \partial g = \{(x, x^*) \in X \times X^* : x^* \in \partial g(x)\}$, $S_\varepsilon(g) := \{(w, w^*) \in X \times X^* : \forall (x, x^*) \in S(g), \langle x^* - w^*, x - w\rangle \geqslant -\varepsilon\}$ and $T_\varepsilon(g) := \{(w, w^*) \in X \times X^* : \forall (x, x^*) \in S(g), \langle x^* - w^*, x - w\rangle \geqslant -\varepsilon\|x-w\|\}$. It follows from [**61**, Property 3.3] that every pair from $S_\varepsilon(g)$ is not far from $S(g)$, that is,

$$S_\varepsilon(g) \subset (S(g) + (8\sqrt{\varepsilon}\mathbb{B}_{X^*} \times 8\sqrt{\varepsilon}\mathbb{B}_{X^*})).$$

It seems that similar results between $T_\varepsilon(g)$ and $S(g)$, more related to (4.2), should also be valid, so it is natural to ask the following questions:

(1) presubdifferentials of which functions $f$ have the property that there is a convex function $g$ such that $\operatorname{gph} \partial f \subset S_\varepsilon(g)$ for a given $\varepsilon \geqslant 0$?
(2) presubdifferentials of which functions $f$ have the property that there is a convex function $g$ such that $\operatorname{gph} \partial f \subset T_\varepsilon(g)$ for a given $\varepsilon \geqslant 0$?

In the one-dimensional setting an answer to question (2) is given in [**38**, Theorem 5, p. 250]. Also Theorem 4.1 can be employed to get some answers to these questions in one-dimension. For example, let $p : \mathbb{R} \to \mathbb{R}$ be a function and $\partial$ be a presubdifferential, $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a $\partial$-eds function on some open interval $U$ relative to $\operatorname{Dom} \partial f$ such that for a dense subset $Q$ of $U$, we have

$$p(q) \in \partial f(q) \text{ for all } q \in Q$$

and

$$\sup_{s \in Q \cap ]-\infty, t]} p(s) < +\infty \text{ for all } t \in U.$$

Putting

$$\sigma(t) := \sup_{s \in Q \cap ]-\infty, t]} p(s),$$

we know (see [47, 12.26 Exercise]) that there is a proper lsc convex function $g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ such that $\sigma(q) \in \partial g(q)$ for every $q \in Q$ (with the convention $\sup \emptyset = -\infty$) and thus, by Theorem 4.1, we get

$$| f(v) - f(u) - (g(v) - g(u)) | \leqslant \sup_{q \in Q \cap \operatorname{Dom} \partial f} d(\sigma(q), \partial f(q)) \, | v - u | \quad \text{for all } u, v \in \operatorname{dom} f \cap U.$$

If the function $f$ is not $\partial$-eds on $U$ relative to $\operatorname{Dom} \partial f$, then there are approximation techniques for several classes of functions, for example, Moreau envelopes, Weierstrass approximation theorem and the above method can be applied to an approximation of $f$.

In general, questions (1) and (2) are open. Answers to them should give us more information how far the functions satisfying one of the inclusions from (1) or (2) are from convex functions. These questions can be extended to larger classes, for example, to DC class; see Proposition 4.2.

We must also add that we do not know results concerning the inclusion

$$T_\varepsilon(g) \subset S(g) + r(\varepsilon)\mathbb{B}_{X^*} \times r(\varepsilon)\mathbb{B}_{X^*}$$

where $r(\cdot)$ is a nonnegative continuous function on $[0, +\infty[$ with $r(0) = 0$, so in our opinion it is an open problem to have a good relation between $T_\varepsilon(g)$ and $S(g)$ in the sense above.

The above integration property can be expanded to the DC function. In view of this, we need the following property, which is an immediate consequence of [55, Corollary 4.10].

Proposition 4.1. *Let $X$ be a Banach space, $U$ be a nonempty open convex subset of $X$ and $f : U \to \mathbb{R} \cup \{+\infty\}$ be a $\partial, \mu$-eds function on $U$ relative to some set $D$ with* $\operatorname{Dom} \partial f \subset D \subset \operatorname{dom} f$, *where $\partial$ is a subdifferential with the exact inclusion sum rule and included in the Clarke subdifferential. Also let $g_2 : U \to \mathbb{R}$ be a continuous convex function. Then the function $f + g_2$ is $\partial, \mu$-eds on $U$ relative to $D$.*

As a simple consequence of Theorem 4.1 and Proposition 4.1 we have the following proposition.

Proposition 4.2. *Let $X$ be a Banach space, $U$ be a nonempty open convex subset of $X$, $\gamma \geqslant 0$ be given, $\partial$ be a subdifferential with the exact inclusion sum rule and included in the Clarke subdifferential, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a $\partial$-eds function on $U$ relative to* $\operatorname{Dom} \partial f$ *and $g : X \to \mathbb{R}$ be DC and continuous on $U$, that is, there are two continuous convex functions $g_1, g_2 : U \to \mathbb{R}$ such that $g \equiv g_1 - g_2$ on $U$. Assume that, for some dense subset $Q$ of $U$, we have*

$$(\partial f(q) + \gamma \mathbb{B}_{X^*}) \cap \partial g(q) \neq \emptyset \text{ and } \partial(f + g_2)(q) = \partial f(q) + \partial g_2(q) \text{ for all } q \in Q. \tag{4.6}$$

*Then $f$ is finite on $U$ and*

$$f(v) - f(u) + \gamma\|v - u\| \geqslant g(v) - g(u) \geqslant f(v) - f(u) - \gamma\|v - u\| \text{ for all } u, v \in U.$$

*Proof.* Observe that, by Proposition 4.1, the function $f + g_2$ is $\partial$-eds on $U$ relative to $\operatorname{Dom} \partial(f + g_2)$. Assumption (4.6) and the inclusion of $\partial$ in $\partial_C$ ensure that $\big(\partial(f + g_2)(q) + \gamma \mathbb{B}_{X^*}\big) \cap \partial g_1(q) \neq \emptyset$ for all $q \in Q$, and hence (4.2) is satisfied for $f + g_2$ and $g_1$ instead of $f$

and $g$, respectively. Theorem 4.1 can be applied to these functions. Hence, $f$ is finite on $U$ and

$$\begin{aligned} & f(v) - f(u) + g_2(v) - g_2(u) + \gamma\|v-u\| \\ & \quad \geqslant g_1(v) - g_1(u) \\ & \quad \geqslant f(v) - f(u) + g_2(v) - g_2(u) - \gamma\|v-u\| \quad \text{for all } u, v \in U, \end{aligned}$$

which implies the statement. □

REMARK 4.3. Let us note that if $\partial$ is a subdifferential with the exact inclusion sum rule such that $\partial f$ contains the Fréchet subdifferential and is included in the Clarke subdifferential, then by Proposition 4.4 recalled below, one sees that

$$\partial f(u) = \partial(f + g_2 - g_2)(u) \subset \partial(f+g_2)(u) + D_F(-g_2)(u) \subset \partial f(u) + D_F g_2(u) - D_F g_2(u),$$

so

$$\partial(f+g_2)(u) = \partial f(u) + \partial g_2(u),$$

at any $u \in U$ where $g_2$ is Fréchet differentiable.

Whenever $g$ is a continuous linear functional, then Theorem 4.1 yields the following proposition.

PROPOSITION 4.3. *Let $U$ be a nonempty open convex subset of a Banach space $X$, $Q$ be a dense subset of $U$, $\partial$ be a presubdifferential, and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a $\partial, \mu$-eds function on $U$ relative to* $\operatorname{Dom}\partial f$. *Let $r, \gamma$ be nonnegative numbers and $a^* \in X^*$ such that for all $x \in Q$*

$$(a^* + r\mathbb{B}_{X^*}) \cap (\partial f(x) + \gamma\mathbb{B}_{X^*}) \neq \emptyset. \tag{4.7}$$

*Then $f$ is finite on $U$ and*

$$|f(x) - f(y) - \langle a^*, x-y\rangle| \leqslant (r+\gamma+\mu)\|x-y\| \quad \text{for all } x, y \in U.$$

*Further, the function $f$ is Lipschitz continuous on $U$ with $(\|a^*\| + \gamma + r + \mu)$ as a Lipschitz constant on $U$.*

*Proof.* Let us put $g(x) := \langle a^*, x\rangle$ for every $x \in X$, $\gamma' := r + \gamma$ and observe that condition (4.2) is satisfied with $\gamma'$ instead of $\gamma$. Hence, the statement of the proposition follows from Theorem 4.1. □

Proposition 4.3 can be used to give sufficient conditions for the Fréchet differentiability at a given point.

COROLLARY 4.2. *Let $U$ be a nonempty open convex subset of a Banach space $X$, $x \in U$ be given, $Q$ be a dense subset of $U$, $\partial$ be a presubdifferential, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a $\partial$-eds function on $U$ relative to* $\operatorname{Dom}\partial f$ *and $\sigma : Q \cup \{x\} \to X^*$ be a continuous mapping relative to $Q \cup \{x\}$ (the continuity means that $\sigma$ is $\tau_{Q\cup\{x\}}$-to-norm continuous at $x$, where $\tau_{Q\cup\{x\}}$ is the induced topology from the strong topology of $X$). Assume that we have*

$$\sigma(q) \in \partial f(q) \text{ for every } q \in Q. \tag{4.8}$$

*Then $f$ is strictly Fréchet differentiable at $x$ with $D_F f(x) = \sigma(x)$.*

*Proof.* Let us put $x^* = \sigma(x)$ and apply Proposition 4.3. For this reason let us fix $\varepsilon > 0$ and take $\delta > 0$ such that $\|\sigma(q) - \sigma(x)\| \leqslant \varepsilon$ for every $q \in Q \cap B(x, \delta)$. It is easy to observe that (4.7) is satisfied with $r := 0$, $\gamma := \varepsilon$, $a^* = \sigma(x)$, $U := B(x, \delta)$ in place of $U$, and $Q' := Q \cap B(x, \delta)$ in place of $Q$. This implies that $f$ is finite on $B(x, \delta)$ and

$$|f(v) - f(u) - \langle x^*, v - u\rangle| \leqslant \varepsilon\|v - u\| \quad \text{for all } u, v \in B(x, \delta).$$

Since we can repeat this reasoning for arbitrary $\varepsilon > 0$, the above inequality yields the strict Fréchet differentiability of $f$ at $x$. □

Before stating the second corollary let us recall that every lsc proper convex function on a Banach space has the property that if it is Fréchet differentiable at a point from the interior of its domain then it is strictly Fréchet differentiable at this point; namely we have the following proposition.

PROPOSITION 4.4. *Let $X$ be a Banach space and $g : X \to \mathbb{R} \cup \{+\infty\}$ be an lsc convex function that is Fréchet differentiable at $x \in \operatorname{int} \operatorname{dom} g$. Then for every $\epsilon > 0$ there is $\delta > 0$ such that*

$$\partial g(y) \subset (D_F g(x) + \varepsilon \mathbb{B}_{X^*}) \text{ for every } y \in B(x, \delta). \tag{4.9}$$

A proof of Proposition 4.4 can be found in [**23**, p. 147, Lemma].

Let us point out that if $f$ is Fréchet differentiable at $x$, then, by (4.9) any selection $\sigma$ of the subdifferential of $f$ satisfying (4.8) is continuous at $x$ relative to $Q \cup \{x\}$. Hence, if we know that $x \in \operatorname{int} \operatorname{dom} f$, then the continuity at $x$ of the selection relative to $Q \cup \{x\}$ is necessary and sufficient for the Fréchet differentiability of $f$ at $x$. The result when compared with [**1**, Corollary 2, p. 460] needs to check the continuity of the subdifferential at $x$ relative to a dense subset, which is less demanding. We can now state the second corollary. It is a simple consequence of Corollary 4.2.

COROLLARY 4.3. *Let $U$ be a nonempty open convex subset of a Banach space $X$, $Q$ be a dense subset of $U$, $\partial$ be a presubdifferential and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a $\partial$-eds function on $U$ relative to $\operatorname{Dom} \partial f$. Assume that $g : X \to \mathbb{R}$ is a convex continuous function on $U$ that is Fréchet differentiable at $x \in U$ and such that*

$$\partial f(q) \cap \partial g(q) \neq \emptyset \text{ for every } q \in Q. \tag{4.10}$$

*Then $f$ is strictly Fréchet differentiable at $x$.*

*Proof.* Let us put $\sigma(x) := D_F g(x)$. By (4.10), for every $q \in Q$ we can choose a point $\sigma(q) \in X^*$ such that $\sigma(q) \in \partial f(q) \cap \partial g(q)$. By (4.9) the selection $\sigma$ is continuous at $x$ relative to $Q \cup \{x\}$. Now we can apply Corollary 4.2 to get the strict Fréchet differentiability of $f$ at $x$. □

As said above, we can derive from Proposition 4.3 the following corollary for the Lipschitz property of an eds function.

COROLLARY 4.4. *Let $U$ be a nonempty open convex subset of $X$ and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a $\partial$-eds function on $U$ relative to $\operatorname{Dom} \partial f$. Then $f$ is Lipschitz continuous on $U$ with $\gamma \geqslant 0$ as a Lipschitz constant whenever there is a dense subset $Q$ of $U$ such that*

$$0 \in \partial f(q) + \gamma \mathbb{B}_{X^*} \quad \text{for all } q \in Q,$$

*that is, the set-valued mapping $\partial f$ admits a selection $\sigma(\cdot)$ on $Q$ such that $\|\sigma(q)\| \leqslant \gamma$ for all $q \in Q$.*

*Further, if $\partial$ satisfies* (P4') *instead of* (P4)*, then the condition is also necessary for the Lipschitz property of $f$ on $U$.*

*Proof.* Applying Proposition 4.3 above with $a^* = 0$ and $r = 0$, we see that the condition is sufficient. Suppose now that $\partial$ satisfies (P4') instead of (P4) and that $f$ is finite and Lipschitz continuous on $U$ with $\gamma$ as a Lipschitz constant. Then, for any fixed $a \in U$, the point $a$ is a minimum on $U$ of the function $f(\cdot) + \gamma\| \cdot - a\|$, and hence by (P4') we have

$$0 \in \partial(f + \gamma\| \cdot - a\|)(a) \subset \partial f(a) + \partial(\gamma\| \cdot - a\|)(a).$$

The function $\gamma\| \cdot - a\|$ being convex continuous, (P3) entails $\partial(\gamma\| \cdot - a\|)(a) = \gamma \mathbb{B}_{X^*}$. This completes the proof. □

Theorem 4.1 gives us possibilities to investigate properties of eds functions through selections of their presubdifferentials. In Corollary 4.2 it is shown that all points of continuity of a selection form a set where the function is strictly differentiable. An obvious consequence of the corollary above is that boundedness of a selection yields Lipschitzness of the function.

The next theorem shows that additional properties of the selection $\sigma$ yield much more regularity for the function $f$. This theorem involves, for the function $f$, the $C^{1,\omega(\cdot)}$ property defined as follows.

Definition 4.1. Let

$$\mathcal{M} := \{\omega : [0, +\infty[ \to [0, +\infty[ : \omega(0) = 0, \text{ and } \omega \text{ continuous at } 0\}. \tag{4.11}$$

For $\omega(\cdot) \in \mathcal{M}$ one says that a mapping $G : U \to Y$ from an open set $U$ of $X$ into a normed space $Y$ is $\mathcal{C}^{1,\omega(\cdot)}$ on $U$ when $G$ is Fréchet differentiable on $U$ and, for each point $a \in U$, there exist some neighbourhood $U' \subset U$ of $a$ and some real $M \geqslant 0$ such that

$$\|D_F G(x) - D_F G(y)\| \leqslant M\omega(\|x - y\|) \quad \text{for all } x, y \in U'. \tag{4.12}$$

(Obviously one obtains an equivalent definition with the Gâteaux differentiability and Gâteaux derivative of $G$). When, for some constant $\alpha > 0$, one has $\omega(t) = t^\alpha$, one just says that $G$ is $C^{1,\alpha}$ instead of $C^{1,\omega(\cdot)}$ with $\omega(t) = t^\alpha$.

In the case $\omega(t) = t^\alpha$, inequality (4.12) means that $D_F G$ is locally Hölder continuous on $U$ with power $\alpha$. So the $\mathcal{C}^{1,1}$ property of $G$ on $U$ corresponds to the local Lipschitz continuity of $D_F G$ on $U$. Let us also point out that if $\alpha > 1$ and $U$ is open and convex, then (4.12) implies that $D_F G$ is constant on $U$, so the most interesting case is when $\alpha \in ]0, 1]$.

Theorem 4.2. *Let $U$ be a nonempty open convex subset of $X$, $Q$ be a dense subset of $U$ and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a $\partial$-eds function on $U$ relative to* Dom $\partial f$*. Assume that there exists some selection $\sigma(\cdot)$ of the set-valued mapping $\partial f$ on $Q$ such that*

$$\sigma(y) \in \partial f(x) + \rho(x, y)\mathbb{B}_{X^*} \quad \textit{for all } x, y \in Q, \tag{4.13}$$

*where $\rho : Q \times Q \to [0, +\infty[$ is a function such that $\rho(x, x) = 0$ and $\rho(\cdot, x)$ is continuous at $x$ relative to $Q$ for any $x \in Q$. Then the following hold.*

(a) *The function $f$ is strictly Fréchet differentiable at every $q \in Q$ with $\sigma(q) = D_F f(q)$ for all $q \in Q$.*

(b) *If $\partial f$ is included in the Clarke subdifferential, then $\partial f$ is reduced to the Fréchet derivative of $f$ on $Q$, that is, $\partial f(q) = \{D_F f(q)\}$ for all $q \in Q$.*
*Moreover, if we additionally assume that we are able to extend $\rho$ to $U \times U$ in such a way that it satisfies: (1) $\rho : U \times U \to [0, +\infty[$, (2) $\rho(x, x) = 0$, (3) $\rho(\cdot, \cdot)$ is continuous at $(x, x) \in U \times U$ for any $x \in U$, then we have the following properties whenever $\partial f$ is included in the Clarke subdifferential.*

(c) *If inclusion (4.13) holds, then the function $f$ is of class $\mathcal{C}^1$ on $U$ and $D_F f(x) = \lim_{Q \ni y \to x} \sigma(y)$ for all $x \in U$.*

(d) *If inclusion (4.13) holds on $Q = U$ for $\rho(x, y) = \omega(\|x - y\|)$ with a function $\omega(\cdot) \in \mathcal{M}$, then the function $f$ is $\mathcal{C}^{1,\omega(\cdot)}$ on $U$. The same conclusion holds when $Q \neq U$ but then $\omega(\cdot) \in \mathcal{M}$ is additionally assumed to be upper semicontinuous. (When $\omega(t) = ct^\alpha$ with real constants $c \geqslant 0$ and $\alpha > 1$, this means that $f$ is a continuous affine function on $U$).*

*Proof.* (a) Fix any $a \in Q$ and any real number $\eta > 0$. Choose some real $\delta > 0$ such that $B(a, \delta) \subset U$ and

$$\rho(x, a) < \eta \quad \text{for all } x \in B(a, \delta) \cap Q.$$

By (4.13) we have

$$\sigma(a) \in \partial f(x) + \eta \mathbb{B}_{X^*} \quad \text{for all } x \in B(a, \delta) \cap Q.$$

Applying Proposition 4.3 with $r = 0$ and $a^* = \sigma(a)$ we have

$$|f(x) - f(y) - \langle \sigma(a), x - y \rangle| \leqslant \eta \|x - y\| \text{ for all } x, y \in B(a, \delta). \tag{4.14}$$

This ensures that the function $f$ is strictly Fréchet differentiable at $a$ and $D_F f(a) = \sigma(a)$.

(b) Under the assumption of the inclusion of $\partial f$ in the Clarke subdifferential of $f$, according to $(a)$, for all $x \in Q$, we have $D_F f(x) \in \partial f(x) \subset \partial_C f(x)$. Further, by (4.14) the function $f$ is locally Lipschitz continuous on $U$, which by the strict Fréchet differentiability ensures that $\partial_C f(x) = \{D_F f(x)\}$. We then deduce that $\partial f(x) = \{D_F f(x)\}$.

(c) Let us fix any $x \in U$. It follows from Proposition 4.3 that for any $\delta > 0$ such that $B(x, \delta) \subset U$, we have

$$|f(z) - f(y) - \langle \sigma(q), z - y \rangle| \leqslant \sup_{B(x,\delta) \times B(x,\delta)} \rho(\cdot, \cdot) \|z - y\| \tag{4.15}$$

for all $y, z \in B(x, \delta)$ and $q \in B(x, \delta) \cap Q$. The function $f$ being locally Lipschitz continuous on $U$ (as seen in the proof of (b) above), for $\delta > 0$ sufficiently small we infer from (4.15) and from the definition of $f^0(x; \cdot)$ that, for every $x^* \in \partial_C f(x)$

$$\|x^* - \sigma(q)\| \leqslant \sup_{B(x,\delta) \times B(x,\delta)} \rho(\cdot, \cdot) \text{ for every } q \in B(x, \delta) \cap Q,$$

which ensures the existence of the limit $\lim_{Q \ni q \to x} \sigma(q) =: \tau(x)$ and

$$\partial_C f(x) = \{\tau(x)\}.$$

Using (4.15) again we obtain

$$|f(z) - f(y) - \langle \tau(x), z - y \rangle| \leqslant \sup_{B(x,\delta) \times B(x,\delta)} \rho(\cdot, \cdot) \|z - y\|$$

for all $y, z \in B(x, \delta)$, and hence $f$ is strictly Fréchet differentiable at $x$ with $D_F f(x) = \tau(x)$.

(d) Suppose that $\rho(x, y) = \omega(\|x - y\|)$. If $Q = U$, then, by (4.13) and $(a)$, we have $D_F f(y) \in D_F f(x) + \omega(\|x - y\|) \mathbb{B}_{X^*}$ for all $x, y \in U$, hence $f$ is $C^{1,\omega(\cdot)}$ on $U$.

Suppose now that $Q$ is dense in $U$ with $Q \neq U$. Using $(c)$ we obtain $D_F f(x) = \lim_{Q \ni q \to x} \sigma(q)$ for every $x \in U$. By (4.13) and (a) we have, for all $q, q' \in Q$,

$$\sigma(q') \in \sigma(q) + \omega(\|q' - q\|) \mathbb{B}_{X^*}, \quad \text{that is, } \|\sigma(q') - \sigma(q)\| \leqslant \omega(\|q' - q\|),$$

which, according to the upper semicontinuity of $\omega$ and to the equality $D_F f(u) = \lim_{Q\ni q\to u} \sigma(q)$ for all $u \in U$, implies

$$\|D_F f(y) - D_F f(x)\| \leqslant \omega(\|y - x\|) \quad \text{for all } x, y \in U.$$

This means that $f$ is of class $C^{1,\omega(\cdot)}$ on $U$. The proof is then completed. □

We have the following first corollary.

Corollary 4.5. *Let $U$ be a nonempty open convex subset of $X$, $Q$ be a dense subset of $U$ and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a $\partial$-eds function on $U$ relative to* $\operatorname{Dom}\partial f$*. The following assertions hold.*

(a) *If the set-valued mapping $\partial f$ admits a selection on $U$ that is continuous, then $f$ is of class $\mathcal{C}^1$ on $U$.*

(b) *If $\partial f(\cdot) \subset \partial_C f(\cdot)$ and $\partial f$ admits a selection on $U$ that is locally Lipschitz continuous on $U$, then $f$ is of class $\mathcal{C}^{1,1}$ on $U$.*

(c) *If the set-valued mapping $\partial f$ admits a selection on $Q$ that is continuous relative to $Q$, then $f$ is strictly Fréchet differentiable at every point $q \in Q$ with $D_F f(q) = \sigma(q)$, so it is of class $\mathcal{C}^1$ relative to $Q$.*

*Moreover, assume that $\partial f$ is included in the Clarke subdifferential; then the following assertions hold.*

(d) *If the set-valued mapping $\partial f$ admits a selection on $Q$ that is also a continuous mapping on $U$, then $f$ is strictly Fréchet differentiable at every point $x \in U$ with $D_F f(x) = \sigma(x)$, so it is of class $\mathcal{C}^1$ on $U$.*

(e) *If the set-valued mapping $\partial f$ admits a selection on $Q$ that is also locally Lipschitz continuous on $U$, then $f$ is strictly Fréchet differentiable at every point $x \in U$ with $D_F f(x) = \sigma(x)$ and it is of class $\mathcal{C}^{1,1}$ on $U$.*

*Proof.* (a) Denote by $\sigma(\cdot)$ a selection of $\partial f$ on $U$ that is continuous. Putting $\rho(x, y) := \|\sigma(x) - \sigma(y)\|$ for all $x, y \in U$, we see that $\rho$ is continuous on $U \times U$ and $\rho(x, x) = 0$ for all $x \in U$. Since

$$\sigma(y) \in \partial f(x) + \|\sigma(x) - \sigma(y)\|\mathbb{B}_{X^*} \quad \text{for all } x, y \in U,$$

assertion $(a)$ of the corollary is a consequence of assertion $(a)$ of Theorem 4.2 (with $Q = U$).

(b) Let $\sigma(\cdot)$ be a selection of $\partial f$ which is locally Lipschitz continuous on $U$. Assertions (a) and (b) of Theorem 4.2 with $Q = U$ give that $f$ is differentiable on $U$ with $D_F f(x) = \sigma(x)$ for all $x \in U$. So, $f$ is of class $\mathcal{C}^{1,1}$ on $U$.

(c) Denote by $\sigma(\cdot) : Q \to X^*$ a continuous selection of the set-valued mapping $\partial f$ on $Q$, that is,

$$\sigma(q) \in \partial f(q) \text{ for every } q \in Q.$$

It follows from Corollary 4.2 that, for every $q \in Q$, the function $f$ is strictly Fréchet differentiable at $q$ with $D_F f(q) = \sigma(q)$, which gives the statement.

(d) and (e). Put $\rho(x, y) := \|\sigma(x) - \sigma(y)\|$ for every $x, y \in U$. Observe that (d) and (e) are consequences of (b), (c) and (d) of Theorem 4.2. □

Another corollary involves a truncation of the presubdifferential of $f$.

Corollary 4.6. *Let $U$ be a nonempty open convex subset of $X$ and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a $\partial$-eds function on $U$ relative to* $\operatorname{Dom}\partial f$*. Let $(a, a^*) \in \operatorname{gph}\partial f$ with $a \in U$. Assume that* *there exists* $r > 0$ *such that*

$$\partial f(y) \cap (a^* + r\mathbb{B}_{X^*}) \subset \partial f(x) + \rho(x,y)\mathbb{B}_{X^*} \quad \text{for all } x, y \in U, \tag{4.16}$$

*where* $\rho : U \times U \to [0, +\infty[$ *is a function such that* $\rho(x,x) = 0$ *and* $\rho(\cdot, x)$ *is continuous at* $x$ *for each* $x \in U$*. Then there exists a convex neighbourhood* $U'$ *of* $a$ *for which the following assertions hold.*

(a) *The function* $f$ *is of class* $\mathcal{C}^1$ *on the neighbourhood* $U'$ *of* $a$ *and* $D_F f(x) \in \partial f(x)$ *for all* $x \in U'$*.*

(b) *If* $\partial f$ *is included in the Clarke subdifferential of* $f$*, then one has in addition* $\partial f(x) = \{D_F f(x)\}$ *for all* $x \in U'$*.*

(c) *If* $\partial f$ *is included in the Clarke subdifferential and* (4.16) *holds for* $\rho(x,y) = \omega(\|x-y\|)$ *with a function* $\omega(\cdot) \in \mathcal{M}$*, then* $f$ *is of class* $\mathcal{C}^{1,\omega(\cdot)}$ *on the neighbourhood* $U'$ *of the point* $a$ *(when* $\omega(t) = ct^\alpha$ *with real constants* $c \geqslant 0$ *and* $\alpha > 1$ *it means that* $f$ *is a continuous affine function on* $U'$*).*

*Proof.* (a) By the continuity of the function $\rho(\cdot, a)$ at the point $a$, choose an open convex neighbourhood $U' \subset U$ of $a$ such that $\rho(x,a) < r$ for all $x \in U'$. Observing that $a^* \in \partial f(a) \cap (a^* + r\mathbb{B}_{X^*})$, assumption (4.16) gives that

$$a^* \in \partial f(x) + \rho(x,a)\mathbb{B}_{X^*} \quad \text{for all } x \in U'.$$

For each $x \in U'$, combining the latter inclusion with the inequality $\rho(x,a) < r$, we see that $\partial f(x) \cap (a^* + r\mathbb{B}_{X^*}) \neq \emptyset$. This allows us to choose a mapping $\sigma : U' \to X^*$ with $\sigma(y) \in \partial f(y) \cap (a^* + r\mathbb{B}_{X^*})$ for all $y \in U'$. The mapping $\sigma(\cdot)$ is then a selection of $\partial f$ on $U'$ and, by assumption (4.16), we have

$$\sigma(y) \in \partial f(x) + \rho(x,y)\mathbb{B}_{X^*} \quad \text{for all } x, y \in U'.$$

Applying assertion (a) of Theorem 4.2, we deduce assertion (a) of the corollary.

Assertions (b) and (c) of the corollary follow in a similar way. □

The next corollary involves the Aubin Lipschitz-like property of the presubdifferential.

Definition 4.2. According to [**2**, **47**] a set-valued mapping $M : U \rightrightarrows Y$ from an open set $U$ of $X$ into a normed space $Y$ satisfies the *Aubin Lipschitz-like property at* a point $a \in U$ *for* $b \in M(a)$ provided there are real numbers $r > 0$, $\gamma \geqslant 0$ and a neighbourhood $U' \subset U$ of $a$ such that

$$M(y) \cap B(b,r) \subset M(x) + \gamma\|y - x\|\mathbb{B}_Y \quad \text{for all } x, y \in U'.$$

When, in place of $\|x-y\|\mathbb{B}_Y$ in the second term, we put $\|x-y\|^\alpha\mathbb{B}_Y$ for some $\alpha > 0$, we will say that $M$ satisfies the *Aubin Hölder-like property* with power $\alpha$ at $a$ for $b \in M(a)$.

Corollary 4.7. *Let* $U$ *be a nonempty open convex subset of* $X$ *and* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a* $\partial$*-eds function on* $U$ *relative to* $\operatorname{Dom}\partial f$*. Let* $(a, a^*) \in \operatorname{gph}\partial f$ *with* $a \in U$*. Assume that the set-valued mapping* $\partial f$ *has the Aubin Lipschitz-like property at* $a$ *for* $a^*$ *and that the presubdifferential* $\partial f$ *is included in the Clarke subdifferential. Then* $f$ *is* $\mathcal{C}^{1,1}$ *near the point* $a$*.*

*Proof.* The corollary is a direct consequence of the definition of Aubin Lipschitz-like property and of Corollary 4.6 applied with $\rho(x,y) = \gamma\|x-y\|$ and the neighbourhood $U' \subset U$ above chosen to be open and convex. □

For classes of functions distinguished in Proposition 3.2 we have (a)–(d) of Theorem 4.2 and Corollary 4.6. This is gathered in the corollary below.

COROLLARY 4.8. *Let $U$ be a nonempty open convex set of $X$ and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc function. Assume that* (4.13) (*or* (4.16), *respectively*) *is fulfilled. Then each one of the following conditions ensure properties* (a), (b), (c) *and* (d) *in Theorem* 4.2 (*or Corollary* 4.6, *respectively*):

(a) *$f$ is convex on $U$;*
(b) *$f$ is locally Lipschitz continuous on $U$ and segment-wise essentially smooth on $U$, and the presubdifferential $\partial f$ is included in the Clarke subdifferential of $f$;*
(c) *$f$ is locally Lipschitz continuous on $U$ and directionally subregular on $U$, and the presubdifferential $\partial f$ is included in the Clarke subdifferential of $f$;*
(d) *$\partial$ is a subdifferential included in the Clarke subdifferential with exact subdifferential sum rule* (P4') *and $f$ is DC on $U$;*
(e) *$f$ is approximate convex on $U$ and the presubdifferential $\partial$ is included in the Clarke subdifferential.*

*Proof.* Concerning (a)–(e), the results are direct consequences of Proposition 3.2, Theorem 4.2 and Corollary 4.6. □

Our next result relates the Fréchet differentiability of an eds function $f$ to the inner semicontinuity property of the subdifferential of $f$. Recall that a set-valued mapping $M : X \rightrightarrows Y$ (where $Y$ is a Hausdorff topological space) is *inner (lower) semicontinuous at* $a \in \operatorname{Dom} M := \{x \in X : M(x) \neq \emptyset\}$ *for* $b \in M(a)$ when, for any neighbourhood $V$ of $b$, there exists some neighbourhood $U$ of $a$ such that $M(x) \cap V \neq \emptyset$ for all $x \in U$. When $M$ is *inner semicontinuous at $a$* for all $b \in M(a)$, one says that $M$ is inner semicontinuous at $a$; in particular $M$ is inner semicontinuous at $a$ whenever $M(a) = \emptyset$. Obviously, $M$ is inner semicontinuous at $a$ when, for each open set $V$ of $Y$ with $M(a) \cap V \neq \emptyset$, there exists someneighbourhood $U$ of $a$ such that $M(x) \cap V \neq \emptyset$ for all $x \in U$. Analogously, $M$ is *outer (upper) semicontinuous at $a$*, provided that, for each open set $V$ containing $M(a)$, there exists some neighbourhood $U$ of $a$ such that $M(x) \subset V$ for all $x \in U$. When the set-valued mapping $M$ is both inner and outer semicontinous at $a$, one says it is *continuous* at $a$.

Similarly, the Peano-Painlevé-Kuratowski *inner* (*or inferior*) *limit* $\operatorname*{Lim\,inf}_{x \to a} M(x)$ is defined by $y \in \operatorname*{Lim\,inf}_{x \to a} M(x)$, provided that, for each neighbourhood $V$ of $y$ there exists some neighbourhood $U$ of $a$ such that $M(x) \cap V \neq \emptyset$ for all $x \in U$. One sees that $M$ is inner semicontinuous at $a$ for $b \in M(a)$ if and only if $b \in \operatorname*{Lim\,inf}_{x \to a} M(x)$. Further, when the set-valued mapping $M$ has closed value at $a$, then we have the inclusion $\operatorname*{Lim\,inf}_{x \to a} M(x) \subset M(a)$.

When $Y$ is a topological dual space, we will write ${}^{\|\,\|}\operatorname*{Lim\,inf}_{x \to a} M(x)$ to emphasize that the inner limit is taken for $Y$ endowed with the topology associated with the dual norm.

PROPOSITION 4.5. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a $\partial$-eds function on an open convex set $U \subset X$ relative to* $\operatorname{Dom} \partial f$. *Assume that $\partial f$ is inner semicontinuous* (*for $X^*$ endowed with the strong topology*) *at $a \in U$ for some $a^* \in \partial f(a)$. Then the function $f$ is strictly Fréchet differentiable at the point $a$ with $D_F f(a) = a^*$. Further, the equality ${}^{\|\,\|}\operatorname*{Lim\,inf}_{x \to a} \partial f(x) = \{a^*\}$ holds.*

*Proof.* By the inner (lower) semicontinuity property, for any real $\varepsilon > 0$ there exists some open convex neighbourhood $U'$ of $a$ with $U' \subset U$ such that

$$(a^* + \varepsilon \mathbb{B}_{X^*}) \cap \partial f(x) \neq \emptyset \quad \text{for all } x \in U'.$$

By Proposition 4.3 we then have $U' \subset \operatorname{dom} f$ and

$$|f(x) - f(y) - \langle a^*, x - y\rangle| \leqslant \varepsilon\|x - y\| \quad \text{for all } x, y \in U'.$$

This means that $f$ is strictly Fréchet differentiable at $a$ with $D_F f(a) = a^*$.

Concerning the equality involving the inner limit of $\partial f(x)$ as $x \to a$, observe first that $a^* \in {}^{\|\ \|}\operatorname*{Lim\,inf}_{x\to a} \partial f(x)$ according to the inner semicontinuity assumption. Fix now any $x^* \in {}^{\|\ \|}\operatorname*{Lim\,inf}_{x\to a} \partial f(x)$. Our reasoning above gives that $f$ is Fréchet differentiable at $a$ with $D_F f(a) = x^*$. Therefore, we have $x^* = a^*$ and hence ${}^{\|\ \|}\operatorname*{Lim\,inf}_{x\to a} \partial f(x) = \{a^*\}$. □

The preceding condition is in fact a characterization of differentiability for approximate convex functions, as shown in the next theorem. Similar characterizations were previously established by Asplund and Rockafellar in [**1**, Theorem 3 and Corollary 2] for convex functions through the use of approximate $\varepsilon$-subdifferentials of convex functions. Characterizations in the same line for convex functions also follow directly from a result of Kenderov [**31**].

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function which is finite at $a \in X$ and approximate convex on some open convex set $U \ni a$. Recall (see, for example, [**37**] and [**54**, Proposition 3.4]) that for each $x \in U$ where $f$ is finite, the directional derivative $f'(x;\cdot)$ exists and is a positively homogeneous convex function. Further

$$\partial_C f(x) = \partial_F f(x) = \{x^* \in X^* : \langle x^*, h\rangle \leqslant f'(x; h)\ \ \forall h \in X\}, \tag{4.17}$$

and hence

$$\partial_C f(x) = \partial_F f(x) = \{D_G f(x)\} \tag{4.18}$$

whenever $f$ is Gâteaux differentiable at $x$. Consider now any real $\varepsilon > 0$ and choose by (3.5) some $\delta > 0$ with $a + 2\delta\mathbb{B}_X \subset U$ and such that for all $u, v \in a + 2\delta\mathbb{B}_X$ and all $\lambda \in ]0, 1[$

$$f(\lambda u + (1 - \lambda)v) \leqslant \lambda f(u) + (1 - \lambda)f(v) + \varepsilon\lambda(1 - \lambda)\|u - v\|. \tag{4.19}$$

Fix any $x \in a + \delta\mathbb{B}_X$, any $h \in \mathbb{B}_X$ and any positive $t \leqslant \delta$. Take any $r \in ]0, \delta[$. Writing

$$x = r(r + t)^{-1}(x - th) + t(r + t)^{-1}(x + rh)$$

and using (4.19) with $\lambda = r(r + t)^{-1}$ we obtain after computation for $x \in \operatorname{dom} f$

$$-t^{-1}[f(x - th) - f(x)] \leqslant r^{-1}[f(x + rh) - f(x)] + \varepsilon.$$

Since the directional derivative $f'(x;\cdot)$ exists, we deduce that

$$-\,t^{-1}[f(x - th) - f(x)] \leqslant f'(x; h) + \varepsilon. \tag{4.20}$$

Consider now any $s \in ]0, t[$. Similarly, applying (4.19) with $\lambda = s/t$, $x + th$ in place of $u$, and $x$ in place of $v$, we get after computation (with the use of the inequality $1 - \lambda < 1$)

$$s^{-1}[f(x + sh) - f(x)] \leqslant t^{-1}[f(x + th) - f(x)] + \varepsilon.$$

Consequently, we have

$$f'(x; h) \leqslant t^{-1}[f(x + th) - f(x)] + \varepsilon. \tag{4.21}$$

We are now ready to state and establish the theorem providing several characterizations of Fréchet differentiability of approximate convex functions.

THEOREM 4.3. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an lsc function that is approximate convex on some neighbourhood of a point $a \in \operatorname{int} \operatorname{dom} f$. Assume that $\partial f$ contains the Fréchet subdifferential and is included in the Clarke subdifferential. Then the following are equivalent:*

(a) *the function $f$ is Fréchet differentiable at the point $a$;*

(b) *the function $f$ is strictly Fréchet differentiable at the point $a$;*
(c) $^{\|\,\|}\operatorname*{Lim\,inf}_{x\to a} \partial f(x) \neq \emptyset$;
(d) $\partial f(a) \neq \emptyset$ *and the set-valued mapping* $\partial f$ *is inner semi-continuous at* $a$ *for* $X^*$ *endowed with the topology associated with the dual norm;*
(e) $\partial f(a) \neq \emptyset$ *and the set-valued mapping* $\partial f$ *is continuous at* $a$ *for* $X^*$ *endowed with the topology associated with the dual norm;*
(f) *there are a neighbourhood* $U$ *of* $a$ *and a dense subset* $Q$ *of* $U$ *with* $a \in Q$ *such that for some mapping* $\sigma : Q \to X^*$ *we have* $\sigma$ *is continuous at* $a$ *relative to* $Q$ *and* $\sigma(q) \in \partial f(q)$ *for every* $q \in Q$.

*Proof.* Suppose that $(c)$ holds and fix $a^* \in {}^{\|\,\|}\operatorname*{Lim\,inf}_{x\to a} \partial f(x)$. Then $a^* \in \partial f(a)$ (because $\partial f(a)$ is strongly closed according to (4.17)) and $\partial f$ is inner semicontinuous (for $X^*$ endowed with the strong topology) at $a$ for $a^* \in \partial f(a)$. Further, there exists an open convex neighbourhood $U$ of $a$ such that $f$ is approximate convex on $U$ and hence $\partial$-eds on $U$ relative to $\operatorname{Dom} \partial f$ by Proposition 3.2. Thus the implication $(c) \Rightarrow (b)$ follows from Proposition 4.5.

Let us show the implication $(a) \Rightarrow (e)$ is true. Choose an open convex neighbourhood $U$ of $a$ over which $f$ is approximate convex. Fix any real $\varepsilon > 0$. By the Fréchet differentiability of $f$ at $a$ take a real $t > 0$ such that $a + 2t\mathbb{B}_X \subset U$ and

$$-\varepsilon\|x-a\| \leqslant f(x) - f(a) - \langle D_F f(a), x-a\rangle \leqslant \varepsilon\|x-a\| \quad \forall x \in a + 2t\mathbb{B}_X. \tag{4.22}$$

Then, for any $x \in a + t\mathbb{B}_X$ we have for $h \in X$, with $\|h\| = 1$, according to (4.20) and (4.22), that

$$\begin{aligned}
f'(x;h) &\geqslant t^{-1}[f(x) - f(x-th)] - \varepsilon \\
&= -\varepsilon + t^{-1}[f(x) - f(a)] + t^{-1}[f(a) - f(x-th)] \\
&\geqslant -\varepsilon + t^{-1}[\langle D_F f(a), x-a\rangle - \varepsilon\|x-a\|] + t^{-1}[-\langle D_F f(a), x-th-a\rangle - \varepsilon\|x-th-a\|] \\
&\geqslant -\varepsilon + t^{-1}[\langle D_F f(a), x-a\rangle - \varepsilon t] + t^{-1}[-\langle D_F f(a), x-th-a\rangle - \varepsilon\|th\| - \varepsilon\|x-a\|] \\
&\geqslant -\varepsilon + t^{-1}[\langle D_F f(a), th\rangle - 3\varepsilon t] \\
&\geqslant \langle D_F f(a), h\rangle - 4\varepsilon,
\end{aligned}$$

that is,

$$\langle D_F f(a), h\rangle \leqslant f'(x;h) + 4\varepsilon \quad \text{for all } h \in X, \text{ with } \|h\| = 1$$

or equivalently

$$\langle D_F f(a), h\rangle \leqslant f'(x;h) + 4\varepsilon\|h\| \quad \text{for all } h \in X.$$

The latter inequality means by (4.17) that

$$D_F f(a) \in \partial f(x) + 4\varepsilon\mathbb{B}_{X^*} \quad \text{for all } x \in a + t\mathbb{B}_X. \tag{4.23}$$

On the other hand, for any $x \in a + t\mathbb{B}_X$ we have in a similar way through (4.21) and (4.22) that, for all $h \in X$ with $\|h\| = 1$,

$$\begin{aligned}
f'(x;h) &\leqslant t^{-1}[f(x+th) - f(x)] + \varepsilon \\
&= t^{-1}[f(x+th) - f(a)] + t^{-1}[f(a) - f(x)] + \varepsilon \\
&\leqslant \langle D_F f(a), h\rangle + 4\varepsilon.
\end{aligned}$$

This yieds $f'(x;h) \leqslant \langle D_F f(a), h\rangle + 4\varepsilon\|h\|$ for all $h \in X$ and hence, by (4.17), we have $\partial f(x) \subset D_F f(a) + 4\varepsilon\mathbb{B}_{X^*}$. Combining this with (4.23) and taking the equality $\partial f(a) = \{D_F f(a)\}$ into account, we obtain that $\partial f$ is inner and outer semicontinuous at $a$ for $X^*$ endowed with the dual norm topology. The desired implication $(a) \Rightarrow (e)$ is then established.

Observing that implications $(e) \Rightarrow (d)$, $(d) \Rightarrow (c)$, and $(b) \Rightarrow (a)$ are obvious, we have at this stage the equivalence between $(a)$, $(b)$, $(c)$, $(d)$, and $(e)$.

Let us show the implication $(f) \Rightarrow (b)$. There exists an open convex neighbourhood $U$ of $a$ such that $f$ is approximate convex on $U$ and hence, by Proposition 3.2, $f$ is $\partial$-eds on $U$ relative to $\operatorname{Dom} \partial f$. Assuming $(f)$ we may apply Corollary 4.2 to get (b).

In order to finish the proof, let us argue that the implication $(e) \Rightarrow (f)$ is also valid. So suppose $(e)$ and note that we also have $(b)$ by the equivalences shown above. Note also that there exists an open convex neighbourhood $U$ of $a$ such that $\partial f(x) \neq \emptyset$ for every $x \in U$. Take any selection $\sigma$ of the presubdifferential $\partial f$ on $U$. By $(e)$, $(b)$, and (4.18) we get $(f)$, which completes the proof. □

## 5. *Lipschitz-like properties of subdifferentials of prox-regular functions*

This section is devoted to the study of Lipschitz-like properties of subdifferentials of prox-regular functions. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc extended real-valued function and $(a, a^*) \in \operatorname{gph} \partial_L f$ ( $\operatorname{gph} \partial_L f$ stands for the graph of the Mordukhovich limiting subdifferential). For any real $\lambda > 0$ the Moreau envelope $e_{\lambda,a^*} f$ and proximal (set-valued) mapping $P_{\lambda,a^*} f$ through the direction $a^*$ are defined for all $x \in \mathbb{R}^n$ by

$$e_{\lambda,a^*} f(x) := \inf_{y \in \mathbb{R}^n} [f(y) - \langle a^*, y\rangle + \frac{1}{2\lambda}\|x - y\|^2]$$

and

$$P_{\lambda,a^*} f(x) := \{y \in \mathbb{R}^n : f(y) - \langle a^*, y\rangle + \frac{1}{2\lambda}\|x - y\|^2 = e_{\lambda,a^*} f(x)\},$$

where $\| \ \|$ is the Euclidean norm of $\mathbb{R}^n$. Poliquin and Rockafellar [40] introduced a large class of lsc extended real-valued functions $f$ on $\mathbb{R}^n$ (including convex functions) for which there exists a threshold $\lambda_0$ such that, for each positive real number $\lambda \leqslant \lambda_0$, there is some neighbourhood $U_\lambda$ of the point $a$ over which the envelope $e_{\lambda,a^*} f$ is $\mathcal{C}^{1,1}$ and the proximal mapping $P_{\lambda,a^*} f$ is single valued and Lipschitz continuous. Because of that regularity of the proximal mapping $P_{\lambda,a^*}$, the functions of the class are called *prox-regular at* the point $a$ *for the direction* $a^* \in \partial f(a)$; see also [41] for other results concerning second-order properties of such functions. Bernard and Thibault showed in [5, 4] that the same properties still hold for $e_{\lambda,a^*}$ and $P_{\lambda,a^*}$ when $f$ is any function of the same class but with a Hilbert space $(H, \| \ \|)$ in place of the Euclidean space $(\mathbb{R}^n, \| \ \|)$.

An interesting geometric way to characterize such a function $f$ of that class (see [40]) corresponds to the existence of some reals $\rho, \delta > 0$ such that, for all $(x, s) \in \operatorname{epi} f$ with $\|(x, s) - (a, f(a))\| < \delta$ and $(x^*, -s^*) \in N_L\big(\operatorname{epi} f; (x, s)\big)$ with $\|(x^*, s^*) - (a^*, -1)\| < \delta$, one has

$$\operatorname{Proj}\big(\operatorname{epi} f; (x, s) + \rho(x^*, s^*)\big) = \{(x, s)\},$$

where $\operatorname{Proj}\big(\operatorname{epi} f; \ \big)$ denotes the metric projection on $\operatorname{epi} f$ with respect to the Euclidean norm of $\mathbb{R}^n \times \mathbb{R}$. Translating this analytically for $\mathbb{R}^n$ or the Hilbert space $H$ gives the following definition.

Definition 5.1 ([40]). Let $U$ be an open convex set of a Hilbert space $(H, \| \ \|)$. Let $f : U \to \mathbb{R} \cup \{+\infty\}$ be an lsc function and $(a, a^*) \in \operatorname{gph} \partial_L f$. Following [40], the function $f$ is *prox-regular* at the point $a$ for the subgradient $a^* \in \partial_L f(a)$, provided there are some real numbers $r \geqslant 0$ and $\delta > 0$ such that for all $y \in B(a; \delta) \subset U$ and all $(x, x^*) \in \operatorname{gph} \partial_L f$ with $\|x - a\| < \delta$, $|f(x) - f(a)| < \delta$, and $\|x^* - a^*\| < \delta$, one has

$$f(y) + r\|y - x\|^2 \geqslant f(x) + \langle x^*, y - x\rangle. \tag{5.1}$$

When the property holds for all subgradients $a^* \in \partial_L f(a)$, the function $f$ is said to be prox-regular at $a \in \operatorname{Dom} \partial_L f$.

As observed in [**39**, **6**], primal lower nice functions on Hilbert spaces are prox-regular. When the involved space is neither a Hilbert space nor a space with differentiable norm (outside zero), property (5.1) does not ensure in general the local Lipschitz continuity of the proximal mapping $P_{\lambda,a^*}f$. Nevertheless, as it will be established below the regularity property in (5.1) has other important consequences. Let us thus fix this property in the case of a general Banach space in the definition below, where $\mathcal{M}$ (see (4.11)) denotes the set of functions $\omega(\cdot)$ from $[0,+\infty[$ into $[0,+\infty[$ that are continuous at 0 with $\omega(0) = 0$.

DEFINITION 5.2. Let $X$ be a Banach space, $\partial$ be a presubdifferential and $f : U \to \mathbb{R} \cup \{+\infty\}$ be an lsc function on an open set $U$, and let $(a, a^*) \in \operatorname{gph} \partial f$. When there exist $\varphi(\cdot) \in \mathcal{M}$ and $\delta > 0$ such that (similarly to (5.1)) for all $y \in B(a, \delta) \subset U$ and all $(x, x^*) \in \operatorname{gph} \partial f$ with $\|x - a\| < \delta$, $|f(x) - f(a)| < \delta$, and $\|x^* - a^*\| < \delta$, one has

$$f(y) + \|y - x\|\varphi(\|y - x\|) \geqslant f(x) + \langle x^*, y - x\rangle, \tag{5.2}$$

the function $f$ will be called $\partial^{1,\varphi(\cdot)}$-subregular at $a$ for $a^* \in \partial f(a)$.

Of course any convex function is $\partial^{1,\varphi(\cdot)}$-subregular, for any $\varphi \in \mathcal{M}$. Another important example of $\partial^{1,\varphi(\cdot)}$-subregular functions is given by the class of qualified convexly composite functions, with $\varphi(t) = ct$ for some constant $c \geqslant 0$. An lsc function $f := g \circ G$ is convexly $\mathcal{C}^{1,1}$-composite on an open convex set $U$ of the Banach space $X$, provided $g : Y \to \mathbb{R} \cup \{+\infty\}$ is an lsc convex function from a Banach space $Y$ into $\mathbb{R} \cup \{+\infty\}$ and the mapping $G : X \to Y$ is $\mathcal{C}^{1,1}$ on $U$. The convexly composite function $g \circ G$ is said to be *qualified* at the point $a \in U \cap \operatorname{dom}(g \circ G)$ whenever the Robinson qualification condition holds at $a$, that is,

$$\mathbb{R}_+\big(\operatorname{dom} g - G(a)\big) - DG(a)(X) = Y.$$

Note that the Robinson qualification condition holds for all points of $U$ in some neighbourhood of $a$ (see [**18**]). The prox-regularity of qualified convexly composite functions was first observed by Poliquin and Rockafellar in [**40**, Proposition 2.5] in the context of finite-dimensional spaces. It has been extended to any Hilbert space by Bernard and Thibault in [**5**, Proposition 2.4].

Finally, let us indicate that the class of $\partial^{1,\varphi(\cdot)}$-subregular functions is included in that of weak-regular functions introduced and studied in Jourani [**29**].

When the convergence $f(x) \to f(a)$ is automatically fulfilled as $(x, x^*) \to (a, a^*)$ with $(x, x^*) \in \operatorname{gph} \partial f$, the condition $|f(x) - f(a)| < \delta$ may be obviously removed in the definition of prox-regularity or of $\partial^{1,\varphi(\cdot)}$-subregularity above. This observation led Poliquin and Rockafellar [**40**] to introduce also the concept of subdifferentially continuous functions.

DEFINITION 5.3. The function $f$ is said to be subdifferentially (or $\partial$-subdifferentially) continuous at the point $a \in \operatorname{Dom} \partial f$ for a subgradient $a^* \in \partial f(a)$ when the function from $\operatorname{gph} \partial f$ to $\mathbb{R} \cup \{+\infty\}$ given by $(x, x^*) \mapsto f(x)$ is continuous at $(a, a^*)$ with respect to the induced topology on $\operatorname{gph} \partial f$, that is, for any real $\varepsilon > 0$ there exists some real $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \quad \text{for all } (x, x^*) \in \operatorname{gph} \partial f \text{ with } \|x - a\| < \delta \text{ and } \|x^* - a^*\| < \delta.$$

Any function which is continuous at $a \in \operatorname{Dom} \partial f$ is subdifferentially continuous at $a$ for any subgradient $a^* \in \partial f(a)$. It is also shown in [**5**, Proposition 2.3] that any lsc convex function

$f : U \to \mathbb{R} \cup \{+\infty\}$ is subdifferentially continuous at any $a \in \operatorname{Dom} \partial f$ for any subgradient $a^* \in \partial f(a)$.

In the proposition below it is recalled that qualified convexly composite functions are prox-regular and subdifferentially continuous functions on Hilbert spaces; see [**40**, **5**] for the details. The proof of Proposition 2.4 in [**5**] is still valid in the context of Definition 5.2 for any Banach space, and it shows that any qualified convexly composite function $f$ as above is $\partial^{1,\varphi(\cdot)}$-subregular at $a$ with $\varphi(t) = ct$ for some real constant $c \geqslant 0$.

PROPOSITION 5.1. *Let $\partial$ be a presubdifferential included in the Clarke subdifferential and let $f = g \circ G : U \to \mathbb{R} \cup \{+\infty\}$ be an lsc convexly $\mathcal{C}^{1,1}$-composite function as above that is qualified at $a \in \operatorname{Dom} \partial f$, where $U$ is an open convex subset of the Banach space $X$. Then, for any subgradient $a^* \in \partial f(a)$, the function $f$ is subdifferentially continuous at $a$ for $a^*$ and also $\partial^{1,\varphi(\cdot)}$-subregular at $a$ for $a^*$. More precisely, inequality (5.2) holds with $\varphi(t) = ct$, for some real constant $c \geqslant 0$.*

For several other properties of prox-regular functions we refer the reader to [**24**, **33**, **40**, **41**, **47**] for the finite-dimensional setting and to [**4**–**6**, **8**] for infinite dimensional spaces.

The theorem below concerning $\partial^{1,\varphi(\cdot)}$-subregular functions establishes in particular that the Aubin Hölder-like property of the presubdifferential of such a function ensures the $\mathcal{C}^{1,\alpha}$-regularity of the function.

THEOREM 5.1. *Let $\partial$ be a presubdifferential included in the Clarke subdifferential and $X$ be a Banach space. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an lsc function that is $\partial^{1,\varphi(\cdot)}$-subregular at a point $a$ for some $a^* \in \partial f(a)$ and $\varphi(\cdot) \in \mathcal{M}$. Assume that $f$ is subdifferentially continuous at $a$ for $a^*$ and that the set-valued mapping $\partial f$ satisfies the Aubin uniform-like continuity property with modulus $\omega(\cdot) \in \mathcal{M}$ at $a$ for $a^*$, that is, there are some real $\rho > 0$ and some neighbourhood $U$ of $a$ such that*

$$\partial f(x) \cap B(a^*, \rho) \subset \partial f(y) + \omega(\|x - y\|)\mathbb{B}_{X^*} \quad \text{for all } x, y \in U. \tag{5.3}$$

*Then $f$ is of class $\mathcal{C}^{1,\omega(\cdot)}$ near $a$ (when $\omega(t) = ct^{\alpha}$ with $\alpha > 1$ it means that $f$ is a continuous affine function near $a$).*

The proof of the theorem uses the next two lemmas.

LEMMA 5.1. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function that is lsc at $a$ and subdifferentially continuous at $a$ for some $a^* \in \partial f(a)$. Assume that, for some $\omega(\cdot) \in \mathcal{M}$, one has*

$$d(a^*, \partial f(x)) \leqslant \omega(\|x - a\|) \quad \text{for all } x \text{ near } a. \tag{5.4}$$

*Then $f$ is finite near $a$ and continuous at $a$.*

*Proof.* Note first that $f(a)$ is finite since $\partial f(a) \neq \emptyset$. Fix any $\varepsilon > 0$. The subdifferential continuity property gives some $\delta > 0$ such that

$$\|x - a\| < \delta, \|x^* - a^*\| < \delta \text{ and } x^* \in \partial f(x) \Rightarrow |f(x) - f(a)| < \varepsilon. \tag{5.5}$$

By the continuity of $\omega(\cdot)$ at 0 with $\omega(0) = 0$ take some positive real number $\eta < \delta$ such that $\omega(t) < \delta$ for all nonnegative real numbers $t < \eta$. For any $x \in B(a, \eta)$ inequality (5.4) gives some $x^* \in \partial f(x)$ with $\|x^* - a^*\| < \delta$, and hence $|f(x) - f(a)| < \varepsilon$ according to (5.5). This corresponds to the continuity of $f$ at $a$. □

The second lemma establishes the local Lipschitz continuity whenever the function $f$ is (in addition to the assumptions of the lemma above) $\partial^{1,\varphi(\cdot)}$-subregular.

LEMMA 5.2. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function which is finite at $a$ and continuous at $a$. Assume that the function $f$ is $\partial^{1,\varphi(\cdot)}$-subregular for some $\varphi(\cdot) \in \mathcal{M}$ at $a$ for some $a^* \in \partial f(a)$ and that (5.4) is fulfilled with $a, a^*$. Then $f$ is finite and locally Lipschitz continuous near $a$.*

*Proof.* Fix $\delta > 0$ such that $d(a^*, \partial f(x)) \leqslant \omega(\|x - a\|)$ for all $x \in B(a, \delta)$ and

$$\langle x^*, y - x\rangle \leqslant f(y) - f(x) + \|y - x\|\varphi(\|y - x\|) \tag{5.6}$$

for all $y \in B(a, \delta)$ and $(x, x^*) \in \operatorname{gph} \partial f$ with $x \in B(a, \delta)$ and $\|x^* - a^*\| < \delta$. By the continuity of $\omega(\cdot)$ at 0 with $\omega(0) = 0$, take some positive real number $\eta < \delta$ such that $\omega(t) < \delta$ for all nonnegative real numbers $t < \eta$. Fix any $x \in B(a, \eta)$ and according to the inequalities $d(a^*, \partial f(x)) \leqslant \omega(\|x - a\|) < \delta$ choose some $\sigma(x) \in \partial f(x)$ such that $\|\sigma(x) - a^*\| < \delta$. By (5.6) we have, for all $y \in B(a, \eta)$,

$$\begin{aligned} f(y) &\geqslant f(x) + \langle \sigma(x), y - x\rangle - \|y - x\|\varphi(\|y - x\|) \\ &\geqslant f(x) - \|\sigma(x)\|\,\|y - x\| - \|y - x\|\varphi(\|y - x\|) \\ &\geqslant f(x) - (\delta + \|a^*\|)\|y - x\| - \|y - x\|\varphi(\|y - x\|). \end{aligned}$$

This is easily seen to ensure the finiteness and local Lipschitz continuity of $f$ near $a$, because $\varphi(\|y - x\|) \leqslant 1$ for $x$ and $y$ close enough to $a$. □

We can now prove the theorem relative to the Aubin property of the presubdifferential of a $\partial^{1,\varphi(\cdot)}$-subregular function.

*Proof of Theorem* 5.1. According to the Aubin uniform-like continuity property of $\partial f$ and to the $\partial^{1,\varphi(\cdot)}$-subregularity of $f$, fix $\delta, \rho > 0$ such that

$$\partial f(y) \cap B(a^*, \rho) \subset \partial f(x) + \omega(\|y - x\|)\mathbb{B}_{X^*} \quad \forall x, y \in B(a, \delta) \tag{5.7}$$

and such that, for all $y \in B(a, \delta)$, $x \in B(a, \delta)$, $x^* \in \partial f(x)$ with $\|x^* - a^*\| < \delta$,

$$f(y) \geqslant f(x) + \langle x^*, y - x\rangle - \|y - x\|\varphi(\|y - x\|). \tag{5.8}$$

By Lemmas 5.1 and 5.2 we may suppose that $f$ is finite and Lipschitz continuous on $B(a, \delta)$ with some Lipschitz constant $K \geqslant 0$. Choose a positive number $\eta < \delta$ such that $\omega(t) < \rho$ for all nonnegative numbers $t < \eta$. By (5.7) observing for each $x \in B(a, \eta)$ that

$$a^* \in \partial f(x) + \omega(\|x - a\|)\mathbb{B}_{X^*},$$

we see that $\partial f(x) \cap B(a^*, \rho) \neq \emptyset$, and hence we can choose some $\sigma(x) \in \partial f(x) \cap B(a^*, \rho)$. Now fix any $x \in B(a, \eta)$ and any $h \in X$. According to the Lipschitz property of $f$ on $B(a, \eta)$ and to the definition of the Clarke directional derivative, take two sequences $t_k \downarrow 0$ and $x_k \to x$ such that $x_k + t_k h \in B(a, \eta)$ for all $k$ and such that

$$f^0(x; h) = \lim_k t_k^{-1}[f(x_k + t_k h) - f(x_k)].$$

Putting $y_k := x_k + t_k h$, inclusion (5.7) gives some $y_k^* \in \partial f(y_k)$ such that

$$\|y_k^* - \sigma(x_k)\| \leqslant \omega(\|y_k - x_k\|) = \omega(t_k\|h\|).$$

By (5.8) we have for all integers $k$

$$\begin{aligned} f(x_k) &\geqslant f(y_k) + \langle y_k^*, x_k - y_k\rangle - \|x_k - y_k\|\varphi(\|x_k - y_k\|) \\ &\geqslant f(y_k) + \langle \sigma(x_k), x_k - y_k\rangle - t_k\|h\|\omega(t_k\|h\|) - t_k\|h\|\varphi(t_k\|h\|), \end{aligned}$$

and hence

$$t_k^{-1}[f(x_k+t_kh)-f(x_k)] \leqslant \langle \sigma(x_k),h\rangle + \big(\omega(t_k\|h\|)+\varphi(t_k\|h\|)\big)\|h\|. \tag{5.9}$$

The sequence $(\sigma(x_k))_k$ being bounded (because $\|\sigma(x_k)\| \leqslant K$), fix $u^*$ as one of its $w^*$-cluster points. Using (5.8) with $x_k$ and $\sigma(x_k)$ in place of $x$ and $x^*$ and passing to the limit along a bounded subnet of $(\sigma(x_k))_k$ $w^*$-converging to $u^*$, we obtain, for all $y \in B(a,\eta)$,

$$f(y) \geqslant f(x) + \langle u^*, y-x\rangle - \|y-x\| \sup_{s\in ]0,\,2\|y-x\|\,]} \varphi(s). \tag{5.10}$$

By the continuity of $\varphi$ at zero ($\varphi(t) \underset{t\downarrow 0}{\longrightarrow} 0$, see the definition of $\mathcal{M}$) it follows from (5.10)) that

$$\langle u^*, h\rangle \leqslant d^- f(x;h). \tag{5.11}$$

On the other hand, using the convergences $\omega(t)\to 0$ and $\varphi(t)\to 0$ as $t\downarrow 0$ we easily deduce from (5.9) that

$$f^0(x;h) \leqslant \langle u^*, h\rangle,$$

and hence taking (5.11) into account we deduce $f^0(x;h) \leqslant d^- f(x;h)$, that is, $f^0(x;h) = d^- f(x;h)$. The Lipschitz function $f$ on $B(a,\eta)$ is then directionally subregular on $B(a,\eta)$, hence it follows from Corollary 4.8(c) that $f$ is $\partial$-eds on $B(a,\eta)$. Since

$$\sigma(y) \in \partial f(x) + \omega(\|x-y\|)\mathbb{B}_{X^*} \quad \text{for all } x,y \in B(a,\eta),$$

Theorem 4.2(d) ensures that $f$ is $\mathcal{C}^{1,\omega(\cdot)}$ on $B(a,\eta)$ and the proof is completed. □

The first corollary concerns qualified convexly composite functions. It is a direct consequence of Theorem 5.1 and Proposition 5.1.

Corollary 5.1. *Let $\partial$ be a presubdifferential included in the Clarke subdifferential and $U$ be a nonempty open convex set of $X$. Let $f : U \to \mathbb{R}\cup\{+\infty\}$ be an lsc convexly $\mathcal{C}^{1,1}$-composite function that is qualified at a point $a \in \operatorname{Dom}\partial f$. Assume that the set-valued mapping $\partial f$ satisfies the Aubin uniform-like continuity property with modulus $\omega(\cdot)\in\mathcal{M}$ at $a$ for some $a^*\in\partial f(a)$. Then $f$ is $\mathcal{C}^{1,\omega(\cdot)}$ near $a$.*

The case of prox-regular functions follows also easily from Theorem 5.1.

Corollary 5.2. *Let $X$ be a Hilbert space, $\partial$ be a presubdifferential included in the Clarke subdifferential, and $U$ be a nonempty open set of $X$. Let $f : X \to \mathbb{R}\cup\{+\infty\}$ be an lsc function which is prox-regular at a point $a\in U$ for some $a^*\in\partial f(a)$. Assume that $f$ is subdifferentially continuous at $a$ for $a^*$ and that the set-valued mapping $\partial f$ satisfies the Aubin uniform-like continuity property with modulus $\omega(\cdot)\in\mathcal{M}$ at $a$ for $a^*$. Then the function $f$ is of class $\mathcal{C}^{1,\omega(\cdot)}$ near the point $a$.*

The next corollary (which is a direct consequence of the previous one) considers the case where the presubdifferentail of the prox-regular function is Aubin Lipschitz-like continuous. It has been first established by Levy and Poliquin [**32**] in finite-dimensional spaces and recently extended to Hilbert spaces by Bačák et al [**3**].

Corollary 5.3 ([**32**, Theorem 3.1]; [**3**, Theorem 5.4]). *Let $X$ be a Hilbert space, $\partial$ be a presubdifferential included in the Clarke subdifferential, and $U$ be a nonempty open set of $X$. Let $f : X \to \mathbb{R}\cup\{+\infty\}$ be an lsc function that is prox-regular at a point $a\in U$ for some*

*$a^* \in \partial f(a)$. Assume that $f$ is subdifferentially continuous at $a$ for $a^*$ and that the set-valued mapping $\partial f$ satisfies the Aubin Lipschitz-like property at $a$ for $a^*$. Then the function $f$ is of class $\mathcal{C}^{1,1}$ near the point $a$.*

## 6. *Characterizations of Lipschitz and $\mathcal{C}^{1,\omega(\cdot)}$ properties through paraconvexity*

In the last two above sections, assertion $(c)$ of Corollary 4.6 and Theorem 5.1 give a characterization of $\mathcal{C}^{1,\alpha}$ property of $\partial$-eds functions and $\partial^{1,\varphi(\cdot)}$-subregular functions, respectively, via the Aubin Hölder-like property of $\partial f$. Our aim in this section is to provide for any function another characterization of its $\mathcal{C}^{1,\alpha}$ property in terms of the concepts of paraconvexity. Let us first recall the notions of paraconvexity and strong-paraconvexity introduced by S. Rolewicz [**48**] in 1979.

Definition 6.1. Let $X$ be a normed space, $\gamma > 0$ be a given positive number and $f : X \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function. Following [**48**] (or [**49**], respectively) we say that the function $f$ is $\gamma$-paraconvex (or strongly $\gamma$-paraconvex, respectively) on a convex set $U$ of $X$, provided there exists $C \geqslant 0$ such that, for all $t \in ]0,1[$ and $x, y \in U$, we have

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) + C\|x-y\|^\gamma \tag{6.1}$$

(or

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) + C\min\{t, 1-t\}\|x-y\|^\gamma), \text{ respectively .} \tag{6.2}$$

More generally, for $\psi(\cdot) \in \mathcal{M}$, according to [**50**] we say that $f$ is $\psi(\cdot)$-paraconvex (or strongly $\psi(\cdot)$-paraconvex, respectively) on $U$ when, for some constant $C \geqslant 0$, we have, for all $t \in ]0,1[$ and $x, y \in U$, that

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) + C\psi(\|x-y\|) \tag{6.3}$$

(or

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) + C\min\{t, 1-t\}\psi(\|x-y\|)), \text{ respectively .} \tag{6.4}$$

Since $t(1-t) \leqslant \min\{t, 1-t\} \leqslant 2t(1-t)$ for all $t \in [0,1]$, we see that $f$ is strongly $\psi(\cdot)$-paraconvex on $U$ if and only if for some real $\rho \geqslant 0$ we have for all $t \in ]0,1[$ and $x, y \in U$

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) + \rho t(1-t)\psi(\|x-y\|). \tag{6.5}$$

When the latter is fulfilled for $\psi(t) = t\omega(t)$ with $\omega(\cdot) \in \mathcal{M}$, the function $f$ is also called $\omega(\cdot)$-*semiconvex* in the literature (see, for example, [**15**]).

Theorem 2 in Rolewicz [**49**] and Proposition 2.1 in Jourani [**28**] tell us that, for $\gamma > 1$, the function $f$ is $\gamma$-paraconvex on $U$ if and only if it is strongly $\gamma$-paraconvex. Such an equivalence does not hold for $\psi(\cdot)$-paraconvexity (see [**50**]).

We say that $f$ is paraconvex or semiconvex around a point $a \in \operatorname{dom} f$ when it is paraconvex or semiconvex on a convex neighbourhood of $a$. Let us note that, following Rolewicz [**50**], $f$ is approximate convex at $a \in \operatorname{dom} f$ in some uniform way if and only if $f$ is $\psi(\cdot)-$paraconvex for some function $\psi(\cdot)$ satisfying $\lim_{t\to 0}(\psi(t)/t) = 0$. Further when $X$ is a Hilbert space, it is not difficult to show that $f$ is strongly 2-paraconvex (or equivalently 2-paraconvex) on $U$ with $\rho$ given by (6.5) and $\psi(t) = t^2$ if and only if the function $f + \rho\|\ \|^2$ is convex on $U$ (see [**49**]).

Let us note that, for $\gamma = 1$, we obtain the following characterization of locally Lipschitz continuous functions in terms of the strong paraconvexity.

Proposition 6.1. *Let $U$ be an open convex set of a normed space $X$ and $f : X \to \mathbb{R}$ be a function. The following hold.*

(a) *If the function $f$ is Lipschitz continuous on $U$, then $f$ and $-f$ are strongly $1-$paraconvex on $U$.*
(b) *If $f$ is strongly $1-$paraconvex and bounded on $U + r\mathbb{B}_X$ for some real $r > 0$, the function $f$ is Lipschitz continuous on $U$.*
(c) *The function $f$ is Lipschitz continuous around a point $a \in U$ if and only if $f$ is bounded around $a$, and $f$ and $-f$ are strongly $1-$paraconvex around $a$.*

*Proof.* (a) Let $K$ be a Lipschitz constant of $f$ over $U$ and fix any $x, y \in U$ and $t \in [0,1]$. Then we have

$$f(y + t(x-y)) - f(y) + t(f(y) - f(x)) \leqslant Kt\|x-y\| + Kt\|x-y\| = 2Kt\|x-y\|$$

and

$$f(y + t(x-y)) - f(x) + (1-t)(f(x) - f(y)) \leqslant 2K(1-t)\|x-y\|,$$

hence

$$f(y + t(x-y)) - tf(x) - (1-t)f(y) \leqslant 2K\min\{t, 1-t\}\|x-y\|,$$

and this translates the strong $1-$paraconvexity of $f$ on $U$. Since $f$ is Lipschitz continuous on $U$ if and only if $-f$ is Lipschitz continuous on $U$, we also obtain the strong $1-$paraconvexity of $-f$ on $U$.

(b) As in [**28**, Proposition 2.2] we follow the standard method of convex functions. Let $\mu$ be an upper bound of $|f|$ on $U + r\mathbb{B}_X$. Fix $x, y \in U$ with $x \neq y$. Put $z := y + r((y-x)/\|y-x\|)$ and observe that $z \in U + r\mathbb{B}_X$. Further, for $\rho \geqslant 0$ given by (6.5) and for $t := (\|y-x\|)/(r + \|y-x\|)$ we have

$$f(y) = f(tz + (1-t)x) \leqslant tf(z) + (1-t)f(x) + \rho t\|z-x\|,$$

hence

$$f(y) - f(x) \leqslant t(f(z) - f(x)) + \rho\|y-x\| \leqslant \left(\rho + \frac{2\mu}{r}\right)\|y-x\|,$$

which says that $f$ is Lipschitz continuous on $U$.

(c) Assertion $(c)$ is a direct consequence of $(a)$ and $(b)$. □

Now, we state a Rolewicz's characterization (Theorem 4 in [**49**]) of $C^{1,\alpha}$ functions on normed spaces in terms of strong paraconvexity or equivalently semiconvexity. For the convenience of the reader, a simple proof will be given for the more general $C^{1,\omega(\cdot)}$ property. Note that, when $X$ is a Hilbert space and $\alpha = 1$, this Rolewicz result has been independently obtained in [**25**] through Moreau decomposition techniques. In the same Hilbert setting and with $\alpha = 1$ another proof has been given recently in [**3**] via the Alexandrov theorem on almost everywhere twice differentiability of convex functions.

In the proof below we use the following description of $\partial_C f$ established in [**28**, **50**] for strongly $\psi(\cdot)$-paraconvex functions: For $f$ Lipschitz continuous and satisfying (6.5) on $U = B(a,\delta)$ with $\rho \geqslant 0$ and $\psi(t) = t\omega(t)$ where $\omega(\cdot) \in \mathcal{M}$, one has for all $x \in U$

$$\partial_C f(x) = \{x^* \in X^* : \langle x^*, h\rangle \leqslant f(x+h) - f(x) + \rho\,\psi(\|h\|)\ \forall h \in B(0, \delta - \|x-a\|)\}. \quad (6.6)$$

THEOREM 6.1. *Let $f : U \to \mathbb{R}$ be a function from an open subset $U$ of a normed space $X$ into $\mathbb{R}$ and let $\psi(\cdot)$ be convex and nondecreasing and $\psi(t) = t\omega(t)$ for $t \geqslant 0$, with $\omega \in \mathcal{M}$. The following hold.*

(a) *If $f$ and $-f$ are strongly $\psi(\cdot)-$paraconvex around $a$ and $f$ is bounded around $a$, then $f$ is of class $\mathcal{C}^{1,\omega(2\cdot)}$ around $a$.*

(b) *If $f$ is of class $\mathcal{C}^{1,\omega(\cdot)}$ around $a$, then $f$ and $-f$ are strongly $\psi(\cdot)-$paraconvex around $a$ and $f$ is bounded around $a$.*

*In particular, $f$ is $C^{1,\alpha}$ around $a$ for some real $\alpha > 0$ if and only if $f$ and $-f$ are strongly $(1+\alpha)$-paraconvex around $a$ and $f$ is bounded around $a$.*

*Proof.* (a) So suppose that $f$ and $-f$ are $\psi(\cdot)-$paraconvex around $a$ and $f$ is bounded around $a$. Then there exists $\delta > 0$ and $\rho > 0$ such that $f$ is Lipschitz continuous on $B(a, \delta)$ and simultaneously $f$ and $-f$ satisfy relation (6.6) for all $x \in B(a, \delta)$. The nonvacuity of $\partial_C f(x)$ and $\partial_C(-f)(x)$ along with equality (6.6) imply that $\partial_C f(x)$ is a singleton for every $x \in B(a, \delta)$, say $\partial_C f(x) = \{D_G f(x)\}$, and

$$|f(x+h) - f(x) - \langle D_G f(x), h\rangle| \leqslant \rho\psi(\|h\|) \tag{6.7}$$

for all $x \in B(a, \delta)$ and $h \in B(0, \delta - \|x - a\|)$. Fix $t \in ]0, \frac{\delta}{2}[$ and $h \in X$, with $\|h\| = 1$, and let $x, y \in B(a, \frac{\delta}{4})$, $x \neq y$. Taking into account the convexity of the function $u \mapsto \psi(\|u\|)$, relation (6.7) ensures

$$\begin{aligned}
\langle D_G f(x), h\rangle &\geqslant t^{-1}[f(x) - f(x - th)] - \rho\omega(t) \\
&= t^{-1}[f(x) - f(y)] + t^{-1}[f(y) - f(x - th)] - \rho\omega(t) \\
&\geqslant t^{-1}[\langle D_G f(y), x - y\rangle - \rho\psi(\|x - y\|)] \\
&\quad + t^{-1}[-\langle D_G f(y), x - th - y\rangle - \rho\psi(\|x - th - y\|)] - \rho\omega(t) \\
&\geqslant -\rho t^{-1}\psi(\|x - y\|) + \langle D_G f(y), h\rangle - \rho t^{-1}\psi(\|x - th - y\|) - \rho\omega(t) \\
&\geqslant -\rho t^{-1}\psi(\|x - y\|) + \langle D_G f(y), h\rangle - \rho t^{-1}[\|x - y\|\omega(2\|x - y\|) + t\omega(2t)] - \rho\omega(t).
\end{aligned}$$

For $t = \|x - y\|$ we obtain

$$\begin{aligned}
\langle D_G f(x), h\rangle &\geqslant \langle D_G f(y), h\rangle - 2\rho(\omega(\|x - y\|) + \omega(2\|x - y\|)) \\
&\geqslant \langle D_G f(y), h\rangle - 4\rho\omega(2\|x - y\|).
\end{aligned}$$

The last inequality follows from the nondecreasing property of $\psi(\cdot)$. Since $h$ is arbitrary, we get

$$\|D_G f(x) - D_G f(y)\| \leqslant 4\rho\omega(2\|x - y\|) \quad \forall x, y \in B\left(a, \frac{\delta}{4}\right).$$

(b) Suppose that $f$ is of class $\mathcal{C}^{1,\omega(\cdot)}$ around $a$. Then $f$ is Fréchet differentiable around $a$ and there exists $\delta > 0$ and $\rho > 0$ such that

$$\|D_F f(x) - D_F f(y)\| \leqslant \rho\omega(\|x - y\|) \quad \forall x, y \in B(a, \delta).$$

Fix any $x, y \in B(a, \delta)$ and any $t \in ]0, 1[$. We have

$$\begin{aligned}
&f(x + t(y - x)) - f(x) + t(f(x) - f(y)) \\
&= t\int_0^1 \langle D_F f(x + st(y - x)), y - x\rangle\, ds - t\int_0^1 \langle D_F f(x + s(y - x)), y - x\rangle\, ds \\
&= t\int_0^1 \langle D_F(x + st(y - x)) - D_F f(x + s(y - x)), y - x\rangle\, ds \\
&\leqslant \rho t\|x - y\| \int_0^1 \omega(s(1 - t)\|x - y\|)\, ds \\
&= \int_0^1 \frac{\rho t}{s(1 - t)}\psi(s(1 - t)\|x - y\|)\, ds \leqslant \rho t\psi(\|x - y\|).
\end{aligned}$$

The last inequality is due to the convexity of $\psi(\cdot)$ and $\psi(0) = 0$. A similar inequality being true with $1 - t$ in place of $t$ in the last member above, we obtain

$$f(ty + (1-t)x) - tf(y) - (1-t)f(x) \leqslant \rho \min(t, 1-t)\psi(\|x - y\|) \leqslant 2\rho t(1-t)\psi(\|x - y\|),$$

and this completes the proof. □

Paraconvex functions enjoy (see [**15**, **28**, **48–51**] and references therein) several other remarkable properties. In particular for a presubdifferential $\partial$ included in the Clarke subdifferential, it is worth emphasizing that, whenever $f$ is lsc on a Banach space $X$ and $\gamma > 1$, according to [**28**, Theorem 7.1 and Corollary 7.1], $f$ is $\gamma$-paraconvex around $a$ if and only if there exists $\delta > 0$ such that the presubdifferential $\partial f$ is hypomonotone with power $\gamma$ on $B(a, \delta)$, that is, there exists $\rho \geqslant 0$ such that

$$\langle x^* - y^*, x - y\rangle \geqslant -\rho\|x - y\|^\gamma \tag{6.8}$$

whenever $x, y \in B(a, \delta)$, $x^* \in \partial f(x)$ and $y^* \in \partial f(y)$. So, as a direct consequence of this and the above theorem, we have the following result providing one more characterization of $\mathcal{C}^{1,\alpha}$ property in terms of hypomonotonicity of subdifferentials.

THEOREM 6.2. *Let $X$ be a Banach space and $\partial$ be a presubdifferential included in the Clarke subdifferential. Let $f : U \mapsto \mathbb{R}$ be a continuous function from an open subset $U$ of $X$ into $\mathbb{R}$ and let $\alpha > 0$ be a positive real number. Then $f$ is of class $\mathcal{C}^{1,\alpha}$ around $a \in U$ if and only if $\partial f$ and $\partial(-f)$ are hypomonotone with power $\gamma := 1 + \alpha$ around $a$.*

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