The role of locally compact cones in nonsmooth analysis*

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Abstract. This paper is aimed at providing the fundamental role that play locally compact cones in nonsmooth analysis. It is concerned with the following subjects :

. Subdifferential calculus for limiting Fréchet subdifferentials and approximate subdifferentials;

. Relationships between limiting Fréchet subdifferentials, approximate subdifferentials and other subdifferentials considered here;

. Metric inequalities;

. Interrelations between limiting Fréchet normal cones, G-normal cone, approximate normal cone and sequential normal cone constructions considered here;

. Extension of the "liminf" formula;

. Study of the Clarke's tangent cones and the boundaries of closed sets in Banach spaces.

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Introduction.

For our introduction, we begin with some examples of applications which motivate our study of the notion of locally compact cones and their role in nonsmooth analysis. To do this, we let X, Y be Banach spaces and X^* and Y^* their topological duals endowed with the weak-star topology w^* where $\langle \cdot, \cdot \rangle$ means the canonic pairing. The distance function $d(x, C)$ to a set C at x is defined by

$$d(x, C) = \inf_{u \in C} \parallel x - u \parallel .$$

A set $K^* \subset X^*$ is said to be (weak-star) *locally compact* if for each point x^* in K^* there exists a weak-star neighbourhood V of x^* such that $\mathrm{cl}^* V \cap K^*$ is weak-star compact. Here "cl*" denotes the weak-star topological closure. Many important properties of these sets are listed in section 2. We give one of them, because we use it in our examples. If $K^* \subset X^*$ is a locally compact cone and $(x_i^*) \subset K^*$ is a net, then (Loewen [40])

$$x_i^* \xrightarrow{w^*} 0 \Longleftrightarrow \parallel x_i^* \parallel \to 0. \tag{1.1}$$

Example 1. *(Separation theorems).* We know that in finite dimensional spaces every proper convex set has a supporting hyperplane at given point of its boundary. In the infinite dimensional case, this result is no longer true. Indeed in any infinite separable Banach space we construct (Borwein [3]) such a set by taking a sequence (b_n) is dense in the closed unit ball. Let

$$C = \mathrm{clco}\{\frac{b_n}{2^n}, \frac{-b_n}{2^n}\}.$$

Then C is a compact symmetric convex set. Moreover C has no supporting hyperplane at the boundary point $0 \in C$. Indeed if there exists $x^* \in X^*$ such that

$$\langle x^*, u \rangle \leq 0, \quad \forall u \in C$$

we get $x^* = 0$.

In [29], we showed that if there exist a locally compact cone $K^* \subset X^*$ and a neighbourhood W of $x \in C$ such that

$$\partial d(u, C) \subset K^*, \quad \forall u \in W \cap C$$

we have that x is a boundary point of C iff C has a supporting hyperplane at x. Here $\partial f(x)$ denotes the subdifferential in the sense of convex analysis of f at x. In fact we established this result in the nonconvex case for the so called extremal systems by means of an abstract subdifferential.

Example 2. *(Closure of the sum).*Given two closed convex cones $K_1 \subset X$ and $K_2 \subset X$ denote by K_1^0 and K_2^0 their negative polars. We know that in a finite dimensional space the following equality holds true

$$(K_1 \cap K_2)^0 = K_1^0 + K_2^0 \tag{1.2}$$

if $K_1^0 \cap (-K_2^0) = \{0\}$. This result is not valid in infinite dimensional situation. But if we assume that K_1^0 is a locally compact cone and $K_1^0 \cap (-K_2^0) = \{0\}$ then (1.2) holds. Indeed it suffices to show that

$$\mathrm{cl}^*(K_1^0 + K_2^0) = K_1^0 + K_2^0$$

since $(K_1 \cap K_2)^0 = \mathrm{cl}^*(K_1^0 + K_2^0)$. So let $x^* \in \mathrm{cl}^*(K_1^0 + K_2^0)$ there are nets $(u_i^*) \subset K_1^0$ and $(v_i^*) \subset K_2^0$ such that $u_i^* + v_i^* \xrightarrow{w^*} x^*$.

We claim that (u_i^*) is bounded. Suppose for some subnet (u_j^*) of (u_i^*) that $\| u_j^* \| \to \infty$, and set $q_j^* := \frac{u_j^*}{\|u_j^*\|}$ and $p_j^* := \frac{v_j^*}{\|u_j^*\|}$. Extracting subnets if necessary we may assume that $(q_j^*) \xrightarrow{w^*} q^*$ and $(p_j^*) \xrightarrow{w^*} -q^*$ and by (1.1) $q^* \neq 0$ and $q^* \in K_1^0 \cap (-K_2^0)$, a contradiction.

So (u_i^*) is bounded and has a weak-star convergent subnet with limit u^* and hence some subnet of (v_i^*) must weak-star converges to $x^* - u^*$.

Example 3. *(Necessary conditions for optimality).* Consider the following optimization problem

$$(P) \qquad \min\{f(x): \quad g(x) \in D\}$$

where $f : X \to \mathbb{R}$ and $g : X \to Y$ are differentiable mappings and D is a nonempty subset of Y. In the case where Y is finite dimensional the following result holds for any closed set D : if x_0 is a local minimum for (P) then there are $\lambda \geq 0$ and $y^* \in N(D, g(x_0))$ such that

$$(\lambda, y^*) \neq (0,0) \tag{1.3}$$

$$\lambda f'(x_0) + y^* \circ g'(x_0) = 0. \tag{1.4}$$

Here $N(D, g(x_0))$ denotes some normal cone to D at $g(x_0)$ (say, for example, Clarke normal , approximate normal cone, etc$\cdots$). This result is not true in the infinite dimensional case. Let, for example (Brokate [9]), $X = Y = l^2$ be the Hilbert space of square summable sequences, with (e_k) its canonical orthonormal base and let the operator $A : l^2 \to l^2$ be defined by

$$A(\sum x_i e_i) = \sum 2^{1-i} x_i e_i.$$

Then A is not surjective and $\mathrm{Im}(A)$ is a proper dense subspace of l^2. The adjoint A^* is injectif but not surjectif. So let $x^* \notin \mathrm{Im}(A^*)$ and set $f = x^*$, $g = A$ and $D = \{0\}$. Then 0 is only the feasible point and it is the optimum for this problem. Moreover there is no $(\lambda, y^*) \neq (0,0)$ satisfying (1.4).

In infinite dimension, most of the authors assumed that D is a closed convex cone with nonempty interior or $D = D_1 \times \{0\}$, where D_1 is a closed convex cone with nonempty interior and $\{0\} \subset \mathbb{R}^n$. The first result which gives condition in the case of closed sets is due to Jourani and Thibault [30] where it assumed that the system $g(x) \in D$ is metrically regular. This condition is expressed metrically in terms of g and D and implies that λ is not equal to zero. In [27] we showed that relations (1.3) and (1.4) subsist in the case where f is vector-valued and D epi-Lipschitz-like in the sense of Borwein [3]. In [32-33] we gave general conditions ensuring (1.3) and (1.4). More precisely, we may show that *if x_0 is a local minimum for (P) and if there exist a locally compact cone $K^* \subset Y^*$ and a neighbourhood V of $g(x_0)$ such that*

$$\partial_A d(u, D) \subset K^*, \quad \forall u \in V \cap D$$

then there exists $\lambda \geq 0$ and $y^* \in \mathbb{R}_+ \partial_A d(g(x_0), D)$, with $(\lambda, y^*) \neq (0,0)$ such that

$$\lambda \partial_A f(x_0) + \partial_A (y^* \circ g)(x_0)$$

where f and g are locally Lipschitzian mappings at x_0, with g strongly compactly Lipschitzian at x_0 ([30]). Here $\partial_A f(x)$ denotes the approximate subdifferential of f at x (see Ioffe [18-19]).

Our aim in this paper is to explore further properties of locally compact cones and to examine the fundamental role that play these cones in the following situations :

. Subdifferential calculus for limiting Fréchet subdifferentials and approximate subdifferentials;

. Relationships between limiting Fréchet subdifferentials, approximate subdifferentials and other subdifferentials considered here;

. Metric inequalities;

. Normal cones formulae;

. Tangent cones formulae;

. Characterization of interior points of closed sets.

The limiting Fréchet subdifferentials [38] is obtained as a weak-star sequential upper limit of $\varepsilon-$ Fréchet subdifferentials while the approximate subdifferential [18-19] is obtained as a weak-star topological upper limit of (lower) subdifferentials. Both are infinite dimensional extensions of the nonconvex construction by Mordukhovich [41-42] and Ioffe [17]. Note that this finite construction is minimal (as sets) among all possible subdifferentials satisfying one or another set of conditions. This make it the finest possible instrument in certain applications of nonsmooth analysis. The nonconvex limiting Fréchet subdifferential construction in *Asplund* Banach spaces inherits these properties and is always contained in the Clarke subdifferential [10]. The approximate subdifferential is defined for functions on locally convex spaces, and can contain strictly the Clarke subdifferential and limiting Fréchet subdifferentials. In Banach spaces, Ioffe [19] considered functions on Banach spaces and for them he defined a generated approximate subdifferential (G-subdifferential). This later construction is expressed geometrically, in the spirit of the original definition of the Clarke subdifferential of non-Lipschitz functions, in terms of the weak-star closure of the cone generated by the approximate subdifferential of the distance function of the epigraph. It is always contained in the Clarke subdifferential and contains the limiting Fréchet subdifferentials.

Chain rules for these subdifferentials have been established in the papers [36-37], [18-19], [46-48], [31], [34], and references therein. In this paper, we give more calculus rules for limiting Fréchet subdifferentials and approximate subdifferential by using locally compact cones. Our approach benefits from Fabian's characterization of Asplund spaces [15-16] and the definition of the weak trustworthy spaces by Ioffe [20]. We establish more general formulae for limiting Fréchet subdifferentials by mean of metric inequality of sets which is a consequence of the notion of metric regularity. These results include, in particular, those of Kruger [36], Mordukhovich and Shao [47-48].

We investigate relationships between all these subdifferentials and some other ones considered here. Indeed, invoking recent results by Jourani and Thibault [35] which use weak-star sequential compactness theorem by Borwein and Fitzpatrick [4], we show that, under some assumptions expressed in terms of locally compact cones, the limiting Fréchet subdifferentials, approximate subdifferential, G-subdifferential and sequential approximate subdifferential constructions coincide for non-Lipschitz functions on *weakly compactly generated Asplund* spaces.

We establish the corresponding normal cones formulae. We show, in particular, that the notion of the $b-$normal cones to a set, introduced jointly with Thibault in [25], coincides with the previous ones in the case where the set satisfies a condition which is formulated in terms of locally compact cones. Therefore we show that under this condition, the approximate normal cone, which can be in general greater than the Clarke normal cone in infinite dimension, is equal to the so called $G-$normal, i.e., the cone generated by the approximate subdifferential of the distance function and this allows us to show that the Clarke normal cone is the closed convex hull of the approximate normal cone. This result has been established by Ioffe [18-19] for sets which are epi-lipschitz and our condition is automatically satisfied for these sets. The metric inequalities are the key in the establishment of these formulae.

We use the Ekeland variational principle [14] to show that, when the space is Asplund, the Clarke tangent cones [10] contains the limits inferior of the closed convex hull of the weak contingent cones. This result has been established by Borwein and Strojwas [7] in the case where the space is reflexive and by Aubin and Frankowska [2] in the case where the space is uniformly smooth and the norm of its topological dual is Fréchet differentiable off the origin. Note that both situations imply that the space is Asplund. We prove that the equality holds under conditions expressed in terms of locally compact cones. More precisely we show that these conditions ensure that the Clarke tangent cone is exactly the limits inferior of the closed convex hull of the contingent or Bouligand cone and the limits inferior of the closed convex hull of the weak contingent.

Our notation is basically standard. For any Banach space X and its topological dual X^* we denote by B_X and B_{X^*} their closed unit balls. The abbreviations "cl*", "int" and "co" are for weak-star closure, interior and convex hull. As usual, dom$f$ and epi$f$ of an arbitrary extended real-valued function f stand for the domain and the epigraph

$$\mathrm{dom} f = \{x : \quad f(x) < +\infty\}$$

$$\mathrm{epi} f = \{(x, r) : \quad f(x) \leq r\}.$$

The indicator function of an arbitrary set C is defined by $\Psi_C(x) = 0$ if $x \in C$ and $\Psi_C(x) = +\infty$ if $x \notin C$. The negative polar of a set K is the set

$$K^0 = \{x^* \in X^* : \quad \langle x^*, h\rangle \leq 0 \quad \forall h \in K\}.$$

2. Some properties of locally compact cones

Recall that a set K^* in X^* is *weak-star locally compact* (for short locally compact) if every point of K^* lies in a weak-star open set V such that $cl^*(V) \cap K$ is weak-star compact. We begin this section by recalling the following important results concerning these cones.

Proposition 2.1 *(Borwein 3]). Let X be a normed vector space. Let Ω be a closed convex subset of X and suppose that Ω contains zero. The following assertions are equivalent :*
(a) $0 \in$ int $(\Omega + H)$ for some (pre-) compact convex set H ;
(b) $0 \in$ int$(\Omega + \Sigma)$ for some finite dimensional compact convex set Σ ;
(c) the polar set Ω^0 is locally compact.

Proposition 2.2 *(Loewen [40]). Let $K^* \subset X^*$ be a locally compact cone. For a net (x_i^*) in K^*, one has*

$$x_i^* \xrightarrow{w^*} 0 \Leftrightarrow \| x_i^* \| \longrightarrow 0.$$

Remark. In fact this result is given by Loewen in reflexive Banach spaces and for sequences instead of nets. But its proof works in general Banach spaces and for nets.

In [40], Loewen gave the following important example of locally compact cones.

Proposition 2.3 *(Loewen [40]). Let $H \subset X$ be a norm-compact set. Then the following cone*

$$c(H) = \{x^* \in X^* : \| x^* \| \leq \max_{h \in H} \langle x^*, h \rangle\}$$

is weak-star closed and locally compact.

Proposition 2.4 *(Jourani [28]). Let $K \subset X$ be a closed convex cone. Then the following assertions are equivalent :*
(a) K^0 is locally compact ;
(b) there exists a norm compact set $H \subset X$ such that $K^0 \subset c(H)$;
(c) there exists a norm compact set $H' \subset X$ such that $0 \in$ int$(K + H')$.

In the case of reflexive Banach spaces we obtain the following characterization.

Proposition 2.5. *Let K be a closed convex cone of a reflexive Banach space X. Then the following assertions are equivalent :*
(a) $J(K)$ is weak-star locally compact ;
(b) K is weak locally compact ;
(c) there exists a norm compact set $H^* \subset X^*$ such that

$$K \subset c(H^*).$$

Here J denotes the canonical injection from X into X^{**}, where X^{**} denotes the topological bidual of X.

It is easy to verify the following result.

Proposition 2.6. *Let $K \subset X$ be a cone. Then the following are equivalent :*

(a) $intK^0 \neq \emptyset$.

(b) there exists $x^* \in X^*$ such that

$$K \subset \{x \in X : \quad \| x \| \leq \langle x^*, x\rangle\}.$$

Note also that the negative polar of any closed convex cone with nonempty interior is weak-star locally compact.

A subset K of X is called a *Bishop-Phelps cone* if there are some $x^* \in -K^0$, an equivalent norm $\| \|'$ and $\alpha \in (0, 1]$ such that

$$K = \{x \in X : \alpha \| x \|' \leq \langle x^*, x\rangle\}.$$

It is easy to see (via Proposition 2.5) that when X is a reflexive Banach space then the Bishop-Phelps cone is weak locally compact.

Note that this cone may have non interior.

Example. *The cone*

$$C = \{x \in L_2[a, b] : x(t) \geq 0 \ a.e. \ on \ [a, b]\}$$

has an unbounded base

$$B = \{x \in C : \int_a^b x(t)dt = 1\}.$$

So, by Theorem 3.2 in [51], C is not representable as a Bishop-Phelps cone. But if we consider (Jahn [23]) the cone generated by the set

$$B_\alpha = \{x \in B : \int_a^b (x(t))^2 dt \leq \alpha\},$$

then it is representable as a Bishop-Phelps cone and its interior is empty.

We may also establish the following proposition by using the result in [51].

Proposition 2.7. *Let K be a closed convex cone in X with $K \neq \{0\}$. Then the following assertions are equivalent :*

(a) int $K^0 \neq \emptyset$;

(b) K is representable as a Bishop-Phelps cone.

Proof. $(b) \Longrightarrow (a)$: is obvious.

$(a) \Longrightarrow (b)$: Let $x^* \in X^*$ be such that $x^* + B_{X^*} \subset K^0$. Then

$$K \subset \{x \in X : \quad \|x\| \leq \langle -x^*, x\rangle\}.$$

Set $\mathcal{B} = \{x \in K : \langle -x^*, x\rangle = 1\}$. Then $\mathcal{B}$ is a closed bounded base of K and hence, by Theorem 3.2 in [51], K is representable as a Bishop-Phelps cone.

To end up with this section let us establish the following theorem which gives complete characterization of locally compact cones.

Theorem 2.8. *Let $K^* \subset X^*$ be a cone and consider the following conditions :*
(a) K^* is locally compact;
(b) there exists a finite dimensional space $L \subset X$ such that $L^\perp \cap K^* = \{0\}$;
(c) there exist $h_1, \cdots, h_k \in X$ and an integer n such that

$$K^* \subset \{x^* \in X^* : \quad \|x^*\| \leq n \max_{i=1,\cdots,k} |\langle x^*, h_i\rangle|\}.$$

Then $(a) \Rightarrow (b)$. If K^* is closed we have $(a) \Leftrightarrow (c)$.
where $L^\perp$ is an annihilator of L, i.e., $L^\perp = \{x^* \in X^* : \langle x^*, h\rangle = 0, \forall h \in L\}$.

Proof. $(a) \Rightarrow (b)$: Suppose the contrary. Then for all finite dimensional space $L \subset X$ there exists $x^*_L \in L^\perp \cap K^*$ such that $\| x^*_L \| = 1$. So (x^*_L) is a net in K^* which is locally compact and extracting subnet we may assume that $x^*_L \xrightarrow{w^*} x^*$. As $x^*_L \in L^\perp$, for all finite dimensional space $L \subset X$ we conclude that $x^* = 0$. Thus, by Proposition 2.2, $\| x^*_L \| \to 0$ and this contradiction completes the proof of this implication.

$(c) \Rightarrow (a)$: follows from Proposition 2.3.

$(a) \Rightarrow (c)$: since $(a) \Rightarrow (b)$ there exists a finite dimensional space L with basis $h_1, \cdots, h_k$ such that $L^\perp \cap K^* = \{0\}$. We claim that there exists an integer n such that

$$K^* \subset \{x^* \in X^* : \quad \|x^*\| \leq n \max_{i=1,\cdots,k} |\langle x^*, h_i\rangle|\}.$$

Suppose the contrary. Then for all integer n there exists $x^*_n \in K^*$ such that

$$\|x^*_n\| > n|\langle x^*_n, h_i\rangle|, \quad \forall i = 1, \cdots, k.$$

We may assume that $\|x^*_n\| = 1$, for all n, and (x^*_n) has some subnet converging to $x^* \in K^*$ with $x^* \neq 0$ (since K^* is locally compact and closed). Thus for all n

$$\frac{1}{n} > |\langle x^*_n, h_i\rangle|, \quad \forall i = 1, \cdots, k$$

and hence $\langle x^*, h_i\rangle = 0$ for $i = 1, \cdots, k$, or equivalently $x^* \in L^\perp$ and this contradicts the fact that $L^\perp \cap K^* = \{0\}$.

Remark. *Let K^* be a closed convex locally compact cone. Set $K^* = K^0$ where K is a closed convex cone. Then [28] the following are equivalent*
(a) $L^\perp \cap K^* = \{0\}$
(b) $L + K = X$
where L is a finite dimensional subspace of X.

3. Subdifferential calculus

It is well known in the litterature that some spaces can be characterized or expressed in terms of *fuzzy* sum rules related to some subdifferentials. Let us give two examples of such spaces. The first one concerns *Asplund spaces [1], i.e., Banach spaces on which every continuous convex function is Fréchet differentiable at a dense set of points.* We refer the reader, for example to the papers by Fabian [15-16], and references therein. One of the most characterizations is the following one which is expressed in terms of the Fréchet ε-subdifferential $\partial_\varepsilon^F f(x)$ of some extended real-valued function f on X at x

$$\partial_\varepsilon^F f(x) = \{x^* \in X^* : \liminf_{h \to 0} \frac{f(x+h) - f(x) - \langle x^*, h\rangle}{\| h \|} \geq -\varepsilon\}$$

if $x \in$dom f and $\partial_\varepsilon^F f(x) = \emptyset$ if $x \notin$ dom$f$. The following assertions are equivalent (Fabian [15-16])

(*i*) *X is Asplund;*

(*ii*) *for any $\varepsilon \geq 0, \delta > 0, \gamma > 0$ and any extended real-valued lower semicontinuous functions f_1 and f_2 on X and $x_0 \in$ dom $f_1 \cap$ dom f_2 with f_1 locally Lipschitzian at x_0, one has*

$$\begin{aligned}\partial_\varepsilon^F (f_1 + f_2)(x_0) \subset \{\hat{\partial}_F f_1(x_1) + \hat{\partial}_F f(x_2) : \quad & x_i \in x_0 + \delta B_X, \\ & |f_i(x_i) - f_i(x_0)| < \delta, i = 1, 2\} + (\varepsilon + \gamma) B_{X^*}.\end{aligned}$$

where $\hat{\partial} f(x)$ is equal to $\partial_\varepsilon^F f(x)$ for $\varepsilon = 0$.

Note that this class of spaces includes all spaces with Fréchet differentiable renorms as well as those of separable duals (see Diesel [13]). The second example, which is also very interesting, is the *weak-trusworthy* spaces (for short WT-spaces) introduced by Ioffe [20]. The definition of these spaces is given in terms of the Dini ε-subdifferential $\partial_\varepsilon^- f(x)$ of extended real-valued functions f on X at x

$$\partial_\varepsilon^- f(x) = \{x^* \in X^* : \langle x^*, h\rangle \leq \liminf_{\substack{t \to 0^+ \\ u \to h}} t^{-1}(f(x+tu) - f(x)) + \varepsilon \| h \|; \forall h \in X\}$$

if $x \in$ dom f and $\partial_\varepsilon f(x) = \emptyset$ if $x \notin$ dom f. *A space X is said to be a WT-space if for each extended real-valued lower semicontinuous functions f_1 and f_2 on X, $x \in domf_1 \cap domf_2$ and $\varepsilon > 0$*

$$\partial_\varepsilon^- (f_1 + f_2)(x) \subset \limsup_{\substack{x_i \xrightarrow{f_i} x \\ i=1,2}} (\partial_\varepsilon^- f_1(x_1) + \partial_\varepsilon^- f_2(x_2)). \tag{3.1}$$

This class of spaces contains the previous one and every Banach space with Gateaux differentiable renorms is a WT-space (see Ioffe [20]). So these fuzzy sum rules give us an idea in the introduction by Kruger and Mordukhovich [38] of the notion of the *limiting*

Fréchet ε-subdifferential $\partial_F f(x_0)$ and by Ioffe [18-19] of the notion of the *approximate subdifferential* $\partial_A f(x_0)$ which have exact calculus rules

$$\partial_F f(x_0) = \text{seq-}\limsup_{\substack{x \xrightarrow{f} x_0 \\ \varepsilon \longrightarrow 0^+}} \partial_\varepsilon^F f(x)$$

and

$$\partial_A f(x_0) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{\substack{x \xrightarrow{f} x_0 \\ \varepsilon \to 0^+}} \partial_\varepsilon^- f_{x+L}(x)$$

where $f_S(x) = f(x)$ if $x \in S$ and $f_S(x) = +\infty$ otherwise and $\mathcal{F}(X)$ denotes the collection of all finite dimensional subspaces of X. In the case where X is Asplund, we easily show that when f_1 is locally Lipschitzian at x_0 and f_2 is lower semicontinuous

$$\partial_F(f_1 + f_2)(x_0) \subset \partial_F f_1(x_0) + \partial_F f_2(x_0). \tag{3.2}$$

When X is a WT-space we obtain (Ioffe [18]) the following simple definitions of approximate subdifferentials for an extended real-valued lower semicontinuous function f on X

$$\partial_A f(x_0) = \limsup_{\substack{x \xrightarrow{f} x_0 \\ \varepsilon \to 0^+}} \partial_\varepsilon^- f(x).$$

As for the limiting Fréchet ε-subdifferential, the approximate subdifferential obeys to the following : if f_1 is locally Lipschitzian at x_0 and f_2 is l.s.c. then

$$\partial_A(f_1 + f_2)(x_0) \subset \partial_A f_1(x_0) + \partial_A f_2(x_0). \tag{3.3}$$

To see the difficulty in the case where neither f_1 nor f_2 is locally Lipschitzian let us give the proof of (3.3) in WT-spaces. For $x^* \in \partial_A(f_1 + f_2)(x_0)$, there are nets $x_i \xrightarrow{f_1+f_2} x_0$, $\varepsilon_i \to 0^+$, $x_i^* \xrightarrow{w^*} x^*$ such that

$$x_i^* \in \partial_{\varepsilon_i}^-(f_1 + f_2)(x_i)$$

and, by (3.1), there are nets $u_i \xrightarrow{f_1} x_0$, $v_i \xrightarrow{f_2} x_0$, $u_i^* \in \partial_{\varepsilon_i}^- f_1(u_i)$, $v_i^* \in \partial_{\varepsilon_i}^- f_2(v_i)$ such that

$$u_i^* + v_i^* \xrightarrow{w^*} x^*.$$

So since f_1 is locally Lipschitz, (u_i^*) is bounded and as the unit ball of X^* is weak-star compact, extracting subnet we may assume that $u_i^* \xrightarrow{w^*} u^*$ and hence $v_i^* \xrightarrow{w^*} x^* - u^*$. Thus $x^* \in \partial_A f_1(x_0) + \partial_A f_2(x_0)$.

Our aim is to give weaker assumptions ensuring the boundedness of (u_i^*). In other word, we are concerning with finding (weaker) assumptions ensuring the following inclusions

$$\limsup_{\substack{u \xrightarrow{C} x_0 \\ v \xrightarrow{D} x_0}} (F(u) + G(v)) \subset \limsup_{u \xrightarrow{C} x_0} F(u) + \limsup_{v \xrightarrow{D} x_0} G(v)$$

where F and G are multivalued mappings defined respectively on metric spaces C and D. But first let us give a nonsmooth characterization of Asplund spaces. For this, we consider the sets (Ioffe [19])

$$S_1 = \{(x, \alpha, \beta) \in X \times \mathbb{R} \times \mathbb{R} : f_1(x) \leq \alpha\}$$

$$S_2 = \{(x, \alpha, \beta) \in X \times \mathbb{R} \times \mathbb{R} : f_2(x) \leq \beta\}$$

$$S = \{(x, \alpha, \beta) \in X \times \mathbb{R} \times \mathbb{R} : f_1(x) + f_2(x) \leq \alpha + \beta\}$$

for extended real-valued functions f_1 and f_2 on X.

Lemma 3.1. *Let $q > 0$, $(x_i, \alpha_i, \beta_i) \in S_i$ and $(x_i^*, \alpha_i^*, \beta_i^*) \in \hat{\partial}^F d((x_i, \alpha_i, \beta_i), S_i)$, $i = 1, 2$, with $|\alpha_1^*| \geq q$ (resp. $|\beta_2| \geq q$). Then $\alpha_1 = f_1(x_1)$ (resp. $\beta_2 = f_2(x_2)$.*

We say that f_1 and f_2 satisfies the *metric inequality (MI)* at x_0 if there exist $r > 0$ and $a > 0$ such that

$$d((x, \alpha, \beta), S_1 \cap S_2) \leq a[d((x, \alpha, \beta), S_1) + d((x, \alpha, \beta), S_2)]$$

for all $x \in x_0 + rB_X, \alpha \in f_1(x_0) + rB_{\mathbb{R}}$ and $\beta \in f_2(x_0) + rB_{\mathbb{R}}$.

In order to state our nonsmooth characterization of Asplund spaces we recall the following result.

Proposition 3.2. *(Jourani and Thibault [31]). Let C be a nonempty closed subset of X and $x \notin C$. Then for all $\varepsilon \in]0, 1[$ and $x^* \in \partial_\varepsilon^F d(x, C)$*

$$1 - \varepsilon \leq \| x^* \| .$$

Proposition 3.3. *Let X be a Banach space. Then the following assertions are equivalent*

(i) X is Asplund.

(ii) For every extended real-valued lower semi-continuous functions f_1 and f_2 on X and $x_0 \in$ $domf_1 \cap$ dom f_2 satisfying the metric inequality (MI) at x_0, we have

$$x^* \in \hat{\partial}(f_1 + f_2)(x_0) \Rightarrow \forall \gamma > 0, \forall \delta > 0, \forall b_1 > a \| x^* \| +1, b_2 > 2a \| x^* \| +3$$

$$\exists x_i \in x_0 + \gamma B_X, f_i(x_i) \in f_i(x_0) + \gamma B_{\mathbb{R}}, \text{ and}$$

$$x_i^* \in \hat{\partial} f_i(x_i), \quad \| x_i^* \| \leq 2b_i, \, i = 1, 2 \text{ such that}$$

$$\| x^* - x_1^* - x_2^* \| \leq 2\delta(1 + b_1 + b_2).$$

Proof. $(i) \Rightarrow (ii)$: Let $x^* \in \hat{\partial}_F(f_1+f_2)(x_0)$. Then for all $\delta > 0$ there exists $r > 0$ such that

$$f_1(x)+f_2(x)-f_1(x_0)-f_2(x_0)-\langle x^*, x-x_0\rangle \geq -(\frac{\delta}{2}) \parallel x-x_0 \parallel, \forall x \in B(x_0,\gamma).$$

So for all $(x,\alpha,\beta) \in S \cap (x_0+rB_X)\times(f_1(x_0)+rB_{\mathbb{R}})\times(f_2(x_0)+rB_{\mathbb{R}})$

$$\alpha+\beta-f_1(x_0)-f_2(x_0)-\langle x^*, x-x_0\rangle \geq -(\frac{\delta}{2}) \parallel x-x_0 \parallel .$$

Taking into account the simple fact that $S_1 \cap S_2 \subset S$ and the metric inequality we get the existence of $s > 0$ such that for all $b_1, b_2 > a \parallel x^* \parallel +3$

$$b_1 d(x,\alpha,\beta,S_1)+b_2 d(x,\alpha,\beta,S_2)+\alpha+\beta-f_1(x_0)-f_2(x_0)-\langle x^*, x-x_0\rangle \geq -(\frac{\delta}{2}) \parallel x-x_0 \parallel$$

for all $x \in x_0+sB_X$, $\alpha \in f_1(x_0)+sB_{\mathbb{R}}$, $\beta \in f_2(x_0)+sB_{\mathbb{R}}$. Since X is Asplund it follows that for all $\gamma > 0$

$$\begin{aligned}(x^*,-1,-1) &\in \hat{\partial}_{\delta/2}(b_1 d(.,S_1)+b_2 d(.,S_2))(x_0,f_1(x_0),f_2(x_0))\\ &\subset \bigcup\{b_1\hat{\partial}_F d(x_1,\alpha_1,\beta_1,S_1)+b_2\hat{\partial}_F d(x_2,\alpha_2,\beta_2,S_2) :\ x_i \in x_0+\gamma B_X,\\ &\qquad \alpha_i \in f_1(x_0)+\gamma B_{\mathbb{R}}, \beta_i \in f_2(x_0)+\gamma B_{\mathbb{R}}\ i=1,2\}+\delta(B_{X^*}\times B_R\times B_R).\end{aligned}$$

So there exist $x_i \in x_0+\gamma B_X, \alpha_i \in f_1(x_0)+\gamma B_{\mathbb{R}}, \beta_i \in f_2(x_0)+\gamma B_{\mathbb{R}}, (x_i^*,\alpha_i^*,\beta_i^*) \in b_i\hat{\partial} d(x_i,\alpha_i,b_i,S_i)$, $i=1,2$, such that

$$\parallel x^*-x_1^*-x_2^* \parallel \leq \delta \tag{3.4}$$

$$\parallel \alpha_1^*+1 \parallel \leq \delta \tag{3.5}$$

$$\parallel \beta_2^*+1 \parallel \leq \delta \tag{3.6}$$

$$\alpha_2^*=0, \quad \beta_1^*=0, \quad \parallel x_i^* \parallel \leq b_i$$

First $(x_1,\alpha_1,\beta_1) \in S_1$ because otherwise, by Proposition 3.2

$$(1-\gamma) \leq \frac{1}{b_1}(\parallel x_1^* \parallel + |\alpha_1^*| + |\beta_1^*|),$$

and since $\parallel x_1^* \parallel \leq \parallel x^* \parallel + \parallel x_2^* \parallel +\varepsilon+\delta+\gamma \leq \parallel x^* \parallel +b_2+\delta+\varepsilon+\gamma$ and b_1 is arbitrary we obtain a contradiction. We also show that $(x_2,\alpha_2,\beta_2) \in S_2$. By lemma 3.1, $\alpha_1 = f_1(x_1)$ and $\beta_2 = f_2(x_2)$ and hence $\frac{x_i^*}{-\alpha_i^*} \in \hat{\partial} f_i(x_i)$ and the proof is terminated since

$$\parallel x^*+\frac{x_1^*}{\alpha_1^*}+\frac{x_2^*}{\alpha_2^*} \parallel \leq \delta + \left|\frac{\alpha_1^*+1}{\alpha_1^*}\right| b_1 + \left|\frac{\alpha_2^*+1}{\alpha_1^*}\right| b_2$$

$$\leq 2\delta(1 + b_1 + b_2).$$

$(ii) \Rightarrow (i)$: It suffices to invoque the following result by Fabian [15-16]

X is Asplund iff

(ii') for every extended-real valued functions f_1 and f_2 in X, $x_0 \in$ dom $f_1 \cap$ dom f_2 with f_2 locally Lipschitzian at x_0 , for all $\varepsilon \geq 0, \gamma > 0$

$$\partial_\varepsilon^F(f_1 + f_2)(x_0) \subset \{\hat{\partial}_F f_1(x_1) + \hat{\partial} f_2(x_2) : \quad x_i \in x_0 + \gamma B_{X^*} \\ f_i(x_i) \in f_i(x_0) + \gamma B_{\mathbb{R}}\} + (\varepsilon + \gamma)B_X^*$$

and to show that $(ii) \Rightarrow (ii')$.

Theorem 3.3. *Let X be Asplund space and f_1 and f_2 two extended real-valued lower semi-continuous functions on X satisfying the metric inequality (MI) at $x_0 \in domf_1 \cap domf_2$. Then*

$$\partial_F(f_1 + f_2)(x_0) \subset \partial_F f_1(x_0) + \partial_F f_2(x_0).$$

Let U and V be topological spaces and $F : U \rightrightarrows V$ be a multivalued mapping. The singular multivalued mapping (or the recession multivalued mapping) F^∞ of F at x_0 is defined by

$$F^\infty(x_0) = \limsup_{\substack{x \xrightarrow{U} x_0 \\ \lambda \to 0^+}} \lambda F(x).$$

Note that if F is bounded on some neighbourhood of x_0, then $F^\infty(x_0) = \{0\}$.

Theorem 3.4. *Let C and D be metric spaces and $F : C \rightrightarrows X^*$, $G : D \rightrightarrows X^*$ be multivalued mappings. Suppose that*

(a) there exist a locally compact cone $K^* \subset X^*$, a bounded set $B^* \subset X^*$ and a neighbourhood V of x_0 such that

$$F(x) \subset K^* + B^*, \forall x \in V \cap C$$

(b) $F^\infty(x_0) \cap (-G^\infty(x_0)) = \{0\}$.

Then

$$\limsup_{\substack{v \xrightarrow{D} x_0 \\ u \xrightarrow{C} x_0}} (F(u) + G(v)) \subset \limsup_{u \xrightarrow{C} x_0} F(u) + \limsup_{v \xrightarrow{D} x_0} G(v).$$

If in addition X is Asplund

$$seq - \limsup_{\substack{u \xrightarrow{C} x_0 \\ v \xrightarrow{D} x_0}} (F(u) + G(v)) \subset \ seq - \limsup_{u \xrightarrow{C} x_0} F(u) + \ seq - \limsup_{v \xrightarrow{D} x_0} G(v).$$

Proof. Let $x^* \in \limsup\limits_{\substack{u \xrightarrow{C} x_0 \\ v \xrightarrow{D} x_0}} (F(u) + G(v))$. Then there are nets $u_i \xrightarrow{C} x_0, v_i \xrightarrow{D} x_0$, $u_i^* \in F(u_i)$ and $v_i^* \in G(v_i)$ such that $u_i^* + v_i^* \xrightarrow{w^*} x^*$.

We claim that (u_i^*) is bounded. So suppose the contrary and without loss of generality we assume $\| u_i^* \| \to +\infty$. Set $x_i^* = \frac{u_i^*}{\|u_i^*\|}$, $y_i^* = \frac{v_i^*}{\|u_i^*\|}$. Since (x_i^*) is bounded we may assume $x_i^* \xrightarrow{w^*} x^*$ and hence $y_i^* \xrightarrow{w^*} -x^*$. Now, by assumptions $x_i^* \in K^* + \frac{1}{\|u_i^*\|} B^*$ and since K^* is locally compact we get $x^* \neq 0$. It follows that

$$x^* \in F^\infty(x_0) \cap (-G^\infty(x_0))$$

and this contradiction completes the proof of the first part of the theorem for the second one it suffices to use the fact that the closed unit ball of the dual of an Asplund space is sequentially weak-star compact.

Now we use this result and Proposition 3.2 to give the following chain rules for approximate subdifferentials and limiting Fréchet subdifferentials.

Theorem 3.5. *Let f_1 and f_2 be two extended real-valued lower semicontinuous functions on X and let $x_0 \in$ dom $f_1 \cap$ dom f_2. Suppose that*

(i) X is a WT-space (resp. an Asplund space) ;

(ii) there exist a locally compact cone K^*, a bounded set B^* and $r > 0$ such that

$$\partial_\varepsilon^- f_1(x) \subset K^* + B^*, \forall x \in x_0 + rB_X$$

$$(\text{ resp } \hat{\partial}_F d(x, \alpha, epi f_1) \subset K^* \times \mathbb{R}, \forall (x, \alpha) \in (x_0 + rB_X) \times (f_1(x_0) + rB_{\mathbb{R}}) \cap epi f_1)$$

(iii) $\partial_A^\infty f_1(x_0) \cap (-\partial_A^\infty f_2(x_0)) = \{0\}$. (resp. $\partial_F^\infty f_1(x_0) \cap (-\partial_F^\infty f_2(x_0)) = \{0\}$). Then

$$\partial_A (f_1 + f_2)(x_0) \subset \partial_A f_1(x_0) + \partial_A f_2(x_0),$$

$$(\text{ resp. } \partial_F (f_1 + f_2)(x_0) \subset \partial_F f_1(x_0) + \partial_F f_2(x_0)).$$

Proof. Let us prove the second inclusion. Set $C =$ graph (f_1), $D =$ graph (f_2), $F(x, \alpha) = \hat{\partial}_F f_1(x), G(x, \alpha) = \hat{\partial}_F f_2(x)$. The assumption (ii) implies that

$$F(x, \alpha) \subset K^*, \quad \forall (x, \alpha) \in (x_0 + rB_x) \times (f_1(x_0) + rB_{\mathbf{R}}) \cap C,$$

and (iii) ensures that

$$F^\infty(x_0, f_1(x_0)) \cap (-G^\infty(x_0, f_2(x_0)) = \{0\}.$$

So Theorem 3.4 implies that

$$\text{seq-}\limsup_{\substack{(v,\beta)\xrightarrow{D}(x_0,f_2(x_0))\\(u,\alpha)\xrightarrow{C}(x_0,f_1(x_0))}} [F(u,\alpha)+G(v,\beta)) \subset$$

$$\text{seq-}\limsup_{(u,\alpha)\xrightarrow{C}(x_0,f_1(x_0))} F(u,\alpha) + \text{seq-}\limsup_{(v,\beta)\xrightarrow{D}(x_0,f_2(x_0))} G(v,\beta)$$

or equivalently

$$\text{seq-}\limsup_{\substack{v\xrightarrow{f_2}x_0\\u\xrightarrow{f_1}x_0}} (\hat{\partial}_F f_1(u) + \hat{\partial}_F f_2(v)) \subset \text{seq-}\limsup_{u\xrightarrow{f_1}x_0} \hat{\partial}_F f_1(u) + \text{seq-}\limsup_{v\xrightarrow{f_2}x_0} \hat{\partial}_F f_2(v).$$

The proof is terminated by using Theorem 2.9 in [47] and Proposition 3.2.

Remark. *The results of Theorem 3.4 remain true if we replace X^* by any reflexive Banach space, the local weak-star compactness by the local weak-compactness and the weak-star limit superior by the weak limit superior.*

4. Relationships between limiting Fréchet subdifferentials and approximate subdifferentials

In this section we establish a connection between limiting Fréchet subdifferentials, approximate subdifferential and the following sequential constructions of approximate subdifferential considered in [47]

$$\partial_A^{\sigma_1} f(x_0) = \text{ seq} - \limsup_{\substack{\varepsilon\longrightarrow 0+\\ x\xrightarrow{f}x_0}} \partial_\varepsilon^- f(x)$$

and

$$\partial_A^{\sigma_2} f(x_0) = \text{ seq } - \limsup_{x\xrightarrow{f}x_0} \partial^- f(x).$$

Using some recent results of Borwein and Fitzpatrick [4], Mordukhovich and Shao [47] showed that in Asplund space and for a locally Lipschitz function f at x_0

$$cl^*(\partial_F f(x_0)) = cl^*(\partial_A^{\sigma_1} f(x_0)) = \partial_A f(x_0). \tag{4.1}$$

They showed that in weakly compactly generated Asplund spaces the following equalities hold

$$\partial_F f(x_0) = \partial_A^{\sigma_1} f(x_0) = \partial_A f(x_0) \tag{4.2}$$

for a locally Lipschitz function f at x_0.

Recall that X is weakly compactly generated (WCG) if there exists a weakly compact set K such that $X = \text{cl}(\text{span}(K))$. Clearly all reflexive Banach spaces and all separable

Banach spaces are weakly compactly generated. For the case of Asplund spaces, there are precise characterization of the WCG property (see [12-13]) which implies, in particular, the existence of a Fréchet differentiable renorms and the weak-star sequential compactness of the closed unit ball of its topological dual.

Using Proposition 3.1 in [35], we easily show that

$$\partial_A f(x_0) = \partial_A^{\sigma_1} f(x_0) \tag{4.3}$$

provided that X is a W C G space and there exist a locally compact cone K^* and $r > 0$ such that

$$\partial_\varepsilon^- f(x) \subset K^* + \rho(\varepsilon) B_{X^*}, \forall x \in x_0 + rB_X, \forall \varepsilon \in]0, r[. \tag{4.4}$$

where $\rho(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.

Our aim in this section is to use this result to extend that of Mordukhovich and Shao [47] to the non-Lipschitz case.

Theorem 4.1. *Let X be Asplund space and f be an extended real-valued lower semi-continuous function on X. Let K^* be a closed and locally compact cone and let $r > 0$. Then*

$$cl^*(\partial_A^{\sigma_1} f(x_0)) = cl^*(\partial_A^{\sigma_2} f(x_0)) = cl^*(\partial_F f(x_0)) \tag{4.5}$$

provided that

$$\partial_F d(x, \alpha;\ epi\ f) \subset K^* \times \mathbb{R}, \forall (x, \alpha) \in (x_0 + rB_X) \times (f(x_0) + rB_{\mathbb{R}}) \cap\ epif. \tag{4.6}$$

If X is WCG and

$$\partial_\varepsilon^- \psi_{\ epif}\ (x, \alpha) \subset K^* \times \mathbb{R}, \forall (x, \alpha) \in (x_0 + rB_X) \times (f(x_0) + rB_{\mathbb{R}} \cap\ epif\ . \tag{4.7}$$

we have

$$\partial_A f(x_0) = \partial_F f(x_0) = \partial_A^{\sigma_1} f(x_0) = \partial_A^{\sigma_2} f(x_0). \tag{4.8}$$

Note that (4.6) and (4.7) are equivalent in Asplund spaces (see Theorem 5.6).

Proof. (1) It suffices to show in (4.5) that

$$\partial_A^{\sigma_1} f(x_0) \subset cl^*(\partial_F f(x_0))$$

because, by Theorem 2.9 in [47],

$$\partial_F f(x_0) = \ \text{seq} - \limsup_{x \xrightarrow{f} x_0} \hat{\partial}_F f(x) \tag{4.9}$$

and

$$\partial_F f(x_0) \subset \partial_A^{\sigma_2} f(x_0) \subset \partial_A^{\sigma_1} f(x_0).$$

So let $x^* \in \partial_A^{\sigma_1} f(x_0)$. Then there are sequences $x_n \xrightarrow{f} x_0$, $\varepsilon_n \to 0^+$ and $x_n^* \xrightarrow{w^*} x^+$ such that

$$x_n^* \in \partial_{\varepsilon_n}^- f(x_n), \text{ for all } n.$$

Let V be a weak-star neighbourhood of 0. Then by Theorem 2.8, there exists a finite dimensional subspace L of X such that $L^\perp \subset V$ and

$$L^\perp \cap K^* = \{0\}. \tag{4.10}$$

Since $x_n^* \in \partial_{\varepsilon_n}^- f(x_n)$, the function

$$x \to f(x) + \psi_{x_n+L}(x) - \langle x_n^*, x - x_n \rangle + 2\varepsilon_n \parallel x - x_n \parallel$$

attains a local minimum at x_n (Lemma 1 in [17]). It follows from (4.7) and (4.10) that $\partial_F f(x_n) \cap L^\perp = \{0\}$ and Theorem 3.5 implies the existence of $u_n^* \in \partial_F f(x_n)$ and $v_n^* \in L^\perp$ such that

$$u_n^* + v_n^* \xrightarrow{w^*} x^*.$$

Note that, by (4.10), (u_n^*) is bounded. So since the closed unit ball in X^* is weak-star sequentially compact (because X is Asplund), we may assume that $u_n^* \xrightarrow{w^*} u^*$ and hence $v_n^* \to x^* - u^*$. Thus

$$x^* \in \partial_F f(x_0) + L^\perp \subset \partial_F f(x_0) + V$$

and hence $x^* \in \mathrm{cl}^* \partial_F f(x_0)$. For the second part it suffices to use (4.5), (4.7), (4.3) and the following consequence of Proposition 3.1 in [35]

$$\partial_F f(x_0) = \limsup_{\substack{\varepsilon \to 0^+ \\ x \xrightarrow{f} x_0}} \partial_\varepsilon^F f(x).$$

5. Metric inequality

In section 3 we used metric inequalities for functions to characterize Asplund spaces. This concept terms out to be important for many applications some of which are considered in this paper. The aim of this section is to present conditions ensuring this metric inequality for sets and functions. Our results in this section are expressed in terms of an abstract subdifferential ∂, called *presubdifferential* [55], satisfying the following conditions.

Let $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}$ and $h : Y \to \mathbb{R}$ be locally Lipschitzian functions with g convex.

a_1) $\partial f(x) \subset X^*$;

a_2) $\partial f(x) = \partial g(x)$ *whenever f is convex and f and g coincide around x. Here the subdifferential of g is taken in the sense of convex analysis;*

a_3) $0 \in \partial f(x)$ *whenever f attains a local minimum at x;*

a_4) *for $h(x,y) = f(x) + h(y)$, $\partial h(x,y) \subset \partial f(x) \times \partial h(y)$;*

a_5) $\partial(f+g)(x) \subset \partial f(x) + \partial g(x)$;

a_6) $\partial f(x) = \limsup_{u \xrightarrow{f} x} \partial f(u)$ (resp. $\partial f(x) = seq - \limsup_{u \xrightarrow{f} x} \partial f(u)$ in the case where the space is Asplund).

The approximate subdifferential and the Clarke subdifferential are presubdifferentials and in Asplund spaces the limiting Fréchet subdifferentials is a presubdifferential.

Theorem 5.1. *Let C be a closed subset of X containing x and let H^* be a weak-star closed subset of X^* be such that*

$$\partial d(C, x) \cap H^* = \{0\}. \tag{1}$$

Suppose that there exist a locally compact cone $K^* \subset X^*$ and $r > 0$ such that

$$\partial d(C, u) \subset K^*, \forall u \in C \cap (x + rB_X). \tag{2}$$

Then there exist $s > 0$ and $a > 0$ such that for each family $\mathcal{D}$ of closed subsets D of X containing 0 and satisfying

$$\partial d(w, D) \subset -H^*, \forall w \in D \cap sB_X \tag{3}$$

we have

$$d(u, C \cap (v + D)) \leq ad(u - v, D) \tag{5.1}$$

for all $u \in C \cap (x + \frac{1}{a}B_X), v \in x + \frac{1}{a}B_X$ and $D \in \mathcal{D}$.

Proof. Suppose the contrary. Then for all integer n there exists a family $\mathcal{D}_n$ of closed sets D containing 0 and satisfying (3) and such that (5.1) does not hold. So there exist $u_n \in C \cap (x + \frac{1}{n}B_X)$, $v_n \in (x + \frac{1}{n}B_X)$ and $D_n \in \mathcal{D}_n$ such that

$$d(u_n, C \cap (v_n + D_n)) > nd(u_n - v_n, D_n)$$

Let $w_n \in D_n$ with

$$\| u_n - v_n - w_n \| < d(u_n - v_n, D_n) + \frac{1}{n} \min \left(\frac{1}{n}, d(u_n, C \cap (v_n + D_n)) - nd(u_n - v_n, D_n)\right). \tag{5.2}$$

Consider the function $f_n(u, w) = \| u - v_n - w \|$ and set $\varepsilon_n^2 = f_n(u_n, w_n)$, $\lambda_n = \min(n\varepsilon_n^2, \varepsilon_n)$. Note that $\varepsilon_n^2 > 0$ and $\varepsilon_n \to 0^+$. The function f_n verifies

$$f_n(u_n, w_n) \leq \inf_{(u,w) \in C \times D_n} f_n(u, w) + \varepsilon_n^2$$

and hence, by Ekeland's variational principle [14], there exist $u'_n \in C$ and $w'_n \in D_n$ such that (for $s_n = \frac{\varepsilon_n^2}{\lambda_n} = \max\left(\frac{1}{n}, \varepsilon_n\right)$)

$$\| u_n - u'_n \| + \| w_n - w'_n \| \leq \lambda_n. \tag{5.3}$$

$$f_n(u'_n, w'_n) \leq f_n(u,w) + s_n(\| u - u'_n \| + \| w - w'_n \|), \quad \forall (u,w) \in C \times D_n.$$

So by Proposition 2.3.4 in [10]

$$f_n(u'_n, w'_n) \leq f_n(u,w) + s_n(\| u - u'_n \| + \| w - w'_n \|) + (1 + s_n)(d(u,C) + d(w, D_n))$$

for (u,w) near (u'_n, w'_n). Thus, by $a_1) - a_5)$ of the definition of the presubdifferential ∂,

$$(0,0) \in \partial \| \,.\, - v_n - \,.\, \| (u'_n, w'_n) + (1 + s_n)(\partial d(C, u'_n) \times \partial d(D, w'_n)) + s_n(B_{X^*} \times B_{X^*}).$$

Since $u'_n - v_n - w'_n \neq 0$ (because otherwise $u'_n - v_n \in D_n$ and hence by (5.2) and (5.3) $\| u_n - u'_n \| \leq \lambda_n \leq n\varepsilon_n^2 < d(u_n, C \cap (v_n + D_n)) \leq \| u_n - u'_n \|$, a contradiction)

$$\partial \| \,.\, - v_n - \,.\, \| (u'_n, w'_n) \subset \{(x^*, -x^*) : \quad \| x^* \| = 1\}.$$

So there exists $x_n^* \in X^*$, $\| x_n^* \| = 1$, $u_n^* \in \partial d(C, u'_n)$ and $w_n^* \in \partial d(D_n, w'_n)$ such that

$$\| x_n^* + (1 + s_n)u_n^* \| \leq s_n$$

$$\| x_n^* - (1 + s_n)w_n^* \| \leq s_n.$$

Since $\| x_n^* \| = 1$, then extracting subnet if necessary we assume that $x_n^* \xrightarrow{w^*} x^*$ and hence $u_n^* \xrightarrow{w^*} x^*$ and $w_n^* \xrightarrow{w^*} -x^*$. Now, by (2) and Proposition 2.2, $x^* \neq 0$. Thus, by (2) and $a_6)$, $x^* \in \partial d(C,x)$ and, by (3), $-x^* \in -H^*$ and this contradicts (1) and the proof is terminated.

The proof of the following two theorems is similar.

Theorem 5.2. *Let C and D be closed subsets of X, with $x \in C \cap D$, be such that*

$$(1') \qquad \partial d(C,x) \cap (-\partial d(D,x)) = \{0\}.$$

Suppose that there exist a locally compact cone K^* and $r > 0$ such that (2) of Theorem 5.1 holds. Then there exists $a > 0$ such that

$$d(u, C \cap (v + D)) \leq a d(u - v, D)$$

for all $u \in C \cap (x + \frac{1}{a}B_X)$ and $v \in \frac{1}{a}B_X$.

Theorem 5.3. (*Uniform metric inequality*). *Let $K^* \subset X^*$ be a locally compact cone and H^* be a closed subset of X^*, with*

$$K^* \cap H^* = \{0\},$$

and let $x \in X$. Then there exist $s > 0$ and $a > 0$ such that for all family $\mathcal{C}$ of closed sets $C \subset X$ containing x and satisfying

$$\partial d(C,u) \subset K^*, \forall u \in C \cap (x + sB_X)$$

and all family $\mathcal{D}$ of closed sets $D \subset X$ containing 0 and satisfying

$$\partial d(D,u) \subset -H^*, \forall u \in D \cap sB_X$$

we have

$$d(u, C \cap (v+D)) \leq ad(u-v, D)$$

for all $C \in \mathcal{C}$, $D \in \mathcal{D}$, $u \in C \cap (x + \frac{1}{a}B_X)$ and $v \in x + \frac{1}{a}B_X$.

Corollary 5.4. *Let C be a closed subset of X containing x. Suppose that there exist a locally compact cone $K^* \subset X^*$ and $r > 0$ such that (2) of Theorem 5.1 holds. Then there exist a finite dimensional subspace L of X and $a > 0$ such that*

$$L^{\perp} \cap K^* = \{0\} \tag{1”}$$

and for all $L' \in \mathcal{D}_L$ (:= the set of all closed subspaces of X containing L), $u \in C \cap (x + \frac{1}{a}B_X)$ and $v \in x + \frac{1}{a}B_X$

$$d(u, C \cap (v + L')) \leq ad(u-v, L').$$

In the following corollary we give conditions ensuring the metric inequality for functions.

Corollary 5.5. *Let f_1 and f_2 be extended real-valued lower semicontinuous functions on X, with $f_1(x)$, $f_2(x) \in \mathbb{R}$, satisfying*

$$\partial_A^{\infty} f_1(x) \cap (-\partial_A^{\infty} f_2(x)) = \{0\}. \tag{5.4}$$

Suppose that there exist a locally compact cone $K^* \subset X^*$ and $r > 0$ such that

$$\partial_A d(\text{ epi } f_1, u, \alpha) \subset K^* \times \mathbb{R}, \forall (u,\alpha) \in (x + rB_X) \times (f_1(x) + B_{\mathbb{R}}). \tag{5.5}$$

Set $S_1 = \{(x,\alpha,\beta) \in X \times \mathbb{R} \times \mathbb{R} : f_1(x) \leq \alpha\}$ and $S_2 = \{(x, \alpha, \beta \in X \times \mathbb{R} \times \mathbb{R} : f_2(x) \leq \beta\}$. Then there exists $a > 0$ such that

$$d(u,\alpha,\beta, S_1 \cap S_2) \leq a\ d(u,\alpha,\beta,S_2)$$

for all $(u,\alpha,\beta) \in S_1 \cap (x, f_1(x), f_2(x)) + r(B_X \times B_{\mathbb{R}} \times B_{\mathbb{R}})$.

Proof. It is enough to show that (5.4) yieds

$$\partial_A d(x, f_1(x), f_2(x); S_1) \cap (-\partial_A d(x, f_1(x), f_2(x), S_2)) = \{0\}$$

and (5.5) implies

$$\partial_A d(S_1, u, \alpha, \beta) \subset K^* \times \mathbb{R} \times \mathbb{R}, \forall (u,\alpha,\beta) \in S_1 \cap (x, f_1(x), f_2(x)) + r(B_X \times B_{\mathbb{R}} \times B_{\mathbb{R}})$$

and to apply Theorem 5.2.

Theorem 5.6. *Let C be a closed subset of X, with $x \in C$ and K^* be a locally compact cone. Consider the following assertions*

(i) *there exists a neighbourhood V_1 of x such that*

$$\partial_A d(C, u) \subset K^*, \forall u \in V_1 \cap C;$$

(ii) *there exist a neighbourhood V_2 of x and $r_2 > 0$ such that*

$$\partial_\varepsilon^- \psi_C(u) \subset K^* + 2\varepsilon B_{X^*}, \forall u \in V_2 \cap C \text{ and } \varepsilon \in]0, r_2[;$$

(iii) *there exists a neighbourhood V_3 of x such that*

$$\partial_F d(C, u) \subset K^*, \forall u \in V_3 \cap C;$$

(iv) *there exists a neighbourhood V_4 of x such that*

$$\partial^- \Psi_C(u) \subset K^*, \forall u \in V_4 \cap C.$$

Then

a) $(i) \Rightarrow (ii)$*;*

b) if K^* is closed and X is a WT-space, $(ii) \Rightarrow (i)$.

c) if K^* is closed and X is Asplund, $(i) \iff (ii) \iff (iii) \iff (iv)$.

Note that b) holds in any Banach space (see Remark following Theorem 6.2).

Proof. a) By Theorem 2.8, there exists a finite dimensional subspace L of X such that $L^\perp \cap K^* = \{0\}$. Corollary 5.4 yields the existence of $a > 0$ and a neighbourhood V' of x, with $V' \subset V_1$ such that for all finite dimensional space L' of X containing L

$$d(u, C \cap (v + L')) \leq ad(u - v, L') \tag{5.6}$$

for all $u, v \in C \cap V'$. Let $r > 0$, $V_2 \subset V'$, $\varepsilon \in]0, r[$, $u \in V_2 \cap C$ and $x^* \in \partial_\varepsilon^- \psi_C(u)$. Then, by Lemma 1 in [17], we have for all L' satisfying (5.6), the function

$$z \to -\langle x^*, z - u \rangle + 2\varepsilon \parallel z - u \parallel$$

attains a local minimum at u on $C \cap (u + L')$ and hence the function

$$z \to -\langle x^*, z - u \rangle + 2\varepsilon \parallel z - u \parallel + 2a(2\varepsilon + \parallel x^* \parallel)(d(C, z) + d(L' + u, z))$$

attains a local minimum at u. Thus by subdifferential calculus

$$x^* \in 2a(2\varepsilon + \parallel x^* \parallel)\partial_A d(C, u) + 2\varepsilon B_{X^*} + L'^\perp$$

and hence

$$x^* \in 2a(2\varepsilon + \parallel x^* \parallel)\partial_A d(C, u) + 2\varepsilon B_{X^*}$$

which implies, by (i), that $x^* \in K^* + 2\varepsilon B_{X^*}$.

b) It suffices to see that, when X is a WT-space

$$\partial_A d(C,u) \subset \limsup_{\substack{\varepsilon\to 0^+ \\ z \xrightarrow{C} u}} \partial_\varepsilon^- \psi_C(z).$$

c) As in a) we show that $(iii) \Rightarrow (ii)$, and since $\partial_F d(C,u) \subset \partial_A d(C,u)$, we have $(i) \Rightarrow (iii)$. Similary (as in a)) we show that $(iii) \Rightarrow (iv)$ and since

$$\partial_F d(C,u) \subset \limsup_{z \xrightarrow{C} u} \partial^- \Psi_C(z)$$

we obtain $(iv) \Rightarrow (iii)$, this terminates the proof.

Remark. *It follows from this theorem and Theorem 2.8 that C is normally compact in the sense of Loewen [40] and Mordukhovich and Shao [47], that is,*

$$\partial_F d(C,u) \subset \{x^* \in X^* : \| x^* \| \leq \max_{j=1,\cdots,m} \langle x^*, h_i \rangle\}, \quad \forall u \in V$$

iff (i) of Theorem 5.6 holds for some locally compact closed cone K^* provided that the space X is Asplund.

Remark. *The difference between Theorems 5.1 and 5.3 is that the real a in Theorem 5.3 is not depending on C but only on x.*

6. Normal cones

Let C be a nonempty closed subset of X containing x. The contingent cone $K(C,x)$ to C at x is the set of all $h \in X$ for which there exist sequences $t_n \to 0^+$ and $h_n \to h$ such that

$$x + t_n h_n \in C, \quad \forall n.$$

Using the approximate subdifferential of Lipschitz continuous distance functions, Ioffe [19] introduced the generated normal cone (G-normal) to C at x given by

$$N_G(C,x) = cl^*(\hat{N}_G(C,x))$$

where the construction

$$\hat{N}_G(C,x) = \mathbb{R}_+ \partial_A d(C,x)$$

is called the nucleus of $N_G(C,x)$. To these cones, he associated the $G-$subdifferential $\partial_G f(x)$ and the G-nucleus $\hat{\partial}_G f(x)$

$$\partial_G f(x) = \{x^* \in X^* : (x^*, -1) \in N_G(\text{ epi} f; x, f(1))\}.$$

$$\hat{\partial}_G f(x) = \{x^* \in X^* : (x^*, -1) \in \hat{N}_G(\text{ epi} f; x, f(x))\}$$

if $x \in \text{ dom} f$ and $\partial_G f(x) = \emptyset$ and $\hat{\partial}_G f(x) = \emptyset$ whenever $x \notin \text{ dom} f$.

In a joint work, with Thibault (see Chapter 3, p. 20), we considered the notion of b-approximate normal cone $N_A^b(C,x)$ to C at x in the following manner : $x^* \in N_A^b(C,x)$ iff for each $L \in \mathcal{F}(X)$ there exist nets $\varepsilon_i \to 0^+$, $x_i \xrightarrow{C} x$, $x_i^* \xrightarrow{w^*} x^*$ and $c > 0$ satisfying for all i

$$\| x_i^* \| \leq c \text{ and } x_i^* \in \partial_{\varepsilon_i}^- \psi_{S \cap (x_i + L)}(x_i).$$

It is a very slight variant of Ioffe's original definition and it is always contained in the approximate normal cone $N_A(C,x)$ to C at x

$$N_A(C,x) = \partial_A \psi_C(x).$$

Following Kruger and Mordukhovich [38], the limiting Fréchet normal cone to C at x is the set

$$N_F(C,x) = \partial_F \psi_C(x).$$

This cone can be characterized in terms of the subdifferential of the distance function (see Thibault [54]) as follows

$$N_F(C,x) = \mathbb{R}_+ \partial_F d(C,x).$$

The reader can easily see that the following inclusions are always true

$$N_F(C,x) \subset \hat{N}_G(C,x) \subset N_A^b(C,x) \subset N_A(C,x).$$

To the σ_1-approximate subdifferential and σ_2-approximate subdifferential we can associate the corresponding cones

$$N_A^{\sigma_1}(C,x) = \partial_A^{\sigma_1} \psi_C(x) \text{ and } N_A^{\sigma_2}(C,x) = \partial_A^{\sigma_2} \psi_C(x).$$

It is also easy to see that

$$N_A^{\sigma_2}(C,x) \subset N_A^{\sigma_1}(C,x) \subset N_A^b(C,x) \subset N_A(C,x).$$

Our aim in this section is to give conditions ensuring equalities of these cones. We begin with the relationship between $N_A^b(C,x)$ and $\hat{N}_G(C,x)$.

Theorem 6.1. *Suppose that there exist a locally compact cone $K^* \subset X^*$ and a neighbourhood V of x such that*

$$\partial_A d(C,u) \subset K^*, \forall u \in V \cap C. \tag{6.1}$$

Then

$$N_A^b(C,x) = \hat{N}_G(C,x).$$

Proof. Since K^* is locally compact, Theorem 2.8 implies the existence of a finite dimensional space $L \subset X$ such that

$$L^\perp \cap K^* = \{0\}.$$

Let $a > 0$ be as in Corollary 5.4 and V be a weak-star neighbourhood of 0 in X^* containing $L'^{\perp}$ for some finite dimensional space $L' \subset X$ containing L. Let $x^* \in N_A^b(C, x)$. Then there are nets $x_i \xrightarrow{C} x_0, \varepsilon_i \to 0^+, x_i^* \xrightarrow{w^*} x^*$ and $c > 0$ such that for all i

$$\| x_i^* \| \leq c \text{ and } \quad x_i^* \in \partial_{\varepsilon_i}^- \psi_{C \cap (x_i + L')}(x_i).$$

Thus for all $\varepsilon > 0$, the function

$$x \to -\langle x_i^*, x - x_i \rangle + (\varepsilon_i + \varepsilon) \| x - x_i \|$$

attains a local minimum on $C \cap (x_i + L')$ at x_i (Lemma 1 in [17]) and hence, by Proposition 2.4.3 in [10], the function

$$z \to -\langle x_i^*, z - x_i \rangle + (\varepsilon_i + \varepsilon) \| z - x_i \| + (c + \varepsilon + \varepsilon_i) d(z, C \cap (x_i + L'))$$

attains a local minimum at x_i. Now, by Corollary 5.4, x_i is a local minimum of the function

$$z \to -\langle x_i^*, z - x_i \rangle + (\varepsilon + \varepsilon_i) \| z - x_i \| + a(c + \varepsilon + \varepsilon_i) d(z, x_i + L') + (a+1)(c + \varepsilon + \varepsilon_i) d(z, C).$$

Thus

$$x_i^* \in (a+1)(c + \varepsilon + \varepsilon_i) \partial_A d(C, x_i) + L'^{\perp} + (\varepsilon + \varepsilon_i) B_{X^*}$$

and hence

$$x^* \in (a+1) c \partial_A d(C, x) + L^{'\perp} \subset (a+1) c \partial_A d(C, x) + V.$$

As $\partial_A d(C, x)$ is weak-star closed, we get $x^* \in (a+1) \subset \partial_A d(C, x)$.

Theorem 6.2. *Let C be a closed subset of X, with $x \in C$, and let K^* be a closed and locally compact cone. Suppose that (6.1) holds. Then*

$$N_A(C, x) = \hat{N}_G(C, x).$$

Proof. Since K^* is locally compact, Theorem 2.8 implies the existence of a finite dimensional space $L \subset X$ such that

$$L^{\perp} \cap K^* = \{0\}.$$

Let $x^* \in N_A(C, x)$. Then, by Proposition 2.1 in [18], there are nets $(x_i) \subset C$, $(x_i^*) \subset X^*$ and $(L_i) \subset F(X)$ such that $x_i \to x$, $x_i^* \xrightarrow{w^*} x^*$, (L_i) is cofinal with $F(X)$ and $x_i^* \in \partial^- \Psi_{C \cap (x_i + L_i)}(x_i)$, for all i. One may suppose that $L \subset L_i$ for all i. By Corollary 5.4 there are i_0, $a > 0$ and $r > 0$ such that

$$d(u, C \cap (x_i + L_i)) \leq a(d(u, C) + d(u - x_i, L_i)) \tag{6.2}$$

for all $i > i_0$ and $u \in x + r B_X$. So since $x_i^* \in \partial^- \Psi_{C \cap (x_i + L_i)}(x_i)$, Lemma 1 in [17] implies that for all $\varepsilon > 0$ the function

$$u \to -\langle x_i^*, u - x_i \rangle + \varepsilon \|u - x_i\| + a(\|x_i^*\| + \varepsilon)[d(u, C) + d(u - x_i, L_i)]$$

attains a local minimum at x_i. Thus using subdifferential calculus rules and letting ε going to zero we obtain

$$x_i^* \in a\|x_i^*\|\partial_A d(C, x_i) + L_i^\perp.$$

So there exist $u_i^* \in a\|x_i^*\|\partial_A d(C, x_i)$ and $v_i^* \in L_i^\perp$ such that

$$x_i^* = u_i^* + v_i^*.$$

We claim that (u_i^*) is bounded. Suppose the contrary and assume without loss of generality that $\|u_i^*\| \to \infty$ and $p_i^* := \frac{u_i^*}{\|u_i^*\|} \xrightarrow{w^*} p^*$ and $q_i^* := \frac{v_i^*}{\|u_i^*\|} \xrightarrow{w^*} -p^*$. As $(u_i^*) \subset K^*$ and K^* is closed and locally compact, it follows that $p^* \neq 0$ and $p^* \in K^*$ and since $L_i^\perp \subset L^\perp$ for all i we get $p^* \in K^* \cap L^\perp$ and this contradict the fact that $K^* \cap L^\perp = \{0\}$. Extracting subnet if necessary we may assume that $u_i^* \xrightarrow{w^*} u^*$. But $v_i^* \in L_i^\perp$, for all i, then $v_i^* \xrightarrow{w^*} 0$ and hence $u^* = x^*$. Relation (6.2) ensures that $K(C \cap (x_i + L_i), x_i) = K(C, x_i) \cap L_i$ and as $\partial^- \Psi_{C\cap(x_i+L_i)}(x_i) = (K(C \cap (x_i + L_i), x_i))^0$ we get $u_i^* \in \partial^- \Psi_{C\cap(x_i+L_i)}(x_i)$ and hence $x^* \in N_A^b(C, x)$ and Theorem 6.1 implies that $x^* \in \hat{N}_G(C, x)$ and the proof is complete.

Remark. *Theorem 6.2 shows that (6.1) is equivalent to*

$$N_A(C, u) \subset K^*, \quad \forall u \in V \cap C.$$

The weak contingent cone (or pseudocontingent cone) $WK(C, x)$ to C at x is the set of all h in X for which there exist sequences $t_n \to 0^+$, and (h_n) converging weakly to h such that

$$x + t_n h_n \in C, \quad \forall n.$$

Note that we always have

$$K(C, x) \subset WK(C, x)$$

and hence if we consider the weak-approximate normal cone

$$N_A^w(C, x) = \text{ seq} - \limsup_{u \xrightarrow{C} x} (WK(C, x))^0$$

we obtain

$$N_A^w(C, x) \subset N_A^{\sigma_2}(C, x)$$

since $\partial^- \psi_C(x) = (K(C, x))^0$.

We have the following connection between this cone and the previous ones.

Corollary 6.3. *Suppose that (6.1) holds and that X is Asplund. Then*

$$N_F(C, x) \subset N_A^w(C, x) \subset N_A^{\sigma_2}(C, x) \subset N_A^{\sigma_1}(C, x) \subset cl^*(N_F(C, x)).$$

Proof. The first inclusion follows from Proposition 3.1 in [7]. For the last inclusion we apply Theorem 4.1, (4.5).

Corollary 6.4. *Suppose in addition to the assumptions of Theorem 6.1 that X is WCG and Asplund. Then*

$$N_F(C,x) = cl^*(N_F(C,x)) = N_A(C,x) = N_G(C,x) = \hat{N}_G(C,x).$$

Proof. Apply Theorem 4.1, (4.7) with $f = \psi_C$ and use Theorem 6.2 to complete the proof.

In conclusion of this section we use the results obtained above to establish relationships between subdifferentials.

Corollary 6.5. *Let f be an extended real-valued lower semicontinuous function on a WCG Asplund space X.If (4.7) holds, then*

$$\partial_F f(x_0) = \partial_A f(x_0) = \partial_G f(x_0) = \hat{\partial}_G f(x_0).$$

7. Tangent cones

Let C be a nonempty closed subset of X containing x. The Clarke's tangent cone $T(C,x)$ to C at x is the set of all h such that for any sequence (x_n) of C, $x_n \to x$ and $t_n \to 0^+$ there exists $h_n \to h$ such that

$$x_n + t_n h_n \in C, \quad \text{for} n \text{large enough.}$$

The Clarke's normal cone $N_c(C,x)$ to C at x is defined by

$$N_c(C,x) = (T(C,x))^0.$$

Cornet [11] has found a topological connection between the Clarke's tangent cone and the contingent cone $K(C,x)$ to C at x. He has shown that if $C \subset \mathbb{R}^m$, then

$$T(C,x) = \liminf_{u \xrightarrow{C} x} K(C,u)$$

here for a multivalued mapping F on C

$$\liminf_{u \xrightarrow{C} x} F(u) = \bigcap_{\varepsilon>0} \bigcup_{\lambda>0} \bigcap_{u \in C \cap (x+\lambda B_X)} (F(u) + \varepsilon B_X)$$

or equivalently $h \in \liminf_{u \xrightarrow{C} x} F(u)$ iff for each sequence $x_n \xrightarrow{C} x$ there exist sequence $h_n \to h$, such that $h_n \in F(x_n)$, for all n.

Using its new characterization of Clarke's tangent cone, Treiman [56-57]showed that the inclusion

$$\liminf_{u \xrightarrow{C} x} K(C,u) \subset T(C,x)$$

is true in any Banach space and equality holds whenever C is epi-Lipschitzian at x in the sense of Rockafellar [53]. But this result does not include the finite dimensional case. In [6], Borwein and Strojwas introduced the concept of compactly epi-Lipschitz sets to show that the previous equality holds for C in this class. A set C is said to be compactly epi-Lipschitz at $x \in C$ if there exist a norm-compact set H and $r > 0$ such that

$$C \cap (x + rB_X) + trB_X \subset C - tH, \quad \forall t \in]0, r[.$$

Note that every epi-Lipschitz set is compactly epi-Lipschitz and every subset of a finite dimensional space is compactly epi-Lipschitz at all its points. In the case where the space is reflexive, these authors obtained the following equality

$$T(C, x_0) = \liminf_{u \xrightarrow{C} x} WK(C, x).$$

They generalize the results of Penot [49] for finite dimensional and reflexive Banach spaces and of Cornet [11]for finite dimensional spaces. Recently Aubin-Frankowska [2 ,p. 134] obtained the following formula

$$T(C,x) = \liminf_{u \xrightarrow{C} x} WK(C,u) = \liminf_{u \xrightarrow{C} x} co(WK(C,u))$$

in the case where the space X is uniformly smooth and the norm of X^* is Fréchet differentiable off the origin.

In their paper [7], Borwein and Strojwas gave the following characterization of reflexive Banach spaces. They showed that the following assertions are equivalent :

(i) $T(C,x) \subset \liminf_{u \xrightarrow{C} x} coWK(C,u)$*, for all closed sets* $C \subset X$*, and* $x \in C$*;*

(ii) X *is reflexive.*

Our aim in this section is to show that (i) holds as equality for more general class of Banach spaces. In the following result we show that (i) holds for every set satisfying (6.1), with $\partial = \partial_A$ or ∂_F or other subdifferentials in Banach spaces.

We begin by showing that the inclusion

$$\liminf_{u \xrightarrow{C} x} \operatorname{clco}(WK(C,x)) \subset T(C,x). \tag{7.1}$$

holds for any closed subset C of an Asplund space.

Proposition 7.1. *Let* X *be an Asplund space. Then (7.1) holds.*

Proof. We will show that if $v \notin T(C,x)$, with $\| v \|= 1$, then $v \notin \liminf_{u \xrightarrow{C} x} \text{clco}(WK(C,x))$.

Since $v \notin T(C,x)$, then, by lemma 1.2.1 in [56], there exist $\varepsilon > 0, x_n \xrightarrow{C} x$, an integer n_o and $\lambda_n > 0$ such that

$$(x_n +]0, \lambda_n](v + \varepsilon B_X)) \cap C = \emptyset, \forall n \geq n_0.$$

We may assume that $\lambda_n \to 0^+$. Set $D = x_n + [0, \frac{\lambda_n}{2}](v + \varepsilon B_X)$. Then $(\lambda_n^3 v + D) \cap C = \emptyset$. Set $f(u,w) = \| u - w - \lambda_n^3 v \| + \| u - x_n \|^2$. By Ekeland's variational principle [14], there exist $u_n \in C$ and $w_n \in D$ such that

$$f(u_n, w_n) + \lambda_n[\| u_n - x_n \| + \| w_n - x_n \|] \leq f(x_n, x_n). \tag{7.2}$$

and

$$f(u_n, w_n) \leq f(u,w) + \lambda_n(\| u - u_n \| + \| w - w_n \|), \forall (u,w) \in C \times D. \tag{7.3}$$

Note that $u_n - w_n - \lambda_n^3 v \neq 0$, and hence

$$\partial_F \| \,.\, - \,.\, - \lambda_n^3 v \| (u_n, w_n) \subset \{(u^*, -u^*) : \quad \| u^* \|= 1\}.$$

By Proposition 2.4.3 in [10]

$$f(u_n, w_n) \leq f(u,w) + \lambda_n(\| u - u_n \| + \| w - w_n \|) + 3d(u,C) + 3d(w,D)$$

for all (u,w) near (u_n, w_n). Thus, by subdifferential calculus there exist $x_n^* \in X^*$, $\| x_n^* \|=$ 1, $a_n^*, d_n^*, b_n^* \in B_{X^*}$ such that

$$x_n^* + \lambda_n b_n^* + 2 \| u_n - x_n \| a_n^* \in 3\partial_F d(C, u_n) \tag{7.4}$$

$$-x_n^* + \lambda_n d_n^* \in 3\partial_F d(D, w_n). \tag{7.5}$$

From (7.5), we get

$$\langle -x_n^*, x_n + \frac{\lambda_n}{2}(v+b) - w_n \rangle \leq \langle -\lambda_n d_n^*, x_n + \frac{\lambda_n}{2}(v+b) - w_n \rangle, \forall b \in \varepsilon B_X$$

which implies that

$$\langle -x_n^*, \frac{\lambda_n}{2}(v+b) \rangle \leq \langle x_n^*, x_n - w_n \rangle + \frac{\lambda_n^2}{2}(\| v \| + \varepsilon) + \lambda_n \| x_n - w_n \|$$

(because of (7.2), $\| w_n - x_n \| \leq \lambda_n^2$ and $\| v \|= 1$), and hence

$$\varepsilon \leq \langle x_n^*, v \rangle + 4\lambda_n + \lambda_n(1 + \varepsilon).$$

As $\| x_n^* \|= 1$ and X is Asplund space, extracting subsequence if necessary we may assume that $x_n^* \xrightarrow{w^*} x^*$, and so

$$\varepsilon \leq \langle x^*, v\rangle \tag{7.6}$$

and this ensures that $x^* \neq 0$. From (7.4) and the sequential weak-star closedness of the limiting Fréchet ε-subdifferential (of Lipschitz functions) we get

$$x^* \in 3\partial_F d(x, C).$$

Using Proposition 3.1 in [7] and the fact that $\partial_F d(C, x) \subset N_F(C, x)$ we get $x^* \in N_A^w(C, x)$ and then there exist sequences $c_n \xrightarrow{C} x$ and $q_n^* \to x^*$ such that $q_n^* \in (WK(C, c_n))^0$. Consequently if $v \in \liminf_{u \xrightarrow{C} x} \mathrm{clco} WK(C, u)$, there exist $q_n \in \mathrm{clco}(WK(C, c_n))$ such that $q_n \to v$. By (7.6) and the fact that $\langle q_n^*, q_n\rangle \to \langle x^*, v\rangle$ it follows that

$$\frac{\varepsilon}{2} < \langle q_n^*, q_n\rangle$$

and this contradiction completes the proof.

The following corollary has been established by Aubin and Frankowska [2] in the case where X is uniformly smooth and the norm of X^* is Fréchet differentiable off the origin. Their proof is based on Edelstein Theorem which states that in Hilbert spaces and some Banach spaces, we can approximate any point by another which has a unique projection of best approximation on a closed subset. The following corollary by Borwein and Strojwas [7] (in which they use a completly different proof) extends their result to the reflexive Banach spaces since this class of spaces contains the previous one (i.e. the class of space which are uniformly smooth and the norm of their topological dual is Fréchet differentiable off the origin). It is obtained as a direct consequence of Proposition 7.1 and the result by Penot [49].

Corollary 7.2. *Suppose that X is reflexive. Then*

$$T(C, x) = \liminf_{u \xrightarrow{C} x} \mathit{clco} WK(C, x) = \liminf_{u \xrightarrow{C} x} WK(C, x).$$

Proof. It suffices to see that (Penot [49])

$$T(C, x) \subset \liminf_{u \xrightarrow{C} x} WK(C, u)$$

and to apply Proposition 7.1.

As a consequence of this corollary and the duality theorem 1.1.8 in [2] we have

$$N(C, x) = \mathrm{clco}(\ \mathrm{seq} - \liminf_{u \xrightarrow{C} x} (WK(C, u))^0)$$

Now we can use the results in section 6 to help pin down an other relationship between the Clarke's tangent cone and the contingent cone.

Proposition 7.3. *Suppose that (6.1) holds. Then*

$$T(C,x) \subset \liminf_{u \xrightarrow{C} x} clcoK(C,u).$$

Proof. We will show that if $v \notin \liminf_{u \xrightarrow{C} x} \mathrm{clco}K(C,u)$, then $v \notin T(C,x)$. Since $v \notin \liminf_{u \xrightarrow{C} x} \mathrm{clco}K(C,u)$, there exist $\varepsilon > 0$ and sequence $x_n \xrightarrow{C} x$ such that

$$(v + \varepsilon B_X) \cap \mathrm{clco}K(C,x_n) = \emptyset, \forall n.$$

By separation theorem there exists $x_n^* \in X^*$, with $\| x_n^* \| = 1$, such that

$$\langle x_n^*, v + \varepsilon b\rangle \leq \langle x_n^*, h\rangle, \forall h \in K(C,x_n) \quad \forall b \in B_X. \tag{7.7}$$

Since $\| x_n^* \| = 1$, we may assume that $x_n^* \xrightarrow{w^*} x^*$ and from (7.7) $x^* \neq 0$. Now by Theorems 6.2 and 6.1 and the fact that

$$N_G(C,x) \subset N_c(C,x)$$

it follows that $-x^* \in N_c(C,x)$. But from (7.7)

$$\langle -x^*, v\rangle > \varepsilon$$

which implies that $v \notin T(C,x)$.

As a consequence of Propositions 7.1 and 7.3 we obtain

Theorem 7.4. *Under assumptions of Propositions 7.1 and 7.3 we have*

$$T(C,x) = \liminf_{u \xrightarrow{C} x} clcoK(C,u) = \liminf_{u \xrightarrow{C} x} clcoWK(C,u).$$

8. Characterization of interior points of sets.

The purpose of this section is to characterize interior points of sets in infinite dimensional spaces by using locally compact cones. The first result in this setting is due to Rockafellar [52]. It states that for a subset C of $\mathbb{R}^n$

$$x \in \text{ int } C \Leftrightarrow x \in C \text{ and } T(C,x) = \mathbb{R}^n.$$

This result is generalized by Borwein and Strojwas [6-7]to infinite dimensional situation for sets which are compactly epi-Lipschitzian (see Section 7). Our aim in this section is to extend both results in infinite dimensional spaces for sets C satisfying (6.1).

Theorem 8.1. *Let C be a nonempty closed proper subset of X, with $x \in C$. Suppose that (6.1) holds. Then the following are equivalent :*

(i) $x \in$ *int* C

(ii) $T(C, x) = X$.

Proof. $(i) \Rightarrow (ii)$: evident.

$(ii) \Rightarrow (i)$: Suppose $x \notin$ int C. Then x is a boundary point of C and hence by Corollary 2.7 in [29], $\partial_A d(x, C) \neq \{0\}$. So let $x^* \in \partial_A d(x, C) \backslash \{0\}$. Then $x^* \in N_c(C, x)$ and hence

$$\langle x^*, u \rangle \leq 0, \forall u \in T(C, x).$$

So, by (ii), $x^* = 0$ and this contradiction completes the proof.

We may rewritte this result in the following manner. If C is a nonempty closed proper subset of X, with $x \in C$, and satisfies (6.1) then x is a boundary point of C iff $N(C, x) \neq \{0\}$.

9. Complements.

Our aim in this section is to show that the method used in the proof of Proposition 7.1 permits us to obtain characterizations of Clarke normal cone. We establish connection between Clarke normal cone $N_c(C, x)$ and the normal cones $N_G(C, x)$, $N_F(C, x)$ and the cone generated by the presubdifferential of the distance function to C at x.

Theorem 9.1. *Let C be a closed subset of X with $x \in C$. Then*

$$N_c(C, x) \subset clco\mathbb{R}_+ \partial d(C, x).$$

Proof. We shall show that if $v \notin T(C, x)$ then $v \notin (\text{clco}\mathbb{R}_+ \partial d(x, C))^0$.The argument will be similar to that used in the proof of Proposition 7.1. As in (7.4) and (7.5) we get the existence of sequences $x_n^* \in X^*$, $\| x_n^* \| = 1$, $a_n^*, d_n^*, b_n^* \in B_{X^*}$ such that

$$x_n^* + \lambda_n b_n^* + 2 \| u_n - x_n \| a_n^* \in 3\partial d(C, u_n)$$

$$-x_n^* + \lambda_n d_n^* \in 3\partial d(D, w_n).$$

As in the proof of Proposition 7.1 we show that $x_n^* \xrightarrow{w^*} x^*$, with $\langle x^*, v \rangle \geq \varepsilon$ for some $\varepsilon > 0$.

As a consequence we obtain the following characterization for Clarke's normal cone obtained by Clarke [10] as the cone generated by the subdifferential of the distance function, by Ioffe [19] as a closed convex hull of the cone generated by the approximate subdifferential of the distance function, and by Kruger-Mordukhovich [38] as a closed convex hull of the cone generated by the limiting Fréchet subdifferentials of the distance function.

Corollary 9.2. *Let C be a closed subset of X with $x \in C$. Then*

a) $N_c(C, x) = cl\, \mathbb{R}_+ \partial_c d(C, x)$.

b) $N_c(C, x) = clco\mathbb{R}_+ \partial_A d(C, x)$.

c) If X is Asplund then $N_c(C,x) = clco\mathbb{R}_+\partial_F d(C,x)$.
d) If K^* is a closed locally compact cone satisfying (6.1) then

$$N_c(C,x) = clcoN_A(C,x).$$

Proof. It suffices to see that $\partial_A d(C,x)$, $\partial_F d(C,x)$ and $\partial_c d(C,x)$ are presubdifferentials and are contained in $N_c(C,x)$ and Theorem 9.1 (for *a*), *b*) and *c*)) and Theorems 6.2 and 9.1 complete the proof.

After we have completed this work, we received a very interesting paper by Ioffe [58] which characterizes condition (6.1) in terms of compactly epi-Lipschitzian sets. So with the help of this result and that of Borwein and Strojwas [6] we can complete Theorem 7.4 as follows : If X is Asplund and if condition (6.1) is satisfied then

$$T(C,x) = \liminf_{u \xrightarrow{C} x} \mathrm{clco} K(C,u) = \liminf_{u \xrightarrow{C} x} \mathrm{clco} WK(C,u) = \liminf_{u \xrightarrow{C} x} K(C,u) = \liminf_{u \xrightarrow{C} x} WK(C,u).$$

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