Lagrange multipliers for multiobjective programs with a general preference

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Dedicated to Boris Mordukhovich in honour of his 60th birthday

Abstract. We consider a nonsmooth multiobjective optimization problems related to a new general preference between infinite dimensional Banach spaces. This preference contains preferences given by generalized Pareto as well as those given by an utility function. We use the concepts of compactly epi-Lipschitzian sets and strongly compactly Lipschitzian mappings to derive Lagrange multipliers of Karush-Kuhn-Tucker type and Fritz-John type in terms of the Ioffe-approximate subdifferentials.
Key words. Nonsmooth analysis, calmness, Lagrange multipliers, preference, multiobjective optimization, scalar optimization.

1 Introduction

In this paper we shall obtain existence of Lagrange multipliers for multiobjective optimization problems in Banach spaces with a general preference. The problem that we consider here is of the form

$$(P)\begin{cases} \min f(x) \\ \text{subject to} \\ x \in C \text{ and } g(x) \in D \end{cases}$$

where $f\colon X \mapsto Z$ and $g\colon X \mapsto Y$ are mappings, X, Y and Z are Banach spaces and $C \subset X$ and $D \subset Y$ are closed sets.

Historically, the concept of preference appeared in the value theory in economics. Many authors in the early studies often defined the preference by an utility function, i.e, given a preference whether its always possible to find an utility function that can determine the preference.
In [8] the author proved, in finite dimension, that a preference $\prec$ determined by a continuous utility function if and only if for any x the sets

$$\{y : x \prec y\} \quad \text{and} \quad \{y : y \prec x\} \quad \text{are closed.} \tag{1}$$

This theorem is not general and besides this it is an existence theorem (i.e do not provide methods for determining utility function) and there are some useful preference that does not satisfy (1).
Always in a historical context, when Kuhn and Tucker proved the Kuhn-Tucker theorem in 1950, which gives necessary conditions for the existence of an optimal solution to a nonlinear programming problem, they launched the theory of nonlinear programming. Karush derived a result that was comparable to the Kuhn-Tucker theorem. Like Karush, Fritz-John looked at the finite-dimensional case and formulated a result that later was acknowleged as a version of Kuhn-Tucker theorem.
For nonsmooth multiobjective optimization problem, necessary optimality conditions can be established under various qualification conditions (constraint qualification conditions). Among well known qualification conditions are the Guignard qualification, Slater qualification, linear objective qualification, Mangasarian-Fromovitz qualification, Robinson qualification, Zowe qualification and Kurcyusz qualification. It has been shown that the last three ones imply the metric regularity, and hence Calmness, of the corresponding set-valued mappings expressing feasibility.
Many works devoted qualification conditions to ensuring the nonvacuity and boudedness of Lagrange multipliers sets of Kuhn-Tucker type ([10], [11], [23], [34], [43] and [40]), or to study differential stability of a marginal function in nonlinear programs ([12], [35] and [38]) and or to obtain subdifferential calculus rules.

Problem (P) is studied in finite or infinite dimensional spaces. For deriving necessary conditions, some authors involved locally Lipschitz functions ([17],[28], [30] and [41]) others consider strongly compactly Lipschitzian functions ([10] and [13]), using Ekeland's variational principle [9] and assuming that D is a closed convex cone with nonempty interior or $D = D_1 \times \{0\}$ with $\{0\} \subset \mathbb{R}^n$ and D_1 a closed convex cone with nonempty interior.

Involving the approximate subdifferential, the authors in [24] derived Lagrange multipliers of Fritz-John type and Kuhn-Tucker type for problem (P)

in Banach spaces by assuming that D is epi-Lipschitz like [3] and [4] and in [27] by assuming that D is compactly epi-Lipschitzian.

In this paper, which appear in the Bellaassali Phd Thesis [1], we give a regularity definition of preferences. We use calmness qualification (i.e. calmness or pseudo-upper Lipschitz continuity of set-valued mappings [36], [39]) and compactly epi-Lipschitzian notion of Borwein and Strojwas [4], [5] and [6], to show the existence of Karush-Kuhn-Tucker Lagrange multipliers in terms of the approximate subdifferential for multiobjective optimization problem with a general preference, and from this result we derive some examples.
The noncalmness case is treated and gives Fritz-John Lagrange multipliers of problem (P). Our result is applied to produce necessary optimality conditions for problems of type

$$\min_{x \in X} F(x)$$

where $F : X \to Y$ is a set-valued mapping with closed graph.

The paper is organized as follows. Section 2 contains the key definitions and some useful results. Section 3 shows the existence of Karush-Kuhn-Tucker Lagrange multipliers for problem (P). Some examples of preferences are given. Section 4 is devoted to Fritz-John Lagrange multipliers of problem (P). Section 5 is concerned with Lagrange multipliers for single-objective programs. Section 6 is concerned with differentiable case.

2 Approximate subdifferentials and preliminaries

Throughout we shall assume that X, Y and Z are Banach spaces, X^*, Y^* and Z^* are their topological duals and $\langle \cdot, \cdot \rangle$ is the pairing between the spaces. We denote by B_X, $B_{X^*}, \cdots$ the closed unit balls of X, $X^*, \cdots$. By $d(\cdot, S)$ we denote the usual distance function to the set S

$$d(x, S) = \inf_{u \in S} \|x - u\|.$$

We write $x \xrightarrow{f} x_0$ and $x \xrightarrow{S} x_0$ to express $x \to x_0$ with $f(x) \to f(x_0)$ and $x \to x_0$ with $x \in S$, respectively.
If f is an extended-real-valued function on X, we write for any subset S of X

$$f_S(x) = \begin{cases} f(x) & \text{if } x \in S, \\ +\infty & \text{otherwise.} \end{cases}$$

The function

$$d^- f(x,h) = \liminf_{\substack{u \to h \\ t \downarrow 0}} t^{-1}(f(x+tu) - f(x))$$

is the lower Dini directional derivative of f at x and the Dini ε-subdifferential of f at x is the set

$$\partial_\varepsilon^- f(x) = \{x^* \in X^* : \langle x^*, h\rangle \le d^- f(x;h) + \varepsilon\|h\|, \forall h \in X\}$$

for $x \in Dom f$ and $\partial_\varepsilon^- f(x) = \emptyset$ if $x \notin Dom f$, where $Dom f$ denotes the effective domain of f. For $\varepsilon = 0$ we write $\partial^- f(x)$.
By $\mathcal{F}(X)$ we denote the collection of finite dimensional subspaces of X. The approximate subdifferential and the singular approximate subdifferential of f at $x_0 \in Dom f$ are respectively defined by the following expressions (see Ioffe [18] and [19])

$$\partial f(x_0) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{x \xrightarrow{f} x_0} \partial^- f_{x+L}(x) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{\substack{x \xrightarrow{f} x_0 \\ \varepsilon \downarrow 0}} \partial_\varepsilon^- f_{x+L}(x)$$

and

$$\partial^\infty f(x_0) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{\substack{x \xrightarrow{f} x_0 \\ \varepsilon \downarrow 0 \\ \lambda \downarrow 0}} \lambda \partial_\varepsilon^- f_{x+L}(x)$$

where
$$\limsup_{\substack{x \xrightarrow{f} x_0 \\ \varepsilon \downarrow 0}} \partial_\varepsilon^- f_{x+L}(x) =$$
$$\{x^* \in X^* : x^* = w^* - lim\, x_i^*, x_i^* \in \partial_{\varepsilon_i}^- f_{x_i+L}(x_i), x_i \xrightarrow{f} x_0,\ \varepsilon_i \downarrow 0\},$$
that is, the set of w^*-limits of all such nets. The approximate normal cone to a closed set $S \subset X$ at $x_0 \in S$ is defined by

$$N_A(S, x_0) = \partial \Psi_S(x_0)$$

where Ψ_S is the indicator function of the set S, that is $\Psi_S(x) = 0$ if $x \in S$ and $\Psi_S(x) = +\infty$ if $x \notin S$. In [18], Ioffe obtained the following geometrical characterizations

$$\partial f(x_0) = \{x^* \in X^* :\ (x^*, -1) \in N_A(\text{epi} f, (x_0, f(x_0)))\}. \tag{2}$$

$$\partial^\infty f(x_0) = \{x^* \in X^* : (x^*, 0) \in N_A(\text{epi} f, (x_0, f(x_0))\}. \tag{3}$$

where epi$f$ denotes the epigraph of f, that is,

$$\text{epi} f = \{(x, r) \in X \times \mathbb{R} : f(x) \le r\}.$$

It is easily seen that the set-valued mapping $x \to \partial f(x)$ is upper semicontinuous in the following sense

$$\partial f(x_0) = \limsup_{x \xrightarrow{f} x_0} \partial f(x)$$

and in [18] and [19] Ioffe has shown that when S is a closed subset of X and $x_0 \in S$

$$\partial d(x_0, S) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{\substack{x \xrightarrow{S} x_0 \\ \varepsilon \downarrow 0}} \partial_\varepsilon^- d_{x+L}(x, S) \cap (1 + \varepsilon) B_{X^*}. \tag{4}$$

The (G-)normal cone $N(S, x_0)$ to S at $x_0 \in S$ is defined by

$$N(S, x_0) = \bigcup_{\lambda > 0} \lambda \partial d(x_0, S).$$

For $x_0 \notin S$ we set $N(S, x_0) = \emptyset$.
In [21], the second author established that the approximate normal cone and the normal cone coincide for compactly epi-Lipschitzian sets (see Section 3 for the definition).
It is also well known that the approximate subdifferential coincides with the limiting Fréchet subdifferential, initiated by Kruger and Mordukhovich (see [32]) and references therein), for locally Lipschitzian function in weakly compactly generated Asplund spaces (see [33]). Note that this result has been extended in [21] to lower semicontinuous functions whose epigraph is compactly epi-Lipschitzian.
In the sequel we shall need the following class of mappings between Banach spaces.

Definition 2.1 *[37]. A mapping $g : X \mapsto Y$ is said to be strongly compactly Lipschitzian (s.c.L.) at a point x_0 if there exist a set-valued mapping $R : X \mapsto 2^{Comp(Y)}$, where $Comp(Y)$ denotes the set of all norm compact subsets of Y, and a function $r : X \times X \to \mathbb{R}_+$ satisfying*
(i) $\lim\limits_{\substack{x \to x_0 \\ h \to 0}} r(x, h) = 0$,

(ii) there exists $\alpha > 0$ *such that*

$$t^{-1}[g(x+th) - g(x)] \in R(h) + \|h\| r(x, th) B_Y$$

for all $x \in x_0 + \alpha B_X, h \in \alpha B_X$ *and* $t \in]0, \alpha[$,
(iii) $R(0) = \{0\}$ *and* R *is upper semicontinuous.*

It can be shown [37] that every s.c.L. mapping is locally Lipschitzian. In finite dimensions the concepts coincide.
Recently we have developped in [26] a chain rule for this class of mappings. Let us note that this chain rule has been obtained before by Ioffe in [19] for maps with compact prederivatives.

Theorem 2.1 *[26]. Let* $g : X \to Y$ *be a s.c.L. mapping at* x_0 *and let* $f : Y \to \mathbb{R}$ *be a locally Lipschitz function at* $g(x_0)$*. Then* $f \circ g$ *is locally Lipschitz at* x_0 *and*

$$\partial(f \circ g)(x_0) \subset \bigcup_{y^* \in \partial f(g(x_0))} \partial(y^* \circ g)(x_0).$$

To complete this section we note the following property of s.c.L. mappings which is a direct consequence of Proposition 2.3 in [27].

Proposition 2.1 *Let* $g : X \to Y$ *be s.c.L. at* x_0 *and let* (y_i^*) *any bounded net of* Y^* *which* w^*-*converges to zero in* Y^* *and let* (x_i) *be a net norm-converging to* x_0 *in* X*. If* $x_i^* \in \partial(y_i^* \circ g)(x_i)$*, then* (x_i^*) w^**-converges to zero in* X^**.*

3 Karush-Kuhn-Tucker Lagrange multipliers

We start by recalling some of the prominent Lipschitz properties formulated for set-valued mappings. Let $M : Y \mapsto X$ be a set-valued mapping. M is said to have the *Aubin property* at some $(\bar{y}, \bar{x})$ in the graph of M, if there exist neighborhoods $\mathcal{Y}$ and $\mathcal{X}$ of $\bar{y}$ and $\bar{x}$ as well as some $K > 0$ such that

$$d(x, M(y)) \leq K\|y - y'\| \quad \forall y, y' \in \mathcal{Y}\ \forall x \in M(y') \cap \mathcal{X}.$$

Fixing one of the $y-$parameters as $\bar{y}$ in the definition of the Aubin property, yields the calmness ([36]) of M at $(\bar{y}, \bar{x})$:

$$d(x, M(\bar{y})) \leq K\|y - \bar{y}\| \quad \forall y \in \mathcal{Y}\ \forall x \in M(y) \cap \mathcal{X}.$$

Obviously, the Aubin property implies calmness whereas the converse is not true (e.g. $M(y) = \{x : x^2 \geq y\}$).

Before giving sufficient conditions for calmness for a special class of set-valued mappings, let us recall the following notion by Borwein and Strojwas [4], [5] and [6]. A set $S \subset X$ is said to be *compactly epi-Lipschitzian* at $x_0 \in S$ if there exist $\gamma > 0$ and a norm compact set $H \subset X$ such that

$$S \cap (x_0 + \gamma B_X) + t\gamma B_X \subset S - tH, \quad \text{for all} \quad t \in]0, \gamma[.$$

Theorem 3.1 *Let $D \subset Y$ and $C \subset X$ be two closed subsets and $g : X \to Y$ be a s.c.L. mapping at $\bar{x} \in C \cap g^{-1}(D)$. Suppose that D is compactly epi-Lipschitz at $g(x_0)$. Suppose also that the following regularity condition holds at $\bar{x}$*

$$[y^* \in \partial d(g(\bar{x}), D) \quad \textit{and} \quad 0 \in \partial(y^* \circ g + d(\cdot, C))(\bar{x})] \Longrightarrow y^* = 0.$$

Then the set-valued mapping $M : Y \mapsto X$ defined by

$$M(y) = \{x \in C : y \in -g(x) + D\}$$

has the Aubin property at the point $(0, \bar{x})$ and hence it is calm at this point or equivalently for some real numbers $a \geq 0$ and $r > 0$

$$d(x, C \cap g^{-1}(D)) \leq a[d(g(x), D) + d(x, C)]$$

for all $x \in \bar{x} + rB_X$.

Proof. See Jourani and Thibault [27].$\diamondsuit$
Note that new "boundary" conditions for calmness for this class of set-valued mappings in finite dimension have been discovered in [15]. Similar conditions have been presented in [14] for convex systems in infinite dimensional situation.
Using the definition of the approximate subdifferential we can easily get the following proposition.

Proposition 3.1 *Suppose that g is s.c.L. at $\bar{x} \in C \cap g^{-1}(D)$ and that the set-valued mapping M in Theorem 3.1 is calm at $(0, \bar{x})$. Then*

$$\partial d(\cdot, M(0)) \subset \bigcup_{y^* \in N(D, g(\bar{x}))} \partial(y^* \circ g)(\bar{x}) + N(C, \bar{x}).$$

Our aim in this section is to show how to use calmness to obtain the existence of Karush-Kuhn-Tucker Lagrange multipliers for multiobjective optimization problems with a general preference. The problem that we will consider is of the form

$$(P)\begin{cases}\min f(x)\\ \text{subject to}\\ x \in C \text{ and } g(x) \in D\end{cases}$$

where $f : X \mapsto Z$ and $g : X \mapsto Y$ are mappings and $C \subset X$ and $D \subset Y$ are closed sets.
Let $\prec$ be a nonreflexive preference for vector in Z. Let $\bar{x}$ be a feasible point for (P). We say $\bar{x}$ is a solution to problem (P) provided that there exists no other feasible point x for (P) such that $f(x) \prec f(\bar{x})$.
For any $z \in Z$ we denote $\mathcal{L}(z) := \{z' \in Z : z' \prec z\}$.
Then $\bar{x}$ is a solution to (P) if and only if $f(C \cap g^{-1}(D)) \cap \mathcal{L}(f(\bar{x})) = \emptyset$.

We need the following regularity assumptions on the preference.

Definition 3.1 *We say that a preference $\prec$ is regular at $z \in Z$ provided that*
(D_1) for any $u \in Z$ near z, $u \in cl\mathcal{L}(u)$;
(D_2) for any u and w near z satisfying $u \prec z$, $w \in cl\mathcal{L}(u)$ we have $w \prec z$;
(D_3) there exists a locally compact cone K^* in Z^* such that for any nets $z_i, z_i' \to z$ in Z

$$N(cl\mathcal{L}(z_i), z_i') \subset K^*.$$

Recall that a set K^* in Z^* is *weak-star locally compact* if every point of K^* lies in a weak-star open set V such that $cl^*(V) \cap K$ is weak-star compact. The first important property of these cones has been established by Loewen in [29] in a reflexive Banach space (but the proof works in any Banach space). He showed that if (z_i^*) is a net in a locally compact cone K^* then

(z_i^*) weak-star converges to 0 iff it converges in norm to 0.

Note that our assertion (D_3) always holds in finite dimensional situation. Our definition of regularity differs from that introduced by Zhu [42]. Indeed the definition in [42] was given in finite dimensional spaces and assumed in addition to (D_1) and (D_2) the following one : for any sequences $z_n, z_n' \to z$

$$\limsup_{n\to\infty} N(cl\mathcal{L}(z_n), z_n') \subset N(cl\mathcal{L}(\bar{z}), \bar{z}).$$

Unfortunately, contrary to what it is indicated in [42], the Zhu's regularity does not hold for a preference determined by an utility function.

Example 3.1 *[2] Let* $u : \mathbb{R}^2 \to \mathbb{R}$ *be an utility function defined by* $u(x, y) = |x| - |y|$. *Then* $N(\mathcal{L}(0), 0) = \{(x, y) \in \mathbb{R}^2 : |x| = |y|\}$ *while* $\cup_{\lambda>0}\lambda\partial u(0, 0) = \cup_{\lambda>0}\lambda([-1, 1] \times \{-1, 1\})$. *So that the preference is not regular in the sense by Zhu [42], but it is regular in the sense of Definition 3.1 (see Proposition 3.2).*

For this reason we introduce the cone

$$\tilde{N}(cl\mathcal{L}(z), z) := w^* - \limsup_{z'', z' \to z} N(cl\mathcal{L}(z''), z').$$

Note that for the preference in Example 3.1 we have

$$\tilde{N}(cl\mathcal{L}(0), 0) = \cup_{\lambda\geq 0}\lambda([-1, 1] \times \{-1, 1\}).$$

Throughout the rest of this section, we make the following standing assumption: *f and g are s.c.L. at* $\bar{x}$. Our main result of this section is the following.

Theorem 3.2 *Let* $\bar{x}$ *be a solution to problem* (P). *Suppose the preference* $\prec$ *is regular at* $f(\bar{x})$ *and the set-valued mapping* $M : Y \mapsto X$, *defined by* $M(y) = \{x \in C : y \in -g(x) + D\}$, *is calm at* $(0, \bar{x})$. *Then there exist* $z^* \in \tilde{N}(cl(\mathcal{L}(f(\bar{x})), f(\bar{x}))$, *with* $z^* \neq 0$, *and* $y^* \in N(D, g(\bar{x}))$ *such that*

$$0 \in \partial(z^* \circ f)(\bar{x}) + \partial(y^* \circ g)(\bar{x}) + N(C, \bar{x}).$$

Proof. Let (θ_k) be a sequence in Z such that

$$\theta_k \prec f(\bar{x}) \quad \text{and} \quad \| \theta_k - f(\bar{x}) \| < \frac{1}{k^2}.$$

Define the sets $\Theta := cl(\mathcal{L}(\theta_k))$ and $\Gamma = C \cap g^{-1}(D)$ and the function

$$h(x, \theta) = \begin{cases} \| f(x) - \theta \| & \text{if } \in B(\bar{x}, s_1), \\ +\infty & \text{otherwise.} \end{cases}$$

where s_1 is such that f and g are Lipschitzian on $B(\bar{x}, s_1)$ with constant $k_f = k_g$. Because of (D_1), $(\bar{x}, \theta_k) \in \Gamma \times \Theta$ and hence

$$h(\bar{x}, \theta_k) \leq \inf_{(x,\theta)\in\Gamma\times\Theta} h(x, \theta) + \frac{1}{k^2}.$$

So, since Γ and Θ are closed and h is lower semicontinuous, Ekeland's variational principle produces $(x_k, \gamma_k) \in \Gamma \times \Theta$ such that

$$\| x_k - \bar{x} \| + \| \theta_k - \gamma_k \| < \frac{1}{k}$$

$$h(x_k,\gamma_k) \le h(x,\theta) + \frac{1}{k}(\parallel x - x_k \parallel + \parallel \theta - \gamma_k \parallel \quad \forall (x,\theta) \in \Gamma \times \Theta.$$

As h is locally Lipschitzian around (x_k,γ_k) we get for (x,θ) near (x_k,γ_k)

$$\begin{aligned} h(x_k,\gamma_k) \le h(x,\theta) &+ \frac{1}{k}(\parallel x - x_k \parallel + \parallel \theta - \gamma_k \parallel) \\ &+(k_f+2)[d(x,\Gamma) + d(\theta,\Theta)] \end{aligned}$$

and hence

$$(0,0) \in \partial h(x_k,\gamma_k) + (k_f+2)[\partial d(x_k,\Gamma) \times \partial d(\gamma_k,\Theta)] + \frac{1}{k}(B_{X^*} \times B_{Z^*}).$$

As $\bar{x}$ is a local solution to problem (P), then by (D_2), and the choice of θ_k one has $\gamma_k \neq f(x_k)$. Since f is s.c.L., Theorem 2.1 implies that

$$\partial h(x_k,\gamma_k) \subset \bigcup_{\|z^*\|=1} (\partial(z^* \circ f)(x_k) \times \{-z^*\}).$$

Thus there are $z_k^* \in Z^*$ with $\parallel z_k^* \parallel = 1$ and $(a_k^*, b_k^*) \in \frac{1}{k}\mathbb{B}$ such that

$$a_k^* \in \partial(z_k^* \circ f)(x_k) + (k_f+2)\partial d(x_k,\Gamma)$$

$$b_k^* + z_k^* \in (k_f+2)\partial d(\gamma_k, cl\mathcal{L}(\theta_k))).$$

Now, extracting subnet if necessary, we may suppose, using (D_3) and Proposition 2.1, $z_k^* \to z^*$, with $z^* \neq 0$,

$$0 \in \partial(z^* \circ f)(\bar{x}) + (k_f+2)\partial d(\bar{x},\Gamma)$$

$$z^* \in \tilde{N}(cl(\mathcal{L}(f(\bar{x}))), f(\bar{x})).$$

So the proof is terminated by applying Proposition 3.1.$\diamondsuit$
In the remainder of this section, we will examine few examples.

Example 3.2 *(Generalized Pareto) Let $K \subset Z$ be a convex cone with K^0 locally compact ($K^0 = \{z^* \in Z^* : \langle z^*, z\rangle \le 0 \quad \forall z \in K\}$). We now define the preference by $z \prec z'$ if and only if $z - z' \in K$ and $z \neq z'$. In this case $\mathcal{L}(z) = \{z' \in Z :\ z' \neq z,\ z' - z \in K\}$. Multiobjective programs with this preference are called generalized Pareto optimization problems. When $Z = \mathbb{R}^m$ and $K = \mathbb{R}^m_-$ (resp. $K =$ int $\mathbb{R}^m_-$) we get Pareto (resp. weak Pareto) optimization problems. It is easy to check that preference defined in this way satisfies assumptions $(D_1) - (D_3)$ in Definition 3.1.*

Thus we have

Corollary 3.1 *Under assumptions of Theorem 3.2, there exist* $z^* \in K^0$*, with* $z^* \neq 0$*, and* $y^* \in N(D, g(\bar{x}))$ *such that*

$$0 \in \partial(z^* \circ f)(\bar{x}) + \partial(y^* \circ g)(\bar{x}) + N(C, \bar{x}).$$

Our next example considers a preference determined by an utility function.

Example 3.3 *(A preference defined by an utility function) Let* $u : Z \mapsto \mathbb{R} \cup \{+\infty\}$ *be a continuous utility function that determines the preference, i.e.,* $z \prec z'$ *if and only if* $u(z) < u(z')$*. We need additional assumptions to ensure the regularity of the preference which we summarize in the following proposition.*

Proposition 3.2 *Let* u *be a continuous utility function determining the preference* $\prec$*. Suppose that the epigraph* epi$u$ *of* u *is compactly epi-Lipschitzian at* $(\bar{z}, u(\bar{z}))$ *and*

$$\liminf_{z \to \bar{z}} d(0, \partial u(z)) > 0. \tag{5}$$

Then the preference is regular at $\bar{z}$ *and*

$$\tilde{N}(cl\mathcal{L}(\bar{z}), \bar{z}) = w^* - \limsup_{z \to \bar{z}} N(cl\mathcal{L}(z), z) = \partial^\infty u(\bar{z}) \bigcup (\bigcup_{\lambda > 0} \lambda \partial u(\bar{z})).$$

Proof. Conditions (D_1) and (D_2) follow from the continuity of u. Since epi$u$ is CEL at $(\bar{z}, u(\bar{z}))$, condition (D_3) follows from [27]. Relation (5) guarantees the existence of two real numbers $a > 0$ and $\alpha > 0$ such that

$$d(z, \{v \in Z : \ u(z) \leq r\}) \leq a \max(0, u(z) - r) \tag{6}$$

for all $z \in B(\bar{z}, \alpha)$ and $r \in B(u(\bar{z}), \alpha)$. Let $z^* \in \tilde{N}(cl\mathcal{L}(\bar{z}), \bar{z})$, with $z^* \neq 0$. Then there are nets $z_i, z_i' \to \bar{z}$, $(\lambda_i) \subset]0, +\infty[$ and (z_i^*) such that

$$\lambda_i z_i^* \to z^*, \quad z_i' \in cl\mathcal{L}(z_i), \quad z_i^* \in \partial d(cl\mathcal{L}(z_i), z_i')).$$

Thus, by (4), we have for each collection (L) of finite dimensional subspaces of Z there are nets $z_{ij} \to z_i$, $z_{ij}^* \to z_i^*$ and $r_j, s_j \to 0^+$ such that

$$z_{ij} \in cl\mathcal{L}(z_i), \quad \|z_{ij}^*\| \leq 1 + s_j$$

and

$$d(z, cl\mathcal{L}(z_i)) - \langle z_{ij}^*, z - z_{ij} \rangle + s_j \|z - z_{ij}\| \geq 0$$

for all $z \in B(z_{ij}, r_j) \cap (z_{ij} + L)$. Since $z^* \neq 0$ we obtain $u(z_i) = u(z_{ij})$, otherwise, by the definition of $cl\mathcal{L}(z_i)$ and the continuity of u, z_{ij} will be an internal point of $cl\mathcal{L}(z_i)$.In this case, we obtain

$$-\langle z_{ij}^*, z - z_{ij}\rangle + s_j\|z - z_{ij}\| \geq 0, \text{ for } z \in z_{ij} + L \text{ near } z_{ij},$$

which implies that $z_{ij}^* \in L^\top + s_j B_{Z^*}$, where $L^\top = \{v^* \in Z^* : \langle v^*, v\rangle = 0\, \forall v \in L\}$. Thus $z_i^* \in L^\top$ and hence $\lambda_i z_i^* \in L^\top$. Since L is arbitrary, we get $z^* = 0$, and this is a contradiction. Now, invoking (6) we get

$$a \max(0, u(z) - u(z_i)) - \langle z_{ij}^*, z - z_{ij}\rangle + s_j\|z - z_{ij}\| \geq 0 \tag{7}$$

for $z \in z_{ij} + L$ near z_{ij}. Consider the function $h_1 : Z \times \mathbb{R} \mapsto \mathbb{R} \cup \{+\infty\}$ defined by $h_1(z, r) = a \max(0, r - u(z_i))$ and let h_2 be the indicator function of epi$u$. By relation (7), we get

$$h_1(z, r) + h_2(z, r) - \langle z_{ij}^*, z - z_{ij}\rangle + s_j\|z - z_{ij}\| \geq 0$$

for $z \in z_{ij} + L$ near z_{ij} and r near $u(z_i)$, which implies that $(z_i^*, 0) \in \partial(h_1 + h_2)(z_i, u(z_i))$. By the subdifferential calculus ([18], [19], [26] and [28]), we get

$$(z_i^*, 0) \in \partial h_1(z_i, u(z_i)) + \partial h_2(z_i, u(z_i))$$

and as $\partial h_1(z_i, u(z_i)) = \{0\} \times [0, a]$, there exists $\beta_i \in [0, a]$ such that $(z^*, -\beta_i) \in \partial h_2(z_i, u(z_i)) = N_A(\text{epi}u, (z_i, u(z_i)))$, or equivalently, either $z_i^* \in \partial^\infty u(z_i)$ or $z_i^* \in \beta_i \partial u(z_i)$. Thus

$$\lambda_i z_i^* \in \partial^\infty u(z_i) \cup \lambda_i \beta_i \partial u(z_i).$$

Having in mind relation (5) we get the boundedness of $(\lambda_i \beta_i)$ and

$$z^* \in \partial^\infty u(\bar{z}) \bigcup \Big(\bigcup_{\lambda > 0} \lambda \partial u(\bar{z})\Big).$$

Conversely let $z^* \in \partial u(\bar{z})$, with $z^* \neq 0$. As epi$u$ is CEL at $(\bar{x}, u(\bar{x}))$, it follows ([21]) that $(z^*, -1) \in \bigcup_{\lambda > 0} \lambda \partial d((\bar{z}, u(\bar{z})), \text{epi}u)$. Then there exist $(w^*, r^*) \in \partial d((\bar{z}, u(\bar{z})), \text{epi}u)$ and $\lambda > 0$ such that $(z^*, -1) = \lambda(w^*, r^*)$. For each collection (L) of finite dimensional subspaces of Z there exist nets $z_i \to \bar{z}$, $r_i^* \to r^*$, $z_i^* \to w^*$ and $r_i, s_i \to 0^+$ such that

$$d((z, r), \text{epi}u) + (2 + s_i) d(z, z_i + L) - \langle z_i^*, z - z_i\rangle - r_i^*(r - u(z_i)) +$$

$$s_i[\|z - z_i\| + |r - u(z_i)|] \geq 0$$

for all $z \in B(z_i, r_i)$ and $r \in B(u(z_i), r_i)$. Thus

$$(2 + s_i)d(z, z_i + L) - \langle z_i^*, z - z_i \rangle + s_i \|z - z_i\| \geq 0$$

for all $z \in cl\mathcal{L}(z_i) \cap B(z_i, r_i)$. By a classical penalty argument ([7]), we have

$$(4 + s_i)d(z, cl\mathcal{L}(z_i)) + (2 + s_i)d(z, z_i + L) - \langle z_i^*, z - z_i \rangle + s_i \|z - z_i\| \geq 0$$

for all z near z_i. Thus

$$z_i^* \in (4 + s_i)\partial d(z_i, cl\mathcal{L}(z_i)) + s_i B_{Z^*} + L^{\perp}$$

and hence $z^* \in \tilde{N}(cl\mathcal{L}(\bar{z}), \bar{z})$. The proof of the inclusion $\partial^{\infty} u(\bar{z}) \subset \tilde{N}(cl\mathcal{L}(\bar{z}), \bar{z})$ is similar because, as previously, $z^* \in \partial^{\infty} u(\bar{z})$ iff $(z^*, 0) \in \bigcup_{\lambda > 0} \lambda \partial d((\bar{z}, u(\bar{z})), \text{epi}u)$.$\diamondsuit$

Using this proposition we have the following corollary of Theorem 3.2.

Corollary 3.2 *Let $\prec$ be a preference determined by an utility function u. Suppose that u satisfies the conditions of Proposition 3.2 with $\bar{z} = f(\bar{x})$. Then under the assumptions of Theorem 3.2 there exist $y^* \in N(D, g(\bar{x}))$ and $z^* \in \partial^{\infty} u(f(\bar{x})) \bigcup [\bigcup_{\lambda > 0} \lambda \partial u(f(\bar{x}))]$, with $z^* \neq 0$, such that*

$$0 \in \partial(z^* \circ f)(\bar{x}) + \partial(y^* \circ g)(\bar{x}) + N(C, \bar{x}).$$

From Theorem 3.2 we can obtain the following corollary, on necessary optimality conditions for set-valued optimization problem

$$\min_{x \in X} F(x) \tag{8}$$

where $F : X \to Y$ is a set-valued mapping with closed graph.
Let $(\bar{x}, \bar{y}) \in GrF$. The point $\bar{x}$ said to be a minimum (with respect to $\bar{y}$) to the problem (8) related to a general preference $\prec$ if and only if

$$F(X) \cap \mathcal{L}(\bar{y}) = \emptyset.$$

Let us recall that the set-valued mapping $D^*F(x, y) : Y^* \to X^*$ defined by:

$$D^*F(x, y)(y^*) = \{x^* \in X^* : (x^*, -y^*) \in N(GrF; (x, y)\}$$

is called the coderivative of F at the point $(x, y) \in GrF$.

Corollary 3.3 *Let $\bar{x}$ be a minimum (with respect to $\bar{y}$) to the problem (8) related to a general preference $\prec$. Suppose that $\prec$ is regular at $\bar{y}$. Then there exists $z^* \in \tilde{N}(cl\mathcal{L}(\bar{y}), \bar{y})$ with $z^* \neq 0$, such that $0 \in D^*F(\bar{x}, \bar{y})(z^*)$.*

Proof. Let $f : X \times Y \to Y$ be a mapping defined by $f(x, y) = y$. Consider the following multiobjective optimization problem

$$\min_{(x,y)\in GrF} f(x, y). \tag{9}$$

Then $\bar{x}$ is a minimum (with respect to $\bar{y}$) to the problem (8), if and only if $(\bar{x}, \bar{y})$ is a solution of problem (9). Using Theorem 3.2, there exist $z^* \in \tilde{N}(cl\mathcal{L}(\bar{y}), \bar{y})$ with $z^* \neq 0$, such that

$$0 \in \partial(z^* \circ f)(\bar{x}, \bar{y}) + N(GrF, (\bar{x}, \bar{y})$$

which gives $0 \in D^*F(\bar{x}, \bar{y})(z^*)$. $\diamondsuit$
In [25] the author got the same result, but for the case of Pareto optimum.

4 Fritz-John Lagrange multipliers

In this section, we consider the case where the set-valued mapping M in Theorem 3.2 is not necessarily calm.

Theorem 4.1 *Let $\bar{x}$ be a solution to problem (P). Suppose the preference $\prec$ is regular at $f(\bar{x})$ and that D is CEL at $g(\bar{x})$. Then there exist $z^* \in \tilde{N}(cl(\mathcal{L}(f(\bar{x})), f(\bar{x}))$, and $y^* \in N(D, g(\bar{x}))$, with $(z^*, y^*) \neq 0$, such that*

$$0 \in \partial(z^* \circ f)(\bar{x}) + \partial(y^* \circ g)(\bar{x}) + N(C, \bar{x}).$$

Proof. Use Theorems 3.1 and 3.2.
We have to note that the conclusion of this theorem holds under assumptions of the previous corollaries without calmness. Namely:

Corollary 4.1 *Let the assumptions of Corollary 3.2 hold without calmness assumption. Then there exist $y^* \in N(D, g(\bar{x}))$ and*
$z^* \in \partial^\infty u(f(\bar{x})) \bigcup [\bigcup_{\lambda>0} \lambda \partial u(f(\bar{x}))]$, with $(y^*, z^*) \neq 0$, such that

$$0 \in \partial(z^* \circ f)(\bar{x}) + \partial(y^* \circ g)(\bar{x}) + N(C, \bar{x}).$$

5 Lagrange multipliers for single-objective programs

In this section we consider problem (P) with $Z = \mathbb{R}$. Our aim here is to give necessary optimality conditions under general assumptions.

Theorem 5.1 *Let $\bar{x}$ be a solution to problem (P) with $Z = \mathbb{R}$ and $\mathcal{L}(r) = r + \mathbb{R}_+$. Suppose that*
1) *f is lower semicontinuous on X and g is s.c.L. at $\bar{x}$;*
2) *the set-valued mapping $M : Y \mapsto X$ defined by $M(y) = \{x \in C : y \in -g(x) + D\}$ is calm at $(0, \bar{x})$;*
3) *either*
i) epif is CEL at $(\bar{x}, f(\bar{x}))$ or
ii) C and D are CEL at $\bar{x}$ and $g(\bar{x})$ respectively and for all $y^* \in \partial d(D, g(\bar{x}))$, with $y^* \neq 0$, we have

$$0 \notin \partial(y^* \circ g)(\bar{x}) + \partial d(C, \bar{x});$$

4) $\partial^\infty f(\bar{x}) \cap (-N(C \cap g^{-1}(D))) = \{0\}$.
Then there exists $y^* \in N(D, g(\bar{x}))$ such that

$$0 \in \partial f(\bar{x}) + \partial(y^* \circ g)(\bar{x}) + N(C, \bar{x}).$$

Proof. The case of $3) - i)$ is established in Jourani [20]. So it suffices to consider case $3) - ii)$. Since $\bar{x}$ is a solution to (P), then $0 \in \partial(f + \Psi_{C \cap g^{-1}(D)})(\bar{x})$, where Ψ_H denotes the indicator function of the set H. It follows from [22] that the set $C \cap g^{-1}(D)$ is CEL at $\bar{x}$. Now using 4) we get ([20]) $0 \in \partial f(\bar{x}) + N(C \cap g^{-1}(D), \bar{x})$. The proof is terminated by applying Proposition 3.1. $\diamondsuit$

6 The differentiable case

In this section, we consider programs with differentiable data. A mapping $h : X \mapsto Y$ is said to be strictly differentiable at $\bar{x}$ if

$$\lim_{x,y \to \bar{x}} \frac{h(x) - h(y) - Dh(\bar{x})(x - y)}{\|x - y\|} = 0.$$

To simplify we assume in our problem (P) that $C = X$. Thus in the differentiable case our previous results may be expressed in a simple way.

Corollary 6.1 *Suppose in addition to the assumptions of Theorem 3.2 (resp. Theorem 4.1) that f and g are strictly differentiable at $\bar{x}$. Then there exist $z^* \in \tilde{N}(cl(\mathcal{L}(f(\bar{x})), f(\bar{x}))$ and $y^* \in N(D, g(\bar{x}))$, with $z^* \neq 0$ (resp. $(z^*, y^*) \neq 0$), such that $z^* \circ Df(\bar{x}) + y^* \circ Dg(\bar{x}) = 0$.*

Corollary 6.2 *In addition to the assumptions of Corollary 3.2 (resp. Corollary 4.1) we suppose that f and g are strictly differentiable at $\bar{x}$. Then there exist $y^* \in N(D, g(\bar{x}))$ and $z^* \in \partial^\infty u(f(\bar{x})) \bigcup [\bigcup_{\lambda > 0} \lambda \partial u(f(\bar{x}))]$, with $z^* \neq 0$ (resp. $(z^*, y^*) \neq 0$), such that $z^* \circ Df(\bar{x}) + y^* \circ Dg(\bar{x}) = 0$.*

Note that if we suppose in Corollary 6.2 that u is strictly differentiable at $f(\bar{x})$ then there exist $y^* \in N(D, g(\bar{x}))$ and $\lambda \geq 0$ with $\lambda \neq 0$ (resp. $(\lambda, y^*) \neq 0$), such that

$$\lambda \nabla u(f(\bar{x})) \circ Df(\bar{x}) + y^* \circ Dg(\bar{x}) = 0.$$

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References

[1] S. Bellaassali, Contributions to multiobjective optimization, Phd Thesis, Université de Bourgogne, France, june 2003.

[2] S. Bellaassali and A. Jourani, Necessary optimality conditions in multiobjective dynamic optimization, SIAM J. Control Optim. **42** (2004), 2043-2061.

[3] J. M. Borwein, Epi-Lipschitz-like sets in Banach space : theorems and examples, Nonlinear Analysis, Vol. **11**, no. 10, pp. 1207-1217, 1987.

[4] J. M. Borwein and H. Strojwas, Tangential approximations, Nonlinear Analysis, Vol. **9**, no. 12, pp. 1347-1366, 1985.

[5] J. M. Borwein and H. M. Strojwas, Proximal analysis and boundaries of closed sets in Banach space, Part I : Theory. Canadian Journal of Mathematics, Vol. **38**, no. 2, pp. 431-452, 1986; Part II : Applications. Canadian Journal of Mathematics, Vol. **39**, no. 2, pp. 428-472, 1987.

[6] J. M. Borwein and H. M. Strojwas, Directionally Lipschizian mappings on Baire spaces, Canadian Journal of Mathematics, Vol. **36**, no. 1, pp. 95-130, 1984.

[7] F.H. Clarke, "Optimization and nonsmooth analysis", Wiley-Interscience, New-York, (1983).

[8] G. Debreu, Mathematiccal economics, Twenty papers of Gerard Debreu, Cambrige Press, pp. 163-172, 1983.

[9] I. Ekeland, On the variational principle, Journal of Mathematical Analysis and Applications, Vol. **47**, pp. 324-353, 1974.

[10] B. El Abdouni and L. Thibault, Lagrange multipliers for Pareto nonsmooth programming problems in Banach space, Optimization, Vol. **26**, no. 3-4, pp. 277-285, 1992.

[11] J. Gauvin, A necessary and sufficient regularity condition to have bounded multipliers in nonconvex programming, Mathematical Programming, Vol. **12**, no. 1, pp. 136-138, 1977.

[12] J. Gauvin and R. Janin, Directional behavior of optimal solutions set in nonlinear mathematical programming, Mathematics of Operations Research, Vol. **13**, no. 4, pp. 629-649, 1988.

[13] B. M. Glover and B. D. Craven, A Fritz-John optimality condition using the approximate subdifferential, Journal of Optimization Theory and Applications, Vol. **82**, no. 2, pp. 253-265, 1994.

[14] R. Henrion and A. Jourani, Boundary-oriented subdifferential characterization of calmness systems, SIAM Journal on Optimization, Vol. **13**, pp. 520-534, 2002.

[15] R. Henrion, A. Jourani and J. Outrata, On calmness of a class of multifunctions, SIAM Journal on Optimization, Vol. **13**, pp. 603-618, 2002.

[16] R. Henrion and J. Outrata, A subdifferential condition for calmness of multifunctions, Journal of Mathematical Analysis and Applications, Vol. **258**, no. 1, pp. 110-130, 2001.

[17] Y. Ishizuka, Optimality conditions for directionally differentiable multiobjective programming problems, Journal of Optimization Theory and Applications, Vol. **72**, no. 1, pp. 91-111, 1992.

[18] A. D. Ioffe, Approximate subdifferentials and applications 2 : Functions on locally convex spaces, Mathematika, Vol. **33**, pp. 111-128, 1986.

[19] A. D. Ioffe, Approximate subdifferentials and applications 3 : Metric theory, Mathematika, Vol. **36**, no. 1, pp. 1-38, 1989.

[20] A. Jourani, Intersection formulae and the marginal function in Banach spaces, Journal of Mathematical Analysis and Applications, Vol. **192**, no. 3, pp. 867-891, 1995.

[21] A. Jourani, The role of locally compact cones in nonsmooth analysis, Communications on Applied Nonlinear Analysis, Vol. **5**, no. 2, pp. 1-35, 1998.

[22] A. Jourani, On a class of compactly epi-Lipschitzian sets, Nonlinear Anal. **54** (2003),471-483

[23] A. Jourani, On constraint qualifications and Lagrange multipliers in nondifferentiable programming problems, Journal of Optimization Theory and Applications, Vol. **81**, no. 3, pp. 533-548, 1994.

[24] A. Jourani, Qualification conditions for multivalued mappings in Banach spaces with applications to nonsmooth vector optimization problems, Mathematical Programming, Vol. **66**, no. 1, Ser. A, pp. 1-23, 1994.

[25] A. Jourani, Necessary conditions for extremality and separation theorems with applications to multiobjective optimization, Optimization, Vol. **44**, no. 4, pp. 327-350, 1998.

[26] A. Jourani and L. Thibault, Approximate subdifferential of composite functions, Bulletin of the Australian Mathematical Society, Vol. **47**, no. 3, pp. 443-455, 1993.

[27] A. Jourani and L. Thibault, Metric regularity for strongly compactly Lipschitzian mappings, Nonlinear Analysis, Vol. **24**, no. 2, pp. 229-240, 1995.

[28] A. Jourani and L. Thibault, Extensions of subdifferential calculus rules in Banach spaces and applications, Canadian Journal of Mathematics, Vol. **48**, no. 4, pp. 834-848, 1996.

[29] P. D. Loewen, Limits of Fréchet normals in nonsmooth analysis, Optimization and nonlinear analysis, Editors A. IOFFE, M. MARCUS and S. REICH, Pitman Research Notes in Mathematics Series, Haifa, Vol. 244, pp. 178-188, 1990.

[30] M. Minami, Weak Pareto-optimal necessary conditions in a nondifferentiable multiobjective program on a Banach space, Journal of Optimization Theory and Applications, Vol. **41**, no. 3, pp. 451-461, 1983.

[31] B. S. Mordukhovich, Approximation methods in problems of optimization and control, Nauka, Moscow, pp. 360-500, 1988.

[32] B.S. Mordukhovich: Variational Analysis and Generalized Differentiation, Vol.1: Basic Theory, Vol. 2: Applications, Springer, Berlin (2005).

[33] B. S. Mordukhovich and Y. Shao, Differential characterizations of covering, metric regularity, and Lipschitzian properties of multifunctions between Banach spaces, Nonlinear Analysis TMA, **25** (1995), 1401-1424.

[34] V. H. Nguyen, J. J. Strodiot and R. Mifflin, On condition to have bounded multipliers in locally Lipschitz programming, Mathematical Programming, Vol. **18**, no. 1, pp. 100-106, 1980.

[35] R. T. Rockafellar, Directional differentiability of the optimal value function in a nonlinear programming problem, Mathematical Programming Study, no. 21, pp. 213-226, 1984.

[36] R. T. Rockafellar and R. Wets, Variational analysis, Springer-Verlag, Berlin, 1998.

[37] L. Thibault, Subdifferential of compactly Lipschitzian vector-valued functions, Annali di Matematica Pura ed Applicata (4), Vol. **125**, pp. 157-192, 1980.

[38] L. Thibault, On subdifferentials of optimal value functions, SIAM Journal on Control and Optimization, Vol. **29**, no. 5, pp. 1019-1036, 1991.

[39] J.J. Ye and X. Y. Ye, Necessary optimality conditions for optimization problems with variational inequaliy constraints, Mathematics of Operation Research, Vol.**22**, (1997), 977-997.

[40] J. J. Ye and Q. Zhu, Multiobjective optimization problems with variational inequality constraints, Mathematical Programming, Vol.**96**, (2003), 139-160.

[41] P. L. Yu, Multicriteria decision making: Concepts, Techniques and Extensions, Plenum Press, New York, 1985.

[42] Q. Zhu, (1999), Hamiltonian necessary conditions for a multiobjective optimal control problem with endpoint constraints, SIAM Journal on Control and Optimization, Vol. **39**, no. 1, pp. 97-112, 2000.

[43] J. Zowe and S. Kurcyusz, Regularity and stability of the mathematical programming problems in Banach spaces, Applied Mathematics and Optimization, Vol. **5**, no. 1, pp. 49-62, 1979.