Noncoincidence of approximate and limiting subdifferentials of integral functionals

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Abstract. For a locally Lipschitz integral functional I_f on $L^1(T, \mathbf{R}^n)$ associated with a measurable integrand f, the limiting subdifferential and the approximate subdifferential never coincide at a point x_0 where $f(t, \cdot)$ is not subdifferentially regular at $x_0(t)$ for a.e. $t \in T$. The coincidence of both subdifferentials occurs on a dense set of $L^1(T, \mathbf{R}^n)$ if and only if $f(t, \cdot)$ is convex for a.e. $t \in T$. Our results allow us to characterize Aubin's Lipschitz-like property as well as the convexity of multivalued mappings between L^1-spaces. New necessary optimality conditions for some Bolza problems are also obtained.

Key Words. Subdifferential, Integral functional, Integrand, Aubin Lipschitz-like property, Bolza problem.

AMS subject classification. 49J52, 49K27.

1 Introduction

Optimal control problems of Bolza type are known to involve a cost function in the integral form

$$J_{\mathcal{L}}(x) := \int_a^b \mathcal{L}(t, x(t), \dot{x}(t))\, dt$$

which has to be minimized under some specific dynamical and endpoints constraints. The present paper is aimed at comparing the approximate (or Ioffe's) subdifferential and the limiting (or Mordukhovich's) subdifferential of such above functionals. We will show that those two types of subdifferentials do not coincide in general in the setting of the above integral functionals. In fact, our analysis will reveal that this noncoincidence holds even for the class of integral functionals

$$I_f(x) := \int_a^b f(t, x(t))\, dt$$

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which do not depend on the velocity $\dot{x}(t)$.
The concept of limiting subdifferential has been introduced first in finite dimensional space by Mordukhovich [29, 30, 31] and then in the context of Banach space by Kruger and Mordukhovich [26, 27] for extended real-valued functions. This subdifferential corresponds to the collection of weak-star *sequential* limiting points of the so-called Fréchet ε-subdifferential.
The other concept of approximate subdifferential has been introduced and developed by Ioffe in [15, 16]. The construction of this subdifferential is based on weak-star *topological* limits of the so-called Dini subdifferentials and Dini $\varepsilon-$subdifferentials.
A number of efficient chain rules are actually available for both concepts in infinite dimensions, some of them requiring certain geometric assumptions on the structure of Banach spaces. We refer the reader to the recent book [33]-[34] and the papers [16]-[17], [18]-[19], [20]- [24], and the references therein for more details and further information.
The limiting subdifferential is always included in the approximate subdifferential, and both concepts coincide in finite dimension and in some special infinite dimensional spaces. The context is different for integral functionals. We show here that there is no coincidence between the limiting and approximate subdifferentials at a point $x_0 \in L^1(T, \mathbf{R}^n)$ for locally Lipschitz integral functional in $L^1(T, \mathbf{R}^n)$ whose integrand is not (subdifferentially) regular at x_0. When the coincidence holds at a regular point, we also give a useful description of the limiting subdifferential in such a case.
The results that we provide for the limiting subdifferential of integral functionals allow us on one hand to obtain characterization of Aubin's Lipschitz-like property as well as the convexity of multivalued mappings between L^1-spaces, and on the other hand to establish new necessary optimality conditions for the following Bolza problem

$$\min \ell(x(a), x(b)) + \int_a^b \mathcal{L}(t, x(t), \dot{x}(t))dt. \tag{1}$$

As we will show in the last section, the coincidence of the limiting subdifferential and the approximate subdifferential at the (local) minimum point z of (1) produces a new optimality condition of the type

$$\begin{aligned}\sup_{\zeta,\xi\in\mathbf{R}^{\mathbf{n}}} \{\langle \dot{p}(t), \zeta\rangle + \quad & \langle p(t), \xi\rangle - \mathcal{L}(t, \zeta, \xi)\} = \\ & \langle \dot{p}(t), z(t)\rangle + \langle p(t), \dot{z}(t)\rangle - \mathcal{L}(t, z(t), \dot{z}(t))\, a.e.t \in [a, b]\end{aligned}$$

for some absolutely continuous p with values in $\mathbf{R}^n$.
The organization of the paper starts with definitions and notations in Section 2. In Section 3, we recall some important cases where the limiting and approximate subdifferentials coincide for certain infinite dimensional spaces. In Section 4 we prove the result mentioned above concerning the noncoincidence at a *fixed* point $x_0 \in L^1(T, \mathbf{R}^n)$ of the limiting and approximate subdifferentials of an integral functional I_f whose integrand $f(t, \cdot)$ is not (subdifferentially) regular at $x_0(t)$

for almost all t provided that the corresponding measure space is *atomless*. We also show that the coincidence of both subdifferentials occurs on a dense set of $L^1(T, \mathbf{R}^n)$ if and only if the measurable integrand is convex, that is, for almost all t the function $f(t, \cdot)$ is convex. In the same section, we give a characterization of the Lipschitz property of I_f in terms of that of $f(t, \cdot)$. In Section 5, we apply the obtained results to characterize Aubin's Lipschitz-like property as well as the convexity of multivalued mappings between L^1-spaces. Finally, the last section is concerned with necessary optimality conditions for Bolza problem (1).

2 Background

In order to make the paper as short as possible, some definitions and the complete wording of the results will not be repeated here, and the reader is referred to [29]-[34] and [15]-[16] if required.

Let X be a Banach space endowed with the norm denoted by $\| \cdot \|$ with which we associate the distance function $d(\cdot, S)$ to a set S. By $B(x, r)$ we denote the open ball centered at x and of radius r. The topological dual space of X will be denoted by X^* and the pairing between X and X^* by $\langle \cdot, \cdot \rangle$.

For a function f and a set S, we write $x \xrightarrow{f} x_o$ and $x \xrightarrow{S} x_o$ to express $x \to x_0$ with $f(x) \to f(x_0)$ and $x \to x_0$ with $x \in S$, respectively.

Let f be an extended real-valued function on X. The limiting (or Mordukhovich's) subdifferential of f at x_0 is the set

$$\partial_L f(x_0) = {}^{\text{seq}} \operatorname*{Lim\,sup}_{x \xrightarrow{f} x_0,\, \varepsilon \downarrow 0} \partial_\varepsilon^F f(x),$$

where ${}^{\text{seq}}\text{Lim}\sup$ stands for the weak-star *sequential* upper limit of subsets in X^*, and where for $\varepsilon \geq 0$

$$\partial_\varepsilon^F f(x) = \{x^* \in X^* : \liminf_{h \to 0} \frac{f(x+h) - f(x) - \langle x^*, h \rangle}{\| h \|} \geq -\varepsilon\}$$

is the Fréchet ε-subdifferential of f at any x where f is finite. We adopt the convention $\partial_\varepsilon^F f(x) = \emptyset$ when $|f(x)| = +\infty$. We also put $\partial^F f(x) = \partial_0^F f(x)$ for $\varepsilon = 0$.

The limiting normal cone to a closed set $S \subset X$ at a point $x_0 \in S$ is given by

$$N_L(S, x_0) = \partial_L \delta_S(x_0),$$

where δ_S denotes the indicator function of S, i.e., $\delta_S(x) = 0$ if $x \in S$ and $\delta_S(x) = +\infty$ otherwise. The theory of Fréchet and limiting subdifferentials are developed, with fairly comprehensive references and remarks, in the paper by Mordukhovich and Shao [32] and in Mordukhovich's books [33, 34].

If f is an extended real-valued function on X, we write for any subset S of X

$$f_S(x) = \begin{cases} f(x) & \text{if } x \in S, \\ +\infty & \text{otherwise.} \end{cases}$$

The function $f^-(x,\cdot)$ with

$$f^-(x,h) = \liminf_{\substack{u\to h\\ r\downarrow 0}} r^{-1}(f(x+ru)-f(x))$$

is the lower Dini directional derivative of f at x with $|f(x)| < +\infty$. The Dini ε-subdifferential of f at x is the set

$$\partial_\varepsilon^- f(x) = \{x^* \in X^* : \langle x^*, h\rangle \le f^-(x;h) + \varepsilon\|h\|, \forall h \in X\}$$

for $x \in \operatorname{Dom} f$ and $\partial_\varepsilon^- f(x) = \emptyset$ if $x \notin \operatorname{Dom} f$, where $\operatorname{Dom} f := \{x \in X : |f(x)| < +\infty\}$ denotes the effective domain of f. For $\varepsilon = 0$ we write $\partial^- f(x)$.
Let $\mathcal{F}(X)$ be the collection of finite dimensional subspaces of X. The approximate (or Ioffe's) subdifferential of f at $x_0 \in \operatorname{Dom} f$ is defined by the following expressions (see Ioffe [15]-[16])

$$\partial_A f(x_0) = \bigcap_{L\in\mathcal{F}(X)} \operatorname*{Lim\,sup}_{x \xrightarrow{f} x_o} \partial^- f_{x+L}(x) = \bigcap_{L\in\mathcal{F}(X)} \operatorname*{Lim\,sup}_{\substack{x \xrightarrow{f} x_o\\ \varepsilon\downarrow 0}} \partial_\varepsilon^- f_{x+L}(x),$$

where the weak-star *topological* upper limit $\operatorname*{Lim\,sup}_{x \xrightarrow{f} x_0} \partial^- f_{x+L}(x)$ is defined by

$$\operatorname*{Lim\,sup}_{x \xrightarrow{f} x_o} \partial^- f_{x+L}(x) = \{x^* \in X^* : x^* = w^* - \lim\, x_i^*, x_i^* \in \partial^- f_{x_i+L}(x_i), x_i \xrightarrow{f} x_0\},$$

that is, the set of w^*-limits of all such nets.
As for the limiting subdifferential, the approximate normal cone to a closed set $S \subset X$ at $x_0 \in S$ is defined by

$$N_A(S,x_0) = \partial_A \delta_S(x_0). \tag{2}$$

In [16], Ioffe gave the following geometrical characterization of the approximate subdifferential

$$\partial_A f(x_0) = \{x^* \in X^* :\ (x^*,-1) \in N_A(\operatorname{epi} f, (x_0, f(x_0)))\}. \tag{3}$$

For weak trusthworthy Banach spaces (including separable Banach spaces), Ioffe established that the approximate subdifferential takes the following simpler form:

$$\partial_A f(x_0) = \operatorname*{Lim\,sup}_{\substack{x \xrightarrow{f} x_o\\ \varepsilon\downarrow 0}} \partial_\varepsilon^- f(x).$$

Through the distance function $d(\cdot,S)$ (also denoted in the paper by d_S) in place of the indicator function in (2), a geometric concept of normal cone has been also considered in [16]. The G-normal cone to a closed set $S \subset X$ at $x_0 \in S$ is defined by

$$N_G(S,x_0) = \mathbf{R}_+ \partial_A d(x_0,S).$$

With this normal cone, taking (3) into account and as for the Clarke subdifferential (see (4)) one associates ([16]) the G-subdifferential of f at x_0 as follows:

$$\partial_G f(x_0) = \{x^* \in X^* : (x^*, -1) \in N_G(\text{epi}f, (x_0, f(x_0)))\}.$$

The limiting subdifferential, the G-subdifferential, and the approximate subdifferential are infinite-dimensional extensions of the nonconvex subdifferential introduced in [29]. We always have the following inclusions:

$$\partial_L f(x_0) \subset \partial_G f(x_0) \subset \partial_A f(x_0).$$

The approximate subdifferential may be bigger than the Clarke's subdifferential $\partial_C f(x_0)$ characterized geometrically in terms of the Clarke's normal cone as follows:

$$\partial_C f(x_0) = \{x^* \in X^* : (x^*, -1) \in N_C(\text{epi}f, (x_0, f(x_0)))\}. \tag{4}$$

The limiting subdifferential and the G-subdifferential are always included in the Clarke's one. When f is finite around x_0 and locally Lipschitz there, the Clarke's subdifferential (see [7]) takes the following form (which is the Clarke's original definition)

$$\partial_C f(x) = \{x^* \in X^* : \langle x^*, h\rangle \leq f^\circ(x; h)\, \forall h \in X\},$$

where

$$f^\circ(x; h) := \limsup_{\substack{r \to 0^+ \\ x \to x_0}} \frac{f(x + rh) - f(x)}{r}.$$

Both Clarke's subdifferential and the approximate subdifferential obey the following sum rule:

$$\partial(f + g)(x_0) \subset \partial f(x_0) + \partial g(x_0)$$

provided that one of the two functions f, g $(: X \mapsto \mathbf{R} \cup \{+\infty\})$ is locally Lipschitz around x_0. This calculus rule also holds for the limiting subdifferential but in Asplund spaces.
We know that Clarke's subdifferential and the approximate subdifferential of locally Lipschitz functions are nonempty sets in any general Banach space and that the limiting subdifferential of a locally Lipschitz function is also a nonempty set provided that the space is *Asplund*.

• *Can we get the same conclusion of nonemptiness for the limiting subdifferential outside Asplund spaces?* Here is an example including both situations.

Example 2.1 *Let $X = L^1[0, 1]$ and let $f(u) = \int_0^1 |\sin u(t)| dt$ and $g = -f$. Then $\partial_L(f + g)(0) = \{0\}$ while $\partial_L f(0) = \{0\}$ and $\partial_L g(0) = \emptyset$. We refer to section 4 for results justifying the two latter equalities.*

3 Relationships between the limiting subdifferential and the approximate subdifferential

In this section, we will provide and review certain cases where some of the subdifferentials considered in the previous section coincide. Let us begin by recalling a connection between limiting subdifferential, approximate subdifferential and the following sequential constructions of approximate-like subdifferential

$$\partial_{A,+}^{\rm seq} f(x_0) = {}^{\rm seq}\mathop{\rm Lim\,sup}_{\substack{\varepsilon \longrightarrow 0+ \\ x \xrightarrow{f} x_0}} \partial_\varepsilon^- f(x)$$

and

$$\partial_A^{\rm seq} f(x_0) = {}^{\rm seq}\mathop{\rm Lim\,sup}_{x \xrightarrow{f} x_0} \partial^- f(x).$$

Mordukhovich and Shao [32] showed that in Asplund space and for a locally Lipschitz function f at x_0

$$\mathrm{cl}^*(\partial_L f(x_0)) = \mathrm{cl}^*(\partial_{A,+}^{\rm seq} f(x_0)) = \mathrm{cl}^*(\partial_A^{\rm seq} f(x_0)) = \partial_A f(x_0),$$

where "cl^*" denotes the weak-star topological closure. They showed that in weakly compactly generated Asplund spaces, one has in fact the following stronger equalities

$$\partial_L f(x_0) = \partial_{A,+}^{\rm seq} f(x_0) = \partial_A^{\rm seq} f(x_0) = \partial_A f(x_0)$$

for a locally Lipschitz function f at x_0.

Recall that X is weakly compactly generated (WCG) if there exists a weakly compact set K such that $X = \mathrm{cl}(\mathrm{span}(K))$. Clearly all reflexive Banach spaces and all separable Banach spaces are weakly compactly generated. For the case where X is an Asplund space, there are precise characterizations of the WCG property (see [8] and [9]) which imply, in particular, the existence of a Fréchet differentiable renorm.

Using Proposition 3.1 in [25], we easily see that

$$\partial_A f(x_0) = \partial_{A,+}^{\rm seq} f(x_0)$$

provided that X is a WCG space and there exist a locally compact cone K^* and a real number $r > 0$ such that

$$\partial_\varepsilon^- f(x) \subset K^* + \rho(\varepsilon) B_{X^*}, \forall x \in x_0 + rB_X, \forall \varepsilon \in]0, r[,$$

where B_{X^*} denotes the closed unit ball of X^*, $\rho(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ and K^* is a weak-star locally compact cone. A set $K^* \subset X^*$ is said to be (weak-star) *locally compact* if for each point x^* in K^* there exists a weak-star neighbourhood V of x^* such that $\mathrm{cl}^* V \cap K^*$ is weak-star compact. Many important properties of these sets are listed in [28] and [18].

In [18], the first author extended the result of Mordukhovich and Shao [32] to the non-Lipschitz case.

Theorem 3.1 *([18]) Let X be an Asplund space and f be an extended real-valued lower semicontinuous function on X. Then*

$$cl^*(\partial_{A,+}^{\mathrm{seq}} f(x_0)) = cl^*(\partial_A^{\mathrm{seq}} f(x_0)) = cl^*(\partial_L f(x_0))$$

provided that there exist a closed and locally compact cone K^* in X^* endowed with its weak-star topology and some real number $r > 0$ such that

$$\partial_L d(x, \alpha; \mathrm{epi} f) \subset K^* \times \mathbf{R}, \forall (x, \alpha) \in (x_0 + rB_X) \times (f(x_0) + rB_{\mathbf{R}}) \cap \mathrm{epi} f. \quad (5)$$

If in addition X is WCG, then

$$\partial_A f(x_0) = \partial_L f(x_0) = \partial_{A,+}^{\mathrm{seq}} f(x_0) = \partial_A^{\mathrm{seq}} f(x_0).$$

Remark 3.1 *Using Theorem 3 in [17] and Theorem 5.6 in [18], one obtains the equivalence between relation (5) and the fact that epif is compactly epi-Lipschitz ([1]-[2]) at $(x_0, f(x_0))$.*

What happens outside Asplund spaces? We will introduce in the following section a large class of functions whose limiting and approximate subdifferentials never coincide.

4 Noncoincidence of the limiting subdifferential and the approximate subdifferential

In this section, we provide a large class of functions over $L^1(T, \mathbf{R}^n)$ for which at a fixed point the limiting subdifferential and the approximate subdifferential never coincide. For this class we will show that :

- The Clarke subdifferential and the approximate subdifferential coincide at this point;
- The Clarke subdifferential is sequentially weak-star closed at this point;
- The limiting subdifferential is never equal to the approximate subdifferential at this point.

We also study the global coincidence at all points of $L^1(T, \mathbf{R}^n)$ between the limiting subdifferential and the approximate subdifferential. We establish that

- such a global coincidence holds for a Lipschitz integral functional if and only if the associated measurable integrand is convex.

Let $(T, \mathcal{T}, \mu)$ be a measure space with an *atomless* [1] $\sigma-$finite positive measure and a μ-complete tribe $\mathcal{T}$. The Lebesgue space of classes of integrable functions defined on T with values in $\mathbf{R}^n$ and endowed with its strong natural topology will be denoted by $L^1(T, \mathbf{R}^n)$. As usual, $\mathbf{R}^n$ is endowed with its Borel tribe $\mathcal{B}$. Throughout, we also assume that the tribe $\mathcal{T}$ is such that the space $L^1(T, \mathbf{R}^n)$ is *separable*.

Let $x_0 \in L^1(T, \mathbf{R}^n)$ and $f : T \times \mathbf{R}^n \mapsto \mathbf{R} \cup \{+\infty\}$ be an extended real-valued function. We use the notation f_t for the function $f(t, \cdot)$, i.e., $f_t(u) = f(t, u)$ for all $u \in \mathbf{R}^n$ and $t \in T$. When the function f is $\mathcal{T} \otimes \mathcal{B}$-measurable, we will say that f is a *measurable integrand*. Then (because of the completeness of $\mathcal{T}$) the function f is a normal integrand in the sense of [4, 35] if and only if (see Corollary 14.34 in [35]) it is a measurable integrand and the functions $f(t, \cdot)$ are lower semicontinuous.

We say that an extended real-valued function g on some Banach space X is *(Dini) subdifferentially regular* at u_0 if

$$\partial^- g(u_0) = \partial_C g(u_0),$$

and when g is locally Lipschitz at u_0, this is equivalent to the equality between the lower Dini directional derivative $g^-(u_0, h)$ and the Clarke's one $g^\circ(u_0, h)$ at u_0 in each direction h. We say that a closed set $S \subset X$ is *(metrically subdifferentially) regular* at $u_0 \in S$ if the distance function $d(\cdot, S)$ is subdifferentially regular at u_0. We also recall that the Fenchel subdifferential of the extended real-valued function g at a point $u_0 \in X$ is the set

$$\partial^{\text{Fen}} g(u_0) = \{u^* \in X^* : \langle u^*, u - u_0 \rangle + g(u_0) \leq g(u), \, \forall u \in X\}.$$

Now, we are able to introduce the following class with which this section will be concerned for a large part: Let $x_0 \in L^1(T, \mathbf{R}^n)$ and set

$$\begin{aligned} \mathcal{C}(x_0) = \{f : &T \times \mathbf{R}^n \mapsto \mathbf{R} \cup \{+\infty\} \textit{ measurable integrand}, \\ & f(t, \cdot) \textit{ is not subdifferentially regular at } x_0(t), \, \mu - \text{a.e. } t \in T\}. \end{aligned}$$

The integral functional associated with any measurable integrand f on $T \times \mathbf{R}^n$ is given by

$$I_f(x) = \int_T f(t, x(t)) \, d\mu \quad \forall x \in L^1(T, \mathbf{R}^n)$$

and the subdifferential of f will be taken with respect to the second variable, that is, $\partial_L f(t, u) = \partial_L f_t(u)$ for any $u \in \mathbb{R}^n$.

If the integral functional I_f is locally Lipschitz on $L^1(T, \mathbf{R}^n)$, the *atomless* property of the measure μ assures us, according to Corollary 3.4 in Giner [12] that

$$\begin{aligned} \partial_C I_f(x_0) &= \partial_A^{\text{seq}} I_f(x_0) \\ &= \partial_A I_f(x_0) \\ &= \{x^* \in L^\infty : \, x^*(t) \in \partial_C f_t(x(w)) \, a.e. \, t \in T\}. \end{aligned} \tag{6}$$

[1] A measure is atomless if every set having positive measure contains a subset having strictly smaller but positive measure.

In fact, the local Lipschitz property of the integral functional is related to the Lipschitzian continuity of the associated integrand as stated in the following theorem which will be involved in several places in the development below. The theorem is contained in Theorem 3.1, Theorem 3.2, and Corollary 3.3 in Giner [11]. For the convenience of the reader, we provide here a quite simple proof. We also mention that Lemma 4.1 below on which the proof of the theorem strongly depends, will be needed later in the paper.

Theorem 4.1 *Let $f : T \times \mathbf{R}^n \to \mathbf{R}$ be a measurable real-valued integrand whose associated integral functional I_f is finite on $L^1(T, \mathbf{R}^n)$ and let $\gamma \geq 0$. Then the following assertions are equivalent:*
(a) The integral functional I_f is γ-Lipschitz on $L^1(T, \mathbf{R}^n)$;
(b) The integral functional I_f is γ-Lipschitz on some ball in $L^1(T, \mathbf{R}^n)$;
(c) The measurable integrand f is γ-Lipschitz, i.e., for almost all $t \in T$ the function $f(t, \cdot)$ is γ-Lipschitz on $\mathbf{R}^n$.

Proof. The implications $(a) \Rightarrow (b)$ and $(c) \Rightarrow (a)$ are obvious. It remains to prove the implication $(b) \Rightarrow (c)$. So suppose that (b) holds, i.e., there exist some $\bar{x}$ in $L^1(T, \mathbf{R}^n)$ and some $\delta > 0$ such that

$$I_f(x) - I_f(x') \leq \gamma \|x - x'\| \quad \forall x, x' \in B_{L^1(T,\mathbf{R}^n)}(\bar{x}, \delta).$$

Then for the measurable integrand $g : T \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ defined by $g(t, u, u') := f(t, u') - f(t, u) + \gamma \|u' - u\|$, the point $(\bar{x}, \bar{x})$ is a local minimum point of I_g in $L^1(T, \mathbf{R}^n \times \mathbf{R}^n)$. According to Lemma 4.1 below, there exists some negligible set $N \subset T$ such that for each $t \in T \setminus N$ one has

$$f(t, u) \leq f(t, u') + \gamma \|u' - u\| \quad \forall u, u' \in \mathbf{R}^n,$$

which completes the proof. ■

The lemma used above is a consequence of a theorem in [10].

Lemma 4.1 *Let $g : T \times \mathbf{R}^n \to \mathbf{R}$ be a measurable integrand and let $x_0 \in L^1(T, \mathbf{R}^n)$ be a point where I_g is finite. If x_0 is a local minimum of I_g in $L^1(T, \mathbf{R}^n)$, then there exists some negligible set $N \subset T$ such that for each $t \in T \setminus N$ one has*

$$g(t, x_0(t)) \leq g(t, u) \quad \forall u \in \mathbf{R}^n.$$

Proof. By Giner's theorem in [10] we know that for any $x \in L^1(T, \mathbf{R}^n)$ we have

$$g(t, x_0(t)) \leq g(t, x(t)) \ \mu - \text{a.e.}\, t. \tag{7}$$

Put $\varphi(t) := \inf_{u \in \mathbf{R}^n} [g(t, u) - g(t, x_0(t))]$. The completeness of the tribe $\mathcal{T}$ assures us that φ is measurable (see Lemma III-39 in [4]) and obviously $\varphi(t) \leq 0$ for all $t \in T$. On the other hand, the integrand h given by $h(t, u) := g(t, u) - g(t, x_0(t))$ being measurable with $I_h(x_0)$ finite, we have

$$\inf_{x \in L^1(T, \mathbf{R}^n)} I_h(x) = \int_T \inf_{u \in \mathbf{R}^n} h(t, u) \, d\mu$$

according to Theorem 14.60 in [35] whose result and proof still hold for any measurable integrand when the tribe is complete thanks to Lemma III-39 and Theorem III-22 in [4]. This tells us according to (7) and to the definition of φ that $\int_T \varphi(t)\,d\mu \geq 0$. Combining this with the fact that $\varphi(\cdot) \leq 0$ yields that $\varphi(t) = 0$ for almost all $t \in T$. Consequently, for almost all t

$$g(t, x_0(t)) \leq g(t, u) \quad \forall u \in \mathbf{R}^n,$$

which finishes the proof. ∎

Now recall that Theorem 3.2 in Chieu [5] says under the *atomless property* of the measure μ that for any $x \in L^1(T, \mathbf{R})$

$$\begin{aligned} \partial_L I_f(x) &= \partial^F I_f(x) \\ &= \{x^* \in L^\infty : x^*(t) \in \partial^{\mathrm{Fen}} f_t(x(t))\, \mu - a.e.\, t \in T\} \end{aligned} \tag{8}$$

provided the integral functional I_f is finite on $L^1(T, \mathbf{R}^n)$.
Note that, based on the definition of the limiting subdifferential, we may easily show that Chieu's Theorem 3.2 [5] is equivalent to Lemma 4.1.

We can now state the theorem concerning the class $\mathcal{C}(x_0)$.

Theorem 4.2 *Let $f \in \mathcal{C}(x_0)$ be such that I_f is finite on $L^1(T, \mathbf{R}^n)$ and Lipschitz on $L^1(T, \mathbf{R}^n)$. Then*

$$\partial_L I_f(x_0) \subsetneq \partial_A^{\mathrm{seq}} I_f(x_0) \tag{9}$$

$$\partial_C I_f(x_0) = \partial_A^{\mathrm{seq}} I_f(x_0) = \partial_A I_f(x_0). \tag{10}$$

Proof. Relation (10) follows from Giner's equality (6). Let us establish the strict inclusion (9). As we said before the inclusion $\partial_L I_f(x_0) \subset \partial_A^{\mathrm{seq}} I_f(x_0)$ is always true. We will show that it is strict, i.e., there exists an element of $\partial_A^{\mathrm{seq}} I_f(x_0)$ which is not contained in $\partial_L I_f(x_0)$. So, suppose the contrary, i.e., $\partial_L I_f(x_0) = \partial_A^{\mathrm{seq}} I_f(x_0)$. Using relation (10), we get

$$\partial_C I_f(x_0) = \partial_L I_f(x_0).$$

Now invoking Chieu's equality (8), we obtain

$$\partial_L I_f(x_0) = \partial^F I_f(x_0) \subset \{x^* \in L^\infty : \, x^*(t) \in \partial^F f(t, x_0(t))\, \mu - \text{a.e.}\}.$$

Further, by the last member of Giner's equality (6) we have

$$\{x^* \in L^\infty : \, x^*(t) \in \partial^F f(t, x_0(t))\, \mu - \text{a.e.}\} \subset \partial_C I_f(x_0),$$

and hence

$$\partial_C I_f(x_0) = \partial^- I_f(x_0) = \{x^* \in L^\infty : \, x^*(t) \in \partial^F f(t, x_0(t))\, \mu - \text{a.e.}\}.$$

Using Theorem 3.1 in [12], we get the subdifferential regularity of $f(t,\cdot)$ at $x_0(t)$ for almost all t and this contradiction completes the proof. ■

Can subdifferential regularity imply coincidence of the limiting subdifferential and the approximate subdifferential? Unfortunately, the following example shows that it is not the case.

Example 4.1 *Consider the function $f_t(u) = f(t,u) = |\sin u|$ and the set $T = [0,1]$. Then*

1. $f_t^-(0,h) = f_t^\circ(0,h) = |h|$, *(hence $f(t,\cdot)$ is (Dini) subdifferentially regular at 0);*
2. $\partial_L I_f(0) = \{0\}$*;*
3. $\partial_A I_f(0) = B_{L^\infty}$.

The equalities concerning $\partial_L I_f(x_0)$ and $\partial_A I_f(x_0)$ follow easily from the equalities between the first and last members in (8) and in (6) respectively.

The following corollary is a direct consequence of Theorem 4.2. It provides a sufficient condition for the regularity of a family of sets.

Corollary 4.1 *Let $M : T \rightrightarrows \mathbf{R}^n$ be a closed-valued and measurable multivalued mapping and $x_0 \in L^1(T,\mathbf{R}^n)$, with $x_0(t) \in M(t)$ for all $t \in T$. Consider the integral functional I_M defined on $L^1(T,\mathbf{R}^n)$ by $I_M(x) = \int_T d(x(t), M(t))\, d\mu$. If $\partial_L I_M(x_0) = \partial_A I_M(x_0)$, then for μ-almost all $t \in T$, the set $M(t)$ is (subdifferentially) regular at $x_0(t)$.*

We point out the following fact concerning the measurable multivalued mapping M. If the set $\Sigma^1(M)$ of its selections belonging to $L^1(T,\mathbf{R}^n)$ is nonempty, we have (see, for example, Theorem 2.2 in [14] for the second equality)

$$\begin{aligned} I_M(x) &= \int_T \inf_{u\in M(t)} d(x(t),u)\, d\mu \\ &= \inf_{y(\cdot)\in\Sigma^1(M)} \int_T d(x(t),y(t))\, d\mu \\ &= d(x,\Sigma^1(M)), \end{aligned}$$

that is, I_M is the distance function to $\Sigma^1(M)$ in $L^1(T,\mathbf{R}^n)$.

What about the converse in Corollary 4.1? Unfortunately, the following example shows that this may not happen.

Example 4.2 *For each $t \in T := [0,1]$, consider the constant set $M(t) = \{(x,r) \in \mathbf{R}^2 : |\sin x| \le r\}$. Then $M(t)$ is regular at $(0,0)$ for each $t \in T$, but $\partial_L I_M(0,0) \neq \partial_A I_M(0,0)$. Proposition 4.1 below provides the justification.*

This example can be extended to more general situations in the following proposition. Its statement also argues the example. Recall first that as usual $\mathbf{1}_{T_0}$ denotes the characteristic function of a subset $T_0 \subset T$ and hence $\mathbf{1}_{T_0}$ is the constant function taking the value 1 at any point of T_0.
In the proposition and its proof, the space $L^1_{\mathbf{R}^n} \times L^1_{\mathbf{R}}$ will be endowed with the norm

$$\|(x,r)\|_{L^1_{\mathbf{R}^n} \times L^1_{\mathbf{R}}} = \|x\|_{L^1_{\mathbf{R}^n}} + \|r\|_{L^1_{\mathbf{R}}}$$

and hence the corresponding dual norm in $L^\infty_{\mathbf{R}^n} \times L^\infty_{\mathbf{R}}$ is

$$\|(x^*,s)\|_{L^\infty_{\mathbf{R}^n} \times L^\infty_{\mathbf{R}}} = \max\{\|x\|_{L^\infty_{\mathbf{R}^n}}, \|r\|_{L^\infty_{\mathbf{R}}}\},$$

where $L^1_{\mathbf{R}^n} := L^1(T,\mathbf{R}^n)$, $L^1_{\mathbf{R}} := L^1(T,\mathbf{R})$, $L^\infty_{\mathbf{R}^n} := L^\infty(T,\mathbf{R}^n)$ and $L^\infty_{\mathbf{R}} := L^\infty(T,\mathbf{R})$. The proof of the proposition also uses the concept of Bouligand contingent cone $K(S,x_0)$ of a subset S of a Banach space X at a point $x_0 \in S$. Recall that a vector $v \in K(S,x_0)$ if and only if there exist a sequence $(v_k)_k$ converging to v and a sequence of positive numbers $s_k \downarrow 0$ such that $x_0 + s_k v_k \in S$ for all integers k.

Proposition 4.1 *Let $f : T \times \mathbf{R}^n \to \mathbf{R}$ be a normal integrand for which I_f is locally Lipschitzian at 0. For $M(t) := \operatorname{epi} f(t,\cdot)$, consider the set $S := \Sigma^1_M$, i.e.,*

$$S = \{(x,r) \in L^1_{\mathbf{R}^n} \times L^1_{\mathbf{R}} : \quad f(t,x(t)) \le r(t), \quad a.e.\, t \in T\}.$$

Suppose that for all $t \in T$, $f(t,0) = 0$ and $f(t,\cdot)$ is subdifferentially regular at 0. Then the following inclusions hold

$$\partial^- I_f(0) \times \{-\mathbf{1}_T\} \subset \mathbf{R}_+ \partial^- d_S(0,0) \tag{11}$$

and

$$\partial^- d_S(0,0) \cap L^\infty_{\mathbf{R}^n} \times \{-\mathbf{1}_T\} \subset \partial^- I_f(0) \times \{-\mathbf{1}_T\}. \tag{12}$$

Further, the following assertions (a) and (b) are equivalent:

(a) $(x^*, -\mathbf{1}_T) \in \mathbf{R}_+ \partial^- d_S(0,0) \Longleftrightarrow (x^*, -\mathbf{1}_T) \in \mathbf{R}_+ \partial^F d_S(0,0)$.

(b) $\partial^- I_f(0) = \partial^F I_f(0) = \{x^* \in L^\infty_{\mathbf{R}^n} : x^*(t) \in \partial^{\mathrm{Fen}} f_t(0) \text{ a.e. } t \in T\}$.

Proof. According to Theorem 4.1 and to the local Lipschitz property of I_f at 0, the function f is a γ-Lipschitz integrand for some $\gamma \ge 0$, that is, for all t outside a negligible set in T and all $u, u' \in \mathbf{R}^n$

$$|f(t,u) - f(t,u')| \le \gamma \|u - u\|. \tag{13}$$

Let us start with the proof of (11). Observe first that the subdifferential regularity assumption of the integrand ensures that

$$x^* \in \partial^- I_f(0) \Longleftrightarrow \langle x^*, h\rangle \le \int_T f_t^-(0, h(t))\, d\mu \quad \forall h \in L^1_{\mathbf{R}^n}$$

or equivalently

$$\langle x^*, h\rangle \le \int_T r(t)\, d\mu - I_f(h) + \int_T f_t^-(0, h(t))\, d\mu \quad \forall (h, r) \in S$$

or equivalently there exists some constant $K > 0$ such that

$$\langle x^*, h\rangle \le \int_T r(t)\, d\mu - I_f(h) + \int_T f_t^-(0, h(t))\, d\mu + K d_S(h, r) \quad \forall h \in L^1_{\mathbf{R}^n}, r \in L^1_{\mathbf{R}},$$

because the functions $h \mapsto I_f(h)$ and $h \mapsto \int_T f_t^-(0, h(t))\, d\mu$ are globally Lipschitz on $L^1_{\mathbf{R}^n}$ according to (13). Taking into account the subdifferential regularity of $f(t, \cdot)$ again and the equality $\lim_{s\to 0^+} \dfrac{I_f(sh)}{s} = \int_T f_t^-(0, h(t))\, d\mu$, we get

$$\langle x^*, h\rangle \le \int_T r(t)\, d\mu + K d_S^-((0,0), (h, r)) \quad \forall h \in L^1_{\mathbf{R}^n}, r \in L^1_{\mathbf{R}},$$

which is equivalent to say that $(x^*, -\mathbf{1}_T) \in K\partial^- d_S(0, 0)$. So, the inclusion of (11) is established.
Let us prove the inclusion of (12).
Take any $(x^*, -\mathbf{1}_T) \in \partial^- d_S(0, 0)$. Then $\|(x^*, -\mathbf{1}_T)\|_{L^\infty_{\mathbf{R}^n} \times L^\infty_{\mathbf{R}}} \le 1$ and $(x^*, -\mathbf{1}_T)$ is in the negative polar of the contingent cone $K(S, (0, 0))$ of S at $(0, 0)$, i.e.,

$$\langle x^*, v\rangle - \int_T \rho(t)\, d\mu \le 0, \quad \forall (v, \rho) \in K(S, (0, 0)). \tag{14}$$

Let $(h, r) \in S$ and $s_k \to 0^+$. Then

$$f(t, s_k h(t)) \le s_k[r(t) + \frac{f(t, s_k h(t))}{s_k} - f(t, h(t))] \quad \text{a.e.}\, t \in T. \tag{15}$$

Set $w_k(t) = \frac{f(t, s_k h(t))}{s_k} - f(t, h(t))$ and $w(t) = f_t^-(0, h(t)) - f(t, h(t))$. Since $|\frac{f(t, s_k h(t))}{s_k}| \le \gamma |h(t)|$, the Lebesgue dominated convergence theorem yields $w_k \to w$ in $L^1_{\mathbf{R}}$, and hence $(h, r + w) \in K(S, (0, 0))$ because (15) means that $s_k(h, r + w_k) \in S$ for all integers n. This allows us to get according to (14)

$$\langle x^*, h\rangle \le \int_T r(t)\, d\mu + \int_T w(t)\, d\mu, \quad \forall (h, r) \in S,$$

that is,

$$\langle x^*, h\rangle \le \int_T r(t)\, d\mu + I_f^-(0, h) - I_f(h), \quad \forall (h, r) \in S. \tag{16}$$

Fix now any $h \in L^1_{\mathbf{R}^n}$ and observe that for $r(t) := f(t, h(t))$ we have $r \in L^1_{\mathbf{R}}$. Consequently, $(h, r) \in S$ and by (16) we have

$$\langle x^*, h\rangle \le I_f^-(0, h),$$

and hence $x^* \in \partial^- I_f(0)$. This finishes the proof of the inclusion in (12) of the proposition.

Let us now establish the equivalence between (a) and (b). Let us start with the implication $(a) \Longrightarrow (b)$. Take any $x^* \in \partial^- I_f(0)$. The inclusion of (11) says that $(x^*, -\mathbf{1}_T) \in \mathbf{R}_+ \partial^- d_S(0,0)$. By (a) we have $(x^*, -\mathbf{1}_T) \in \mathbf{R}_+ \partial^F d_S(0,0)$, that is, $\alpha(x^*, -\mathbf{1}_T) \in \partial^F d_S(0,0)$ for some $\alpha > 0$. Putting $g(t,u,s) := d((u,s), \operatorname{epi} f_t)$ for all $(u,s) \in \mathbf{R}^n \times \mathbf{R}$, the observation after Corollary 4.1 assures us that $\alpha(x^*, -\mathbf{1}_T) \in \partial^F I_g(0,0)$. Chieu's equality (8) tells us that for almost all $t \in T$ we have

$$\alpha(x^*(t), -1) \in \partial^{\mathrm{Fen}} g_t(0,0) = \partial^{\mathrm{Fen}} d_{\operatorname{epi} f_t}(0,0).$$

Since $(0,0) \in \operatorname{epi} f_t$, the definition of Fenchel subdifferential of $d_{\operatorname{epi} f_t}$ at $(0,0)$ easily gives that for almost all $t \in T$

$$\inf_{h \in \mathbf{R}^n} \{ f(t,h) - \langle x^*(t), h \rangle \} \geq 0, \text{ i.e., } x^*(t) \in \partial^{\mathrm{Fen}} f_t(0),$$

and hence in particular $x^* \in \partial^F I_f(0)$. The last two inclusions obviously entail the equalities in (b) of the proposition.
It remains to show that $(b) \Rightarrow (a)$. Suppose that (b) holds. The implication $\Rightarrow$ of (a) being always true, let us prove the reverse one. Fix $(x^*, -\mathbf{1}_T) \in \partial^- d_S(0,0)$. The inclusion of (12) ensures that $x^* \in \partial^- I_f(0)$. By assumption (b), for almost all $t \in T$ we have $x^*(t) \in \partial^{\mathrm{Fen}} f_t(0)$. It is not difficult to see that this implies that for all $(h,r) \in S$

$$\langle x^*, h \rangle \leq \int_T r(t)\, d\mu,$$

and hence

$$\langle x^*, h \rangle - \int_T r(t)\, d\mu \leq d_S(h,r)$$

for all $(h,r) \in L^1_{\mathbf{R}^n} \times L^1_{\mathbf{R}}$ since $\|(x^*, -\mathbf{1}_T)\|_{L^\infty_{\mathbf{R}^n} \times L^\infty_{\mathbf{R}}} \leq 1$. This assures us in particular that $(x^*, -\mathbf{1}_T) \in \partial^F d_S(0,0)$ and completes the proof. ■

Remark 4.1 *This proposition implies that*

$$\partial^- d_S(0) = \partial^F d_S(0) \Longrightarrow \partial^- I_f(0) = \partial^F I_f(0).$$

In the previous part of this section, we address the study of the coincidence of $\partial_L I_f$ and $\partial_A I_f$ at a *fixed* point $x_0 \in L^1(T, \mathbf{R}^n)$. The next part is concerned with the coincidence of those subdifferentials over the whole space $L^1(T, \mathbf{R}^n)$.

We start with the following proposition which will be used in the proof of the result related to the coincidence of the subdifferentials at all points. Recall first that for a multivalued mapping G defined on X, its effective domain is the set $\operatorname{Dom} G := \{u \in X : G(u) \neq \emptyset\}$.

Proposition 4.2 *Let X be a Banach space and $g : X \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function whose restriction to* $\operatorname{Dom} g$ *is continuous and with* $\operatorname{int} \operatorname{Dom} g \neq \emptyset$. *Then the function g is convex if and only if* $\operatorname{Dom} g$ *is a convex set and and there exists a subset $D \subset (\operatorname{Dom} \partial^{\mathrm{Fen}} g) \cap \operatorname{int} \operatorname{Dom} g$ which is dense in* $\operatorname{Dom} g$.

Proof. If the function g is convex, then its effective domain is obviously convex. Further, the continuity of the convex function g on $\operatorname{int}\operatorname{Dom} g$ assures us that its Fenchel subdifferential is nonempty at any point of $\operatorname{int}\operatorname{Dom} g$ and of course $\operatorname{int}\operatorname{Dom} g$ is dense in $\operatorname{Dom} g$ according to the convexity of $\operatorname{Dom} g$.
Suppose now that the property of the proposition holds, i.e., $\operatorname{Dom} g$ is convex and there exist a subset $D \subset \operatorname{int}\operatorname{Dom} g$ which is dense in $\operatorname{Dom} g$ and such that $\partial^{\mathrm{Fen}} g(u) \neq \emptyset$ for all $u \in D$. Let $u \in \operatorname{Dom} g$ and $\lambda \in]0,1[$. Take any $v \in \operatorname{int}\operatorname{Dom} g$. By the convexity assumption of $\operatorname{Dom} g$ we have $\lambda u + (1-\lambda)v \in \operatorname{Dom} g$ and hence there exists a sequence $(y_k)_k$ in D converging to $\lambda u + (1-\lambda)v$. Choose for each integer k some $y_k^* \in \partial^{\mathrm{Fen}} g(y_k)$. For each integer k take the vector v_k given by $y_k - u = (1-\lambda)(v_k - u)$. It is easily seen that $(v_k)_k$ converges to v, and hence deleting a finite number of k if necessary we may suppose that $v_k \in \operatorname{int}\operatorname{Dom} g$. By definition of the Fenchel subdifferential we have

$$\langle y_k^*, v_k - y_k\rangle \leq g(v_k) - g(y_k) \quad \text{and} \quad \langle y_k^*, u - y_k\rangle \leq g(u) - g(y_k)$$

and hence after multiplication by $(1-\lambda)$ and λ respectively we obtain

$$\langle y_k^*, (1-\lambda)v_k - (1-\lambda)y_k\rangle \leq (1-\lambda)g(v_k) - (1-\lambda)g(y_k)$$

and

$$\langle y_k^*, \lambda u - \lambda y_k\rangle \leq \lambda g(u) - \lambda g(y_k).$$

Adding the two latter inequalities yields

$$g(y_k) \leq \lambda g(u) + (1-\lambda)g(v_k),$$

and the continuity of g on $\operatorname{Dom} g$ allows us to write that

$$g(\lambda u + (1-\lambda)v) \leq \lambda g(u) + (1-\lambda)g(v). \tag{17}$$

Finally, fix any $v \in \operatorname{Dom} g$. Take a sequence $(v_k')_k$ in D converging to v. Applying (17) with v_k' and taking the limit yields

$$g(\lambda u + (1-\lambda)v) \leq \lambda g(u) + (1-\lambda)g(v),$$

which completes the proof. ■

Theorem 4.3 *Let $f : T \times \mathbf{R}^n \to \mathbf{R}$ be a real-valued normal integrand such that I_f is finite and continuous on $L^1(T, \mathbf{R}^n)$. Then there exists a dense set $\mathcal{D}$ in $L^1(T, \mathbf{R}^n)$ such that*

$$\partial_L I_f(x) \neq \emptyset \ \forall x \in \mathcal{D} \tag{18}$$

if and only if for μ-almost all $t \in T$ the function $f(t, \cdot)$ is convex.

Proof. The convexity of $f(t, \cdot)$ for almost all $t \in T$ obviously entails the convexity of I_f. Therefore, the function I_f being in addition continuous on $L^1(T, \mathbf{R}^n)$ we get that $\partial_L I_f(x) = \partial^{\mathrm{Fen}} I_f(x) \neq \emptyset$ for any $x \in L^1(T, \mathbf{R}^n)$.

Suppose now that for some dense set $\mathcal{D}$ in $L^1(T, \mathbf{R}^n)$ one has $\partial_L I_f(x) \neq \emptyset$ for all $x \in \mathcal{D}$. Chieu's equality (8) ensures that $\mathcal{D} \subset \operatorname{Dom} \partial^{\mathrm{Fen}} I_f$. Proposition 4.2 then assures us that I_f is convex. So, for all $\lambda \in]0,1[$ and $x, y \in L^1(T, \mathbf{R}^n)$ we have

$$I_f(\lambda x + (1-\lambda) y) \leq \lambda I_f(x) + (1-\lambda) I_f(y).$$

Define the function $g : T \times \mathbf{R}^n \times \mathbf{R}^n \mapsto \mathbf{R}$ by

$$g(t, u, v) = \lambda f(t, u) + (1-\lambda) f(t, v)) - f(t, \lambda u + (1-\lambda) v).$$

Note that g is a measurable integrand and $g(t, u, u) = 0$ for all $u \in \mathbf{R}^n$. Then for all $x \in L^1(T, \mathbf{R}^n)$, the function I_g attains its minimum on $L^1(T, \mathbf{R}^n) \times L^1(T, \mathbf{R}^n)$ at (x, x). By Lemma 4.1 we get

$$\inf_{u,v \in \mathbf{R}^n} g(t, u, v) \geq 0 \text{ a.e. } t \in T.$$

Let $\{\lambda_k\}$ be a countable dense set in $[0,1]$. Then for all k, there exists $N_k \subset T$ such that $\mu(N_k) = 0$ and

$$\inf_{u,v \in \mathbf{R}^n} \{\lambda_k f(t, u) + (1-\lambda_k) f(t, v)) - f(t, \lambda_k u + (1-\lambda_k) v)\} \geq 0, \, \forall t \in T \backslash N_k.$$

Now for the negligible set $N := \bigcup_{k \in \mathbf{N}} N_k$ we see that for all $t \in T \setminus N$ and $k \in \mathbf{N}$

$$\lambda_k f(t, u) + (1-\lambda_k) f(t, v)) \geq f(t, \lambda_k u + (1-\lambda_k) v) \, \forall u, v \in \mathbf{R}^n.$$

Using the lower semicontinuity of $f(t, \cdot)$, we obtain the convexity of $f(t, \cdot)$ for each $t \in T_0$. ■

Remark 4.2 *It follows from the statement of the theorem that the assumptions made there ensure the equivalence between relation (18) and the following one:*

$$\partial_L I_f(x) \neq \emptyset \, \forall x \in L^1(T, \mathbf{R}^n).$$

Note that the local Lipschitzness property of I_f (and hence also, by Theorem 4.1, the global one) is entailed by the convexity and the continuity of I_f.

The result concerning the dense or global coincidence of the limiting subdifferential and the approximate subdifferential on $L^1(T, \mathbf{R}^n)$ can now be stated.

Theorem 4.4 *Let $f : T \times \mathbf{R}^n \to \mathbf{R}$ be a real-valued normal integrand such that I_f is finite and Lipschitz on $L^1(T, \mathbf{R}^n)$. Then there exists a dense set $\mathcal{D}$ in $L^1(T, \mathbf{R}^n)$ such that*

$$\partial_L I_f(x) = \partial_A I_f(x) \, \forall x \in \mathcal{D}$$

if and only if for μ-almost all $t \in T$ the function $f(t, \cdot)$ is convex.

Proof. Since the approximate subdifferential of a (locally) Lipschitz function is nonempty at any point, the equality in the theorem ensures the nonemptiness of $\partial_L I_f$ at any point of $\mathcal{D} \subset L^1(T, \mathbf{R}^n)$ and hence the theorem follows from Theorem 4.3. ■

Concerning decomposable sets of $L^1(T, \mathbf{R}^n)$ we have the following corollary.

Corollary 4.2 *Let $M : T \rightrightarrows \mathbf{R}^n$ be a measurable multivalued mapping with nonempty closed values and admitting at least a selection in $L^1(T, \mathbf{R}^n)$. Then for the integral functional I_M of Corollary 4.1, we have*

$$\partial_L I_M(x) = \partial_A I_M(x) \quad \forall x \in L^1(T, \mathbf{R}^n)$$

if and only if for μ-almost all $t \in T$ the set $M(t)$ is convex.

Proof. Putting $f(t,u) := d(u, M(t))$ for all $t \in T$ and $u \in \mathbf{R}^n$, the function f is a normal integrand which satisfies the assumptions of Theorem 4.4. Then, there exists a measurable subset T_0 whose complement in T is negligible and such that for each $t \in T_0$ the distance function $d(\cdot, M(t))$ is convex. So, the set $M(t)$ being closed, we conclude that $M(t)$ is convex for all $t \in T_0$. ■

Remark 4.3 *The results of the paper can be easily extended as stated to the context of infinite dimensional separable Banach space X instead of $\mathbf{R}^n$ provided $L^1(T, X)$ is separable. But to avoid technicality, we have restricted our study to the finite dimensional situation.*

5 Applications to the Aubin property and to the convexity of multivalued mappings

Let $M : T \times \mathbf{R}^m \rightrightarrows \mathbf{R}^n$ be a multivalued mapping which is $\mathcal{T} \otimes \mathcal{B}(\mathbf{R}^m)$-measurable in the sense of [4, 35], that is,

$$M^{-1}(U) := \{(t,x) \in T \times \mathbb{R}^m : M(t,x) \cap U \neq \emptyset\} \in \mathcal{T} \otimes \mathcal{B}(\mathbf{R}^m)$$

for any open set U in $\mathbb{R}^n$. Assume also that the graph of $M(t, \cdot)$ is closed for almost all $t \in T$ and that $t \mapsto d(0, M(t, x(t))$ is μ-integrable for all $x \in L^1(T, \mathbf{R}^m)$.
For each $x \in L^1(T, \mathbf{R}^m)$ consider

$$\Sigma^1 M(x) := \{y \in L^1(T, \mathbf{R}^n) : y(t) \in M(t, x(t)) \text{ a.e.}\},$$

so that $\Sigma^1 M : L^1(T, \mathbf{R}^m) \rightrightarrows L^1(T, \mathbf{R}^n)$ defines a multivalued mapping with closed graph.

Proposition 5.1 *With the above notations, the following assertions are equivalent:*

(a) The multivalued mapping $\Sigma^1 M$ has the γ-Aubin property at some point $(\bar{x}, \bar{y})$ with $\bar{y}(t) \in M(t, \bar{x}(t))$ a.e. $t \in T$; that is, there exists $r > 0$ such that

$$\Sigma^1 M(x) \cap B_{L^1(T,\mathbf{R}^n)}(\bar{y}, r) \subset \Sigma^1 M(x') + \gamma \|x - x'\| \mathbf{B}_{L^1(T,\mathbf{R}^n)}$$

for all $x, x' \in B_{L^1(T,\mathbf{R}^m)}(\bar{x}, r)$;
(b) The multivalued mapping $\Sigma^1 M$ is γ-Lipschitzian on $L^1(T, \mathbf{R}^m)$; that is,

$$\Sigma^1 M(x) \subset \Sigma^1 M(x') + \gamma \|x - x'\| \mathbf{B}_{L^1(T,\mathbf{R}^n)} \quad \forall x, x' \in L^1(T, \mathbf{R}^m);$$

(c) For almost all $t \in T$ the multivalued mapping $u \rightrightarrows M(t, u)$ is γ-Lipschitzian on $\mathbf{R}^m$.

Proof. For a multivalued mapping $G : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ with nonempty closed values, it is easy to see that G is γ-Lipschitzian if and only if

$$d(v', G(u')) \le d(v, G(u)) + \|v' - v\| + \gamma \|u' - u\|$$

for all $u, u' \in \mathbf{R}^m$ and $v, v' \in \mathbf{R}^n$. By 9.37 in [35], the γ-Aubin property of G at $(\bar{u}, \bar{v}) \in \operatorname{gph} G$ is characterized by the existence of some neighborhoods U and V of $\bar{u}$ and $\bar{v}$ respectively such that

$$d(v', G(u')) \le d(v, G(x)) + \|v' - v\| + \gamma \|u' - u\|$$

for all $u, u' \in U$ and $v, v' \in V$. Then, observing as in the previous section that

$$d_{L^1(T,\mathbf{R}^n)}(y, \Sigma^1 M(x)) = \int_T d(y(t), M(t, x(t))\, d\mu$$

and using the norm on $\mathbf{R}^m \times \mathbf{R}^n$ given by $\|u\| + \gamma \|v\|$ one easily sees that the result follows from Theorem 4.1. ■

We know by Corollary 1.6 in Hiai-Umegaki [14] that $\Sigma^1 M$ takes convex values if and only if for each $x \in L^1(T, \mathbf{R}^m)$ the set $M(t, x(t))$ is convex for almost all $t \in T$. We rely next the convexity of the graph of $M(t, \cdot)$ to the limiting subdifferential of the functions $(x, y) \mapsto \Delta_{\Sigma^1 M}(x, y)$ and $(u, v) \mapsto \Delta_{M(t,\cdot)}(u, v)$ defined by

$$\Delta_{\Sigma^1 M}(x, y) = d_{L^1(T,\mathbf{R}^n)}(y, \Sigma^1 M(x)) \quad \forall (x, y) \in L^1(T, \mathbf{R}^m) \times L^1(T, \mathbf{R}^n)$$

$$\Delta_{M(t,\cdot)}(u, v) := d(v, M(t, u)) \quad \forall (u, v) \in \mathbf{R}^m \times \mathbf{R}^n.$$

There is a connection between this type of functions and the Fréchet normal cone as well as the limiting normal cone to the graph of the multivalued mapping $\Sigma^1 M$. This link has been established by Thibault in [36] for the limiting subdifferential with the use of the Ekeland variational principle, while for Fréchet subdifferential, Thibault [37] gives a simpler and direct proof by using the definition of Fréchet subdifferential. More precisely, Thibault showed that when the graph of a multivalued mapping $G : X \rightrightarrows Y$ (between two Banach spaces X and Y) is closed, then for any $(\bar{x}, \bar{y})$ in this graph,

$$N_L(\text{gph}G, (\bar{x}, \bar{y})) = \mathbf{R}_+ \partial_L \Delta_G(\bar{x}, \bar{y}) \quad \text{and} \quad N^F(\text{gph}G, (\bar{x}, \bar{y})) = \mathbf{R}_+ \partial^F \Delta_G(\bar{x}, \bar{y}).$$

Proposition 5.2 *One has $\partial_L \Delta_{\Sigma^1 M}(x, y) \neq \emptyset$ for all (x, y) in a dense subset of $L^1(T, \mathbf{R}^m) \times L^1(T, \mathbf{R}^n)$ if and only if for almost all $t \in T$ the multivalued mapping $M(t, \cdot)$ has a convex graph.*

Proof. It suffices to apply Theorem 4.3 and to see that the multivalued mapping $M(t, \cdot)$ has a convex graph iff the function $(u, v) \mapsto \Delta_{M(t,\cdot)}(u, v)$ is convex. ■

6 Application to necessary optimality conditions for Bolza problem

Our aim in this section is to give new necessary optimality conditions for Bolza problem (1). The domain over which the minimization occurs is typically the space $W^{1,1}([a, b], \mathbf{R}^n)$ (abbreviated $W^{1,1}$), consisting of all absolutely continuous functions $x\colon [a, b] \mapsto \mathbf{R}^n$ ($\dot{x}$ denotes the derivative (almost everywhere) of x), where a, b are real numbers with $a < b$.
As in [38], we define the functions $\alpha : \mathbf{R}^n \times L^1([a, b], \mathbf{R}^n) \to \mathbf{R}^n \times \mathbf{R}^n$ and $\beta : \mathbf{R}^n \times L^1([a, b], \mathbf{R}^n) \to L^1([a, b], \mathbf{R}^n) \times L^1([a, b], \mathbf{R}^n)$ by putting for all $(\zeta, v) \in \mathbf{R}^n \times L^1([a, b], \mathbf{R}^n)$

$$\alpha(\zeta, v) = (\zeta, \zeta + \int_a^b v(s)\, ds), \quad \beta(\zeta, v) = (t \mapsto \zeta + \int_a^t v(s)\, ds, v).$$

Observe that the spaces $W^{1,1}([a, b], \mathbf{R}^n)$ and $\mathbf{R}^n \times L^1([a, b], \mathbf{R}^n)$ are isomorphic through the bicontinuous bijective linear mapping $J : W^{1,1}([a, b], \mathbf{R}^n) \to \mathbf{R}^n \times L^1([a, b], \mathbf{R}^n)$ given by $J(x) = (x(a), \dot{x})$. So, with the above notations, an element $z \in W^{1,1}([a, b], \mathbf{R}^n)$ is a local solution of problem (1) if and only if the point $(z(a), \dot{z})$ is a local solution with respect to $(\zeta, v) \in \mathbf{R}^n \times L^1([a, b], \mathbf{R}^n)$ of the problem

$$\min(\ell \circ \alpha)(\zeta, v) + (I_{\mathcal{L}} \circ \beta)(\zeta, v) \tag{19}$$

where $I_{\mathcal{L}}(x, y) = \int_a^b \mathcal{L}(t, x(t), y(t))dt$ for all $(x, y) \in L^1([a, b], \mathbf{R}^n) \times L^1([a, b], \mathbf{R}^n)$.
Following [38], the adjoint mappings $\alpha^* : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n \times L^\infty([a, b], \mathbf{R}^n)$ and $\beta^* : L^\infty([a, b], \mathbf{R}^n) \times L^\infty([a, b], \mathbf{R}^n) \to \mathbf{R}^n \times L^\infty([a, b], \mathbf{R}^n)$ of α and β are given by

$$\alpha^*(\zeta, \xi) = (\zeta + \xi, \tilde{\xi}) \quad \text{and} \quad \beta^*(q, p) = (\int_a^b q(s)ds, p - Q)$$

where $\tilde{\xi} : t \mapsto \xi$ is a constant function and $Q(t) = -\int_t^b q(s)ds$ for all $t \in [a, b]$.
We assume that $\mathcal{L} : [a, b] \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is $\mathcal{T} \otimes \mathcal{B} \otimes \mathcal{B}$-measurable.
To avoid technicality, the assumptions will be made directly on $I_{\mathcal{L}}$ instead of $\mathcal{L}$.

Proposition 6.1 *Let z be a local solution of Bolza problem (1) in $W^{1,1}$. Suppose that ℓ and $I_{\mathcal{L}}$ are locally Lipschitz at $(z(a), z(b))$ and $(z, \dot{z})$ respectively. Suppose also that*

$$\partial_A^{\mathrm{seq}} I_{\mathcal{L}}(z, \dot{z}) = \partial_L I_{\mathcal{L}}(z, \dot{z}). \tag{20}$$

Then there exists $p \in W^{1,1}$ *such that*

$$(p(a), -p(b)) \in \partial_L \ell((z(a), z(b))$$

and such that for a.e. $t \in [a,b]$

$$\sup_{\zeta,\xi \in \mathbf{R^n}} \{\langle \dot{p}(t), \zeta\rangle + \langle p(t), \xi\rangle - \mathcal{L}(t,\zeta,\xi)\} = \\ \langle \dot{p}(t), z(t)\rangle + \langle p(t), \dot{z}(t)\rangle - \mathcal{L}(t, z(t), \dot{z}(t)).$$

Proof. We follow some arguments of Proposition 3.4 in [38]. According to (19) and to the analysis above we have

$$(0,0) \in \partial_A(\ell \circ \alpha + I_{\mathcal{L}} \circ \beta)(z(a), \dot{z}).$$

The functions ℓ and $\mathcal{L}$ being locally Lipschitzian near $(z(a), z(b))$ and $(z, \dot{z})$ respectively, we have by approximate subdifferential calculus rules (see [15]) that

$$\partial_A(\ell \circ \alpha + I_{\mathcal{L}} \circ \beta)(z(a), \dot{z}) \subset \alpha^*\big(\partial_A \ell(z(a), z(b))\big) + \beta^*\big(\partial_A I_{\mathcal{L}}(z, \dot{z})\big),$$

and hence there are $(\rho, \sigma) \in \mathbf{R}^n \times \mathbf{R}^n$ and $(q,p) \in L^\infty([a,b], \mathbf{R}^n) \times L^\infty([a,b], \mathbf{R}^n)$ with $(\rho, \sigma) \in \partial_A \ell(z(a), z(b))$ and $(q,p) \in \partial_A I_{\mathcal{L}}(z, \dot{z})$ such that

$$(0,0) = \alpha^*(\rho, \sigma) + \beta^*(q,p) = (\rho + \sigma, \tilde{\sigma}) + (\int_a^b q(s)\, ds, \rho - Q),$$

where as above $\tilde{\sigma}(t) = \sigma$ and $Q(t) = -\int_t^b q(s)\, ds$ for all $t \in [a,b]$. Therefore,

$$\rho + \sigma = -\int_a^b q(s)\, ds \quad \text{and} \quad p(t) + \int_t^b q(s)\, ds = -\sigma \quad \text{for a.e. } t \in [a,b]. \tag{21}$$

The second equality of (21) tells us that p (in fact a representative of its equivalence class) is absolutely continuous with $\dot{p}(t) = q(t)$ for a.e. t, and the continuity of p combined with the second equality of (21) gives $p(b) = -\sigma$ and $p(a) + \int_a^b q(s)\, ds = -\sigma$. Using the latter equality in the first equality of (21) we obtain $\rho = p(a)$. Consequently

$$(p(a), -p(b)) \in \partial_A \ell(z(a), z(b)) \quad \text{and} \quad (p, \dot{p}) \in \partial_A I_{\mathcal{L}}(z, \dot{z}). \tag{22}$$

Theorem 4.2 and relation (20) ensure that $\partial_A I_{\mathcal{L}}(z, \dot{z}) = \partial_L I_{\mathcal{L}}(z, \dot{z})$, and hence relation (8) and the second inclusion of (22) entail that for almost every $t \in [a,b]$ we have

$$(\dot{p}(t), p(t)) \in \partial^{\mathrm{Fen}} \mathcal{L}(t, z(t), \dot{z}(t)),$$

that is, for almost every $t \in [a,b]$

$$\langle \dot{p}(t), \zeta\rangle + \langle p(t), \xi\rangle - \mathcal{L}(t,\zeta,\xi) \le \\ \langle \dot{p}(t), z(t)\rangle + \langle p(t), \dot{z}(t)\rangle - \mathcal{L}(t, z(t), \dot{z}(t))$$

for all $\zeta, \xi \in \mathbf{R}^n$. Further, the function ℓ being defined on the finite dimensional space $\mathbf{R}^n \times \mathbf{R}^n$ we have $\partial_A \ell(z(a), z(b)) = \partial_L \ell(z(a), z(b))$ and hence $(p(a), -p(b)) \in \partial_L \ell(z(a), z(b))$. The proof of the theorem is then complete. ■

Consider now a closed subset C of $\mathbf{R}^n \times \mathbf{R}^n$ and the problem

$$\begin{cases} \text{Minimize } \int_a^b \mathcal{L}(t, x(t), \dot{x}(t))\, dt \\ \text{subject to the constraint } (x(a), x(b)) \in C, \end{cases} \tag{23}$$

where $\mathcal{L}$ satisfies the local Lipschitz property above near z. If z is a local solution of problem (23), adapting the method of Lemma 3.2 in Clarke [6] (see also [38, Lemma 3.5]) one obtains some constant $\gamma > 0$ such that z is a local solution of problem (1) for $\ell(\zeta, \xi) = \gamma d_C(\zeta, \xi)$ for all $(\zeta, \xi) \in \mathbf{R}^n \times \mathbf{R}^n$. So, we deduce the following corollary.

Corollary 6.1 *Let z be a local solution of problem (23) in $W^{1,1}$. Suppose that $I_{\mathcal{L}}$ satisfies the assumptions of Proposition 6.1. Then there exists $p \in W^{1,1}$ such that*

$$(p(a), -p(b)) \in N_L\big(C, (z(a), z(b))\big)$$

and such that for a.e. $t \in [a, b]$

$$\sup_{\zeta, \xi \in \mathbf{R^n}} \{\langle \dot{p}(t), \zeta \rangle + \quad \langle p(t), \xi \rangle - \mathcal{L}(t, \zeta, \xi)\} = \\ \langle \dot{p}(t), z(t) \rangle + \langle p(t), \dot{z}(t) \rangle - \mathcal{L}(t, z(t), \dot{z}(t)).$$

Remark 6.1 *Under the convexity of ℓ, Proposition 6.1 implies that z is a local solution to the Bolza problem (1) if and only if it is a global one.*

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