Mathematical Programming manuscript No.
(will be inserted by the editor)

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Error Bounds for Eigenvalue and Semidefinite Matrix Inequality Systems

Dedicated to Terry Rockafellar in honor of his 70th birthday

Received: date / Revised version: date

Abstract. In this paper we give sufficient conditions for existence of error bounds for systems expressed in terms of eigenvalue functions (such as in eigenvalue optimization) or positive semidefiniteness (such as in semidefinite programming).

1. Introduction

Given Euclidean space E (which is a finite-dimensional real inner-product space) and $f : E \to \mathbb{R} \cup \{+\infty\}$ an extended real-valued lower semicontinuous (l.s.c.) function on E, consider the inequality system

$$f(x) \leq 0. \tag{1}$$

We say that the inequality system (1) has a local (global) *error bound* if the set S of solutions of (1) is nonempty and for some $0 < \epsilon < +\infty$ ($\epsilon = +\infty$) there exists a scalar $a > 0$ such that

$$d(x, S) \leq af(x) \quad \forall x \in f^{-1}(0, \epsilon) := \{x \in E : 0 < f(x) < \epsilon\}, \tag{2}$$

where $d(x, S) = \inf_{u \in S} \|x - u\|$ and $\| \cdot \|$ denotes the Euclidean norm on E. Given some $x_0 \in S$, the system (1) (or the set S) is said to be *metrically regular at* x_0 (or has an error bound near x_0) if for some $\delta > 0$ there exists a scalar $a > 0$ such that

$$d(x, S) \leq af_+(x) \quad \forall x \in B(x_0, \delta), \tag{3}$$

where $f_+(x) = \max(f(x), 0)$ and $B(x_0, \delta)$ denotes the open ball centered at x_0 with radius δ.

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Mathematics Subject Classification (1991): 45J52,49J53,90C22,90C31

Error bounds have important applications in sensitivity analysis of mathematical programming and in convergence analysis of some numerical algorithms. In his celebrated result [10] dating back to 1952, A.J. Hoffman showed that if f is a maximum of a finite number of affine functions in $\mathbb{R}^n$, then the inequality system (1) has a global error bound. Hoffman's result was extended to the linear systems in general Banach space by Ioffe in [11]. For nonlinear inequality systems, the existence of error bounds usually requires some conditions. For error bound results related to a continuous or convex system on $\mathbb{R}^n$, the reader is referred to the survey papers [15,20] and the references therein for a summary of the theory and applications of error bounds. Recently, considerable progress has been made on error bounds for lower semicontinuous functions in general Banach spaces (see, e.g., [3,6,7,12,18,24–26]). The extension to the lower semicontinuous system makes it possible to study the error bound for a system with equality, inequality and abstract constraints such as

$$g_i(x) \leq 0 \quad \forall i = 1, \dots, p, \quad g_i(x) = 0, \forall i = p+1, \dots, I, \quad x \in C,$$

by taking

$$f(x) := \max\{g_1(x), \dots, g_p(x), |g_{p+1}(x)|, \dots, |g_I(x)|, \delta_C\},$$

where δ_C denotes the indicator function of set C. The extension from $\mathbb{R}^n$ to general Banach space makes it possible to study inequality systems involving the sum of the m largest eigenvalues and inequality constraint systems arising from linear semidefinite programming such as in [1,8].

The purpose of this paper is to study error bounds for systems expressed in terms of eigenvalue functions (in particular linear combinations of eigenvalues) and inequality systems arising from *nonlinear* semidefinite programming.

In the rest of this section we describe the eigenvalue and semidefinite matrix inequality systems. Let $\mathcal{S}^n$ denotes the space of real-symmetric matrices of order n endowed with the usual scalar product

$$\langle A, B \rangle = \mathrm{tr}(AB),$$

where $\mathrm{tr}A$ denotes the trace of matrix A. We define the eigenvalue map $\lambda : \mathcal{S}^n \mapsto \mathbb{R}^n$ by

$$\lambda(X) = (\lambda_1(X), \cdots, \lambda_n(X))$$

where $\lambda_1(X), \cdots, \lambda_n(X)$ are the eigenvalues of the matrix X arranged with decreasing order

$$\lambda_1(X) \geq \cdots \geq \lambda_n(X).$$

It is known that

$$\|\lambda(A) - \lambda(B)\| \leq \|A - B\|,$$

i.e., the eigenvalue map is Lipschitzian (see e.g. [2, III. 6.15]).

Let $L : \mathbb{R}^m \mapsto \mathcal{S}^n$ be a linear mapping, that is, there exist $L_1, \cdots, L_m \in \mathcal{S}^n$ such that

$$L(x) = \sum_{i=1}^{m} x_i L_i, \qquad \forall x = (x_1, \cdots, x_m) \in \mathbb{R}^m$$

and let $B \in \mathcal{S}^n$. We define the mapping $\mathcal{A} : \mathbb{R}^m \mapsto \mathcal{S}^n$ by

$$\mathcal{A}(x) = L(x) - B, \quad \forall x \in \mathbb{R}^m.$$

The first eigenvalue inequality system we intend to study is given in $\mathcal{S}^n$ by

$$(f \circ \lambda)(X) \leq 0, \quad X \in \mathcal{S}^n \qquad \text{(I)}.$$

The second one is given in $\mathbb{R}^m$ by

$$(f \circ \lambda \circ \mathcal{A})(x) \leq 0, \quad x \in \mathbb{R}^m \qquad \text{(II)},$$

where $f : \mathbb{R}^n \mapsto \mathbb{R}$ is a locally Lipschitz mapping and $f \circ g$ denotes the composite function of f and g. The first semidefinite inequality system we propose to study is given in $\mathcal{S}^n$ by

$$\begin{aligned} & g_i(X) \leq 0, \quad i = 1, 2, \ldots, p, \\ & g_i(X) = 0, \quad i = p+1, 2, \ldots, I, \qquad \text{(III)} \\ & X \succeq 0, \end{aligned}$$

where g_i are C^1 functions on $\mathcal{S}^n$. The second one is given in $\mathbb{R}^m$ by

$$G(x) \preceq 0, \qquad \text{(IV)}$$

where $G : R^m \to \mathcal{S}^n$ is a C^1 mapping and for $A \in \mathcal{S}^n$, the notation $A \succeq (\preceq) 0$ means that the matrix A is positive (negative) semidefinite.

Systems (I) and (II) arise frequently in eigenvalue optimization. For example, for any integer κ between 1 and n, let $f(x_1, \ldots, x_n) = \sum_{i=1}^{\kappa} x_i$. Then $f \circ \lambda(X)$ is the sum of κ th largest eigenvalues of matrix X as in eigenvalue optimization [19].

Systems (III) and (IV) are the constraint sytems of a nonlinear semidefinite program and its dual program (see, e.g., [23]).

2. Preliminaries

This section contains some background material on nonsmooth analysis and preliminary results which will be used later. We give only concise definitions and results that will be needed in the paper. For more detailed information on the subject our references are Clarke, Ledyaev, Stern and Wolenski [5], Mordukhovich [16,17] and Rockafellar and Wets [21]. Note that in a finite dimensional space, the limiting Fréchet subdifferential in the following definition coincides with the limiting proximal subdifferential as in [5].

Definition 1. *Let $f : E \mapsto \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function and $x_0 \in E$ be such that $f(x_0) < \infty$. For any given $\varepsilon \geq 0$, the Fréchet ε-subdifferential is the set*

$$\partial_\varepsilon f(x) = \{x^* \in E : \liminf_{h \to 0} \frac{f(x+h) - f(x) - \langle x^*, h \rangle}{\| h \|} \geq -\varepsilon\}.$$

The limiting Fréchet subdifferential of f at x_0 is the set

$$\partial f(x_0) := \{x^* \in E : \exists x_k \to x_0, \varepsilon_k \to 0^+, \text{ and } x_k^* \to x^* \text{ with } x_k^* \in \partial_{\varepsilon_k} f(x_k)\}.$$

Proposition 1 (Chain Rule). *(see e.g. [17, Corollary 6.3]) Let E_1, E_2 be two Euclidean spaces and $\phi : E_1 \to E_2$, $f : E_2 \to \mathbb{R}$. Suppose that ϕ is Lipschitz near $x_0 \in E_1$ and f is Lipschitz near $\phi(x_0)$. Then for each $x_0 \in E_1$*

$$\partial(f \circ \phi)(x_0) \subset \bigcup \{\partial\langle \mu, \phi\rangle(x_0) : \mu \in \partial f(\phi(x_0))\},$$

where $\langle \mu, \phi\rangle(x) := \langle \mu, \phi(x)\rangle$ and $\langle \mu, \phi(x)\rangle$ denotes the inner product of $\mu, \phi(x)$ in E_2.

In this paper we mainly rely on the following results. We only quote the results under the assumptions we need in the paper.

Theorem 1. *[26, Theorem 2.2] Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be l.s.c.. Suppose that, for some $x_0 \in X$, $0 < \delta \leq +\infty$, $0 < a < +\infty$ and $0 < \epsilon \leq \delta/(2a)$, the set $B(x_0, \delta/2) \cap f^{-1}(-\infty, \epsilon)$ is nonempty and*

$$\|\xi\| \geq a^{-1} \text{ for all } \xi \in \partial f(x) \text{ and each } x \in B(x_0, \delta) \cap f^{-1}(0, \epsilon).$$

Then $S := \{x \in E : f(x) \leq 0\}$ is nonempty and

$$d(x, S) \leq a f(x) \text{ for all } x \in B(x_0, \frac{\delta}{2}) \cap f^{-1}(0, \epsilon).$$

Moreover, if $x_0 \in S$, then the condition $0 < \epsilon \leq \delta/(2a)$ can be replaced with $0 < \epsilon \leq +\infty$.

Theorem 2. *[13, Corollary 3.13] Let C be a closed subset of E and $x_0 \in S$, where $S := \{x \in C : g_i(x) \leq 0, i = 1, \ldots, p,\ g_i(x) = 0,\ i = p+1, \ldots, I\}$. Suppose that $g : E \to R^I$ is Lipschitz near x_0. If the following constraint qualification is satisfied at x_0 :*

$$\left.\begin{array}{r} \lambda_i \geq 0,\ \lambda_i g_i(x_0) = 0\ \forall i = 1, \ldots, p \\ 0 \in \partial\langle \lambda, g\rangle(x_0) + N_C(x_0) \end{array}\right\} \Rightarrow \lambda = 0,$$

where $N_C(x_0) := \partial\delta_C(x_0)$ denotes the limiting normal cone of C at x_0, then S is metrically regular at x_0.

Theorem 3. *[26, Theorem 4.7(ii)] Let C be a nonempty closed subset of E and $f_i : E \to \mathbb{R}$ be continuously differentiable for each $i \in \mathcal{I}$ where $\mathcal{I}$ is a given finite set. Denote*

$$f(x) = \max\{f_i(x) : i \in \mathcal{I}\} \text{ and } \mathcal{I}(x) := \{i \in \mathcal{I} : f_i(x) = f(x)\} \text{ for } x \in E.$$

Suppose that for some $0 < a$ and each $x \in f^{-1}(0, \infty)$, there exists a vector u_x such that $\|u_x\| = 1$, $u_x \in K_C(x)$ and $\langle \nabla f_i(x), u_x\rangle \leq -a^{-1}$ for each $i \in \mathcal{I}(x)$. Then $S := \{x \in C : f(x) \leq 0\}$ is nonempty and

$$d(x, S) \leq a f_+(x) \text{ for all } x \in C,$$

where $\nabla f(x)$ denotes the gradient of a function f at x, $K_C(x)$ denotes the contingent cone of C at x defined by

$$K_C(x) := \{v \in E : \exists t_n \downarrow 0, v_n \to v \text{ s.t. } x + t_n v_n \in C \ \forall n\}.$$

3. Eigenvalue inequality systems

Let S_I and S_{II} be solution sets of the eigenvalue systems (I) and (II) respectively. Our aim in this section is to prove the following sufficient conditions ensuring the existence of error bounds for systems (I) and (II).

Theorem 4. (i) *Suppose that for some $U_0 \in S_I$, $0 < \delta \leq +\infty$, $0 < \gamma < +\infty$, $0 < \epsilon \leq +\infty$, the following condition holds*

$$|\sum_{i=1}^{n} \mu_i| \geq \gamma \quad \forall \mu \in \partial f(\lambda(X)), \quad X \in B(U_0, \delta) \cap (f \circ \lambda)^{-1}(0, \epsilon).$$

Then

$$d(X, S_I) \leq \frac{\sqrt{n}}{\gamma}(f \circ \lambda)(X), \quad \forall X \in B(U_0, \frac{\delta}{2}) \cap (f \circ \lambda)^{-1}(0, \epsilon).$$

(ii) *Suppose that for some $U_0 \notin S_I$, $0 < \delta \leq +\infty$, $0 < \gamma < +\infty$, $0 < \epsilon \leq \frac{\delta\gamma}{2\sqrt{n}}$, the set $B(U_0, \delta/2) \cap (f \circ \lambda)^{-1}(-\infty, \epsilon)$ is nonempty and*

$$|\sum_{i=1}^{n} \mu_i| \geq \gamma \quad \forall \mu \in \partial f(\lambda(X)), \quad X \in B(U_0, \delta) \cap (f \circ \lambda)^{-1}(0, \epsilon).$$

Then $S_I \neq \emptyset$ and

$$d(X, S_I) \leq \frac{\sqrt{n}}{\gamma}(f \circ \lambda)(X), \quad \forall X \in B(U_0, \frac{\delta}{2}) \cap (f \circ \lambda)^{-1}(0, \epsilon).$$

Theorem 5. *Suppose that there exist $\sigma > 0$ and $d \in \mathbb{R}^m$ such that*

$$\sigma I_n - L(d) \in \mathcal{S}^n_-,$$

where I_n denotes the identity matrix of order n and $\mathcal{S}^n_-$ denotes the cone of negative semidefinite matrices of order n.

(i) If for some $x_0 \in S_{II}$, $0 < \delta \le +\infty, 0 < \gamma < +\infty, 0 < \epsilon \le +\infty$, it holds that for all $\mu \in \partial f(\lambda(\mathcal{A}(x)))$, $x \in B(x_0, \delta) \cap (f \circ \lambda \circ \mathcal{A})^{-1}(0, \epsilon)$,

$$\begin{cases} \lambda_1(\sigma I_n - L(d)) \sum_{\mu_i \ge 0} \mu_i + \lambda_n(\sigma I_n - L(d)) \sum_{\mu_i \le 0} \mu_i \le 0, \\ \sum_{i=1}^{n} \mu_i \ge \gamma, \end{cases} \tag{4}$$

then

$$d(x, S_{II}) \le \frac{\|d\|}{\sigma\gamma}(f \circ \lambda)(\mathcal{A}(x)), \quad \forall x \in B(x_0, \frac{\delta}{2}) \cap (f \circ \lambda \circ \mathcal{A})^{-1}(0, \epsilon).$$

(ii) Suppose that for some $x_0 \notin S_{II}$, $0 < \delta \le +\infty, 0 < \gamma < +\infty, 0 < \epsilon \le \frac{\delta\sigma\gamma}{2\|d\|}$, the set $B(x_0, \delta/2) \cap (f \circ \lambda \circ \mathcal{A})^{-1}(-\infty, \epsilon)$ is nonempty and that for all $\mu \in \partial f(\lambda(\mathcal{A}(x)))$, $x \in B(x_0, \delta) \cap (f \circ \lambda \circ \mathcal{A})^{-1}(0, \epsilon)$, condition (4) holds. Then $S_{II} \ne \emptyset$ and

$$d(x, S_{II}) \le \frac{\|d\|}{\sigma\gamma}(f \circ \lambda)(\mathcal{A}(x)), \quad \forall x \in B(x_0, \frac{\delta}{2}) \cap (f \circ \lambda \circ \mathcal{A})^{-1}(0, \epsilon).$$

The proof of Theorems 4 and 5 is based on the following fundamental lemmas.

Lemma 1. *Let $\mu = (\mu_1, \cdots, \mu_n) \in \mathbb{R}^n$ and $X \in \mathcal{S}^n$. Then each $A \in \partial\langle \mu, \lambda\rangle(X)$ satisfies $tr(A) = \sum_{i=1}^{n} \mu_i$. If $\mu_1 = \mu_2 = \cdots = \mu_n = \mu$, then $\|A\| = \sqrt{n}|\mu| = \frac{|tr(A)|}{\sqrt{n}}$.*

Proof. Consider the function $f(x) := \sum_{i=1}^n \mu_i x_i$ defined on $\mathbb{R}^n$. It is easy to see that f is invariant under coordinate permutations if and only if $\mu_1 = \mu_2 = \cdots = \mu_n = \mu$. In the case when f is invariant the limiting subdifferential of the function $f \circ \lambda$ is given by Lewis in [14] as:

$$\begin{aligned} &\partial(f \circ \lambda)(X) \\ &\quad = \{U^\top (\text{Diag } \mu) U : U \text{ orthogonal}, \;\; U^\top (\text{ Diag } \lambda(X)) U = X, \mu \in \partial f(\lambda(X))\}, \end{aligned}$$

where $U^\top$ denotes the transpose of matrix U and Diag μ denotes the diagonal matrix with diagonal entries $\mu_1, \mu_2, \ldots, \mu_n$. Consequently one has $\|A\| = \sqrt{\sum_{i=1}^n \mu_i^2} = \sqrt{n}|\mu|$.

Now consider the case when f is not invariant. Let $A \in \partial\langle\mu,\lambda\rangle(X)$. Then by the definition of Limiting Fréchet subdifferential, there are sequences $A_k \to A$, $U_k \to X$ and $r_k, \varepsilon_k \to 0^+$ such that

$$\langle\mu, \lambda(U) - \lambda(U_k)\rangle - \langle A_k, U - U_k\rangle + \varepsilon_k\|U - U_k\| \geq 0, \quad \forall U \in B(U_k, r_k).$$

Taking $U = U_k + r_k I$ and $U = U_k - r_k I$ in the last inequality we obtain (because $\lambda(U_k + r_k I) = \lambda(U_k) + r_k\lambda(I)$ and $\lambda(U_k - r_k I) = \lambda(U_k) - r_k\lambda(I)$)

$$\sum_{i=1}^{n} \mu_i - \mathrm{tr}(A_k) + \varepsilon_k\sqrt{n} \geq 0$$

and

$$-\sum_{i=1}^{n} \mu_i + \mathrm{tr}(A_k) + \varepsilon_k\sqrt{n} \geq 0.$$

Passing to the limit in the last two inequalities we get $\mathrm{tr}(A) = \sum_{i=1}^{n} \mu_i$.

■

Lemma 2. *(e.g., [2, III.2.2]) Let $A, B \in \mathcal{S}^n$ and $C \in \mathcal{S}^n_-$. Then for each $j = 1, \cdots, n$, the following inequality hold*

$$\lambda_n(B) \leq \lambda_j(A + B) - \lambda_j(A) \leq \lambda_1(B).$$

Using the above lemma and the definition of the limiting Fréchet subdifferential we get the following lemma.

Lemma 3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz mapping. Suppose that:*

(i) There exist $\sigma > 0$ and $W \in \mathcal{S}^n$ such that

$$\sigma I_n - W \in \mathcal{S}^n_-.$$

(ii) $\lambda_1(\sigma I_n - W) \sum_{\mu_i \geq 0} \mu_i + \lambda_n(\sigma I_n - W) \sum_{\mu_i \leq 0} \mu_i \leq 0, \quad \forall \mu \in \partial f(\lambda(X)).$

Then for each $A \in \partial(f \circ \lambda)(X)$, there exists $\mu \in \partial f(\lambda(X))$ such that

$$\langle A, W\rangle \geq \sigma \sum_{i=1}^{n} \mu_i$$

Proof. Let $A \in \partial(f \circ \lambda)(X)$. By the chain rule (Propositon 1), there exists $\mu \in \partial f(\lambda(X))$ such that $A \in \partial\langle\mu,\lambda\rangle(X)$. Then there are sequences $A_k \to A$, $U_k \to X$ and $\varepsilon_k, r_k \to 0^+$ such that

$$\langle\mu, \lambda(U) - \lambda(U_k)\rangle - \langle A_k, U - U_k\rangle + \varepsilon_k\|U - U_k\| \geq 0$$

for all $U \in B(U_k, r_k)$. In particular for $U = U_k - r_k W$ we get

$$\langle \mu, \lambda(U_k - r_k W) - \lambda(U_k)\rangle + r_k\langle A_k, W\rangle + \varepsilon_k r_k \|W\| \geq 0. \tag{5}$$

By assumption (i), there exists $p^- \in \mathcal{S}^n_-$ such that $-W = p^- - \sigma I_n$. Since

$$\begin{aligned}\lambda(U_k - r_k W) &= \lambda(U_k + r_k p^- - r_k \sigma I_n)\\ &= \lambda(U_k + r_k p^-) - r_k \sigma \lambda(I_n),\end{aligned}$$

(5) becomes

$$\langle \mu, \lambda(U_k + r_k p^-) - \lambda(U_k)\rangle + r_k\langle A_k, W\rangle + \varepsilon_k r_k \|W\| \geq r_k \sigma \sum_{i=1}^n \mu_i. \tag{6}$$

Lemma 2 implies that for each $\mu_i \geq 0$

$$\mu_i(\lambda_i(U_k + r_k p^-) - \lambda_i(U_k)) \leq \mu_i \lambda_1(r_k p^-)$$

and for each $\mu_i \leq 0$

$$\mu_i(\lambda_i(U_k + r_k p^-) - \lambda_i(U_k)) \leq \mu_i \lambda_n(r_k p^-)$$

which implies

$$\sum_{i=1}^n \mu_i(\lambda_i(U_k + r_k p^-) - \lambda_i(U_k)) \leq \sum_{\mu_i \geq 0} \mu_i \lambda_1(r_k p^-) + \sum_{\mu_i \leq 0} \lambda_n(r_k p^-). \tag{7}$$

Under assumption (ii), (6) and (7) imply that

$$\varepsilon_k \|W\| + \langle A_k, W\rangle \geq \sigma \sum_{i=1}^n \mu_i$$

which ensures that

$$\langle A, W\rangle \geq \sigma \sum_{i=1}^n \mu_i.$$

■

Proof of Theorem 4. By virtue of Theorem 1 it suffices to show that for each $X \in B(U_0, \delta) \cap (f \circ \lambda)^{-1}(0, \epsilon)$ and all $A \in \partial(f \circ \lambda)(X)$,

$$\|A\| \geq \frac{\gamma}{\sqrt{n}}.$$

Since $A \in \partial(f \circ \lambda)(X)$, by the chain rule (Proposition 1) there exists $\mu \in \partial f(\lambda(X))$ such that $A \in \partial\langle \mu, \lambda\rangle(X)$. By Lemma 1, $tr(A) = \sum_{i=1}^n \mu_i$. Since

$$|\mathrm{tr}(A)| = |\mathrm{tr}(AI)| \leq \|A\|\|I\| = \sqrt{n}\|A\|,$$

one has

$$\|A\| \geq \frac{|tr(A)|}{\sqrt{n}} = \frac{|\sum_{i=1}^n \mu_i|}{\sqrt{n}} \geq \frac{\gamma}{\sqrt{n}}$$

and the proof of the theorem is completed.

■

Proof of Theorem 5. By virtue of Theorem 1 it suffices to show that for each $x \in B(x_0, \delta) \cap (f \circ \lambda \circ \mathcal{A})^{-1}(0, \epsilon)$ and all $p \in \partial(f \circ \lambda \circ \mathcal{A})(x)$,

$$\|p\| \geq \frac{\sigma\gamma}{\|d\|}.$$

By the chain rule (Proposition 1) there exists $C \in \partial(f \circ \lambda)(\mathcal{A}(x))$ such that

$$\nabla\langle C, \mathcal{A}\rangle(x) = (\mathrm{tr}(CL_1), \cdots, \mathrm{tr}(CL_m)).$$

Lemma 3 implies that there exists $\mu \in \partial f(\lambda(\mathcal{A}(x))$ such that

$$\langle p, d\rangle = \langle C, L(d)\rangle \geq \sigma \sum_{i=1}^{n} \mu_i \geq \sigma\gamma.$$

Thus

$$\|p\| \geq \frac{\sigma\gamma}{\|d\|}$$

and the proof of the theorem is completed.

■

4. Inequality systems involving linear combination of eigenvalues

In this section, we study the eigenvalue inequality systems (I) and (II) under the assumption that $f(x) := \sum_{i=1}^{n} \alpha_i x_i - c$, where α_i, c are constants such that $\sum_{i=1}^{n} \alpha_i \neq 0$. First we show that the eigenvalue inequality system (I) under these assumptions has a global error bound and the optimal constant is given for the cases when $\alpha_1 \geq \cdots \geq \alpha_n \geq 0$ and $c = 0$ and when $\alpha_1 = \alpha_2 = \cdots = \alpha_n > 0$.

Theorem 6. *For any given constants α_i, c such that $\sum_{i=1}^{n} \alpha_i \neq 0$, the set*

$$S_1 := \{X \in \mathcal{S}^n : \sum_{i=1}^{n} \alpha_i \lambda_i(X) \leq c\}$$

is nonempty and

$$d(X, S_1) \leq \frac{\sqrt{n}}{|\sum_{i=1}^{n} \alpha_i|}(\sum_{i=1}^{n} \alpha_i \lambda_i(X) - c), \quad \forall X \notin S_1. \tag{8}$$

If moreover $\alpha_1 \geq \cdots \geq \alpha_n \geq 0$ and $c = 0$ or $\alpha_1 = \alpha_2 = \cdots = \alpha_n > 0$ then $\frac{\sqrt{n}}{\sum_{i=1}^{n} \alpha_i}$ is the smallest constant for which inequality (8) holds.

Proof. Since $\dfrac{\partial f}{\partial x_i}(x) = \alpha_i$, the global error bound results follow from Theorems 4. Hence it suffices to prove that the last property holds.
Notice that if $\alpha_1 \geq \cdots \geq \alpha_n \geq 0$ then function $X \to \sum_{i=1}^n \alpha_i \lambda_i(X) - c$ is convex (see e.g. [14, Lemma 5.2]). It is known that for the convex system $\sum_{i=1}^n \alpha_i \lambda_i(X) \leq c$ and any constant $a > 0$, $\|A\| \geq a^{-1}$ for each $A \in \partial\langle \alpha, \lambda\rangle(X)$ and each $X \notin S_1$ if and only $d(S_1, X) \leq a(\sum_{i=1}^n \alpha_i \lambda_i(X) - c) \quad \forall X \notin S_1$ (see e.g. [25, Theorem 7]). By Lemma 1, if $\alpha_1 = \alpha_2 = \cdots = \alpha_n > 0$ then $\|A\| = \sqrt{n}\alpha_1 = \frac{\sum_{i=1}^n \alpha_i}{\sqrt{n}}$. Hence $\frac{\sqrt{n}}{\sum_{i=1}^n \alpha_i}$ is the smallest constant for which inequality (8) holds in the case when $\alpha_1 = \alpha_2 = \cdots = \alpha_n > 0$.
The rest of the proof follows from the the following Lemma.

■

Lemma 4. *Suppose that* $\alpha_1 \geq \cdots \geq \alpha_n \geq 0$ *and* $c = 0$. *Then the set* S_1 *is convex and the projection onto* S_1 *of the identity matrix* I *is exactly* $0 \in S_1$. *In other words* $d(I, S_1) = \|I\|$.

Proof. Since the function $X \to \sum_{i=1}^n \alpha_i \lambda_i(X)$ is convex S_1 is a convex set. Let $X \in S_1$ be such that $d(I, S_1) = \|I - X\|$. Then X is the projection of I onto S_1 is equivalent to

$$\langle I - X, U - X\rangle \leq 0 \quad \forall U \in S_1$$

which is equivalent, because S_1 is a closed convex cone, to

$$\mathrm{tr}(X) = \mathrm{tr}(XX) \quad \text{and} \quad \mathrm{tr}((I - X)U) \leq 0 \quad \forall U \in S_1.$$

Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of the matrix X and let k be such that $\alpha_i > 0 \ \forall i = 1, \cdots, k \quad \alpha_i = 0 \ \forall k+1, \ldots, n$. Then since $X \in S_1$,

$$\sum_{i=1}^k \alpha_i \lambda_i \leq 0, \quad \sum_{i=1}^n \lambda_i = \sum_{i=1}^n \lambda_i^2.$$

We claim that for each $i = 1, \cdots, n$, $\lambda_i = 0$. Let $\Theta = \{i \in \{1, \cdots, n\} : \quad \lambda_i < 0\}$. We have to prove that $\Theta = \emptyset$. Suppose that $\Theta \neq \emptyset$. Then $\Theta = \{j, \cdots, n\}$ for some $j \leq n$. We have two possibilities:
1) Either $j \leq k$. In this case

$$\alpha_j(\sum_{i=1}^{j-1} \lambda_i) \leq \sum_{i=1}^{j-1} \alpha_i \lambda_i \leq \sum_{i=j}^n \alpha_i(-\lambda_i) \leq \alpha_j(\sum_{i=j}^n (-\lambda_i))$$

and hence

$$\sum_{i=1}^n \lambda_i^2 = \sum_{i=1}^n \lambda_i \leq 0$$

which implies $\lambda_1 = \cdots, \lambda_n = 0$ and this contradicts the fact that $\Theta \neq \emptyset$.

2) Or $j > k$. In this case for all $i = 1, \cdots, j-1$, $\lambda_i \geq 0$ and since $\sum_{i=1}^{k} \alpha_i \lambda_i \leq 0$ we get $\lambda_1 = \cdots = \lambda_k = 0$. As $\lambda_1 \geq \cdots \geq \lambda_n$ we have for all $i = k+1, \cdots, n$, $\lambda_i \leq 0$. Since $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \lambda_i^2$ we obtain that $\lambda_1 = \cdots, \lambda_n = 0$ and this contradicts again the fact that $\Theta \neq \emptyset$.

Both contradictions imply that $\Theta = \emptyset$. Now since $\displaystyle\sum_{i=1}^{k} \alpha_i \lambda_i \leq 0$ we have $\lambda_1 = \cdots, \lambda_n = 0$ and thus $X = 0$ and the proof is completed.

■

Applying Theorem 5 to the system of linear combination of eigenvalues, we derive the following global error bound for this special case of system (II) under a Slater type condition (9) which amounts to saying that there exists a vector $d \in \mathbb{R}^m$ such that $L(d)$ is positive definite.

Theorem 7. *For given constants α_i, c, suppose that $\sum_{i=1}^{n} \alpha_i > 0$ and there exist $\sigma > 0$ and $d \in \mathbb{R}^m$ such that*

$$\begin{cases} \sigma I_n - L(d) \in \mathcal{S}_-^n, \\ \lambda_1(\sigma I_n - L(d)) \displaystyle\sum_{\alpha_i \geq 0} \alpha_i + \lambda_n(\sigma I_n - L(d)) \sum_{\alpha_i \leq 0} \alpha_i \leq 0 \end{cases}. \qquad (9)$$

Then the set $S_2 := \{x \in \mathbb{R}^m : \displaystyle\sum_{i=1}^{n} \alpha_i \lambda_i(\mathcal{A}(x)) \leq c\}$ is nonempty and

$$d(x, S_2) \leq \frac{\|d\|}{\sigma \sum_{i=1}^{n} \alpha_i} \left[\sum_{i=1}^{n} \alpha_i \lambda_i(\mathcal{A}(x)) - c \right], \quad \forall x \notin S_2.$$

In the special case where $\alpha_1 = 1$, $\alpha_i = 0$ $\forall i \neq 1$ and $c = 0$, the above theorem gives the following global error bound result for a linear semidefinite program first given in [7, Corollary 2.1].

Corollary 1. *Suppose that there exist $\sigma > 0$ and $d \in \mathbb{R}^m$ such that*

$$\sigma I_n - L(d) \in \mathcal{S}_-^n. \qquad (10)$$

Then the set $S_3 := \{x \in \mathbb{R}^m : L(x) - B \preceq 0\}$ is nonempty and

$$d(x, S_3) \leq \frac{\|d\|}{\sigma} \lambda_1 \left(L(x) - B \right), \quad \forall x \notin S_3.$$

For an integer κ between 1 and n, consider the function

$$E_\kappa(X) := \text{ sum of the } \kappa\text{th largest eigenvalues of } X.$$

Then it is clear that

$$E_\kappa(X) = \sum_{i=1}^{\kappa} \lambda_i(X) = \sum_{i=1}^{n} \alpha_i \lambda_i(X) \quad \forall X \in \mathcal{S}^n,$$

with $\alpha_i = 1,\ i = 1, \ldots, \kappa$ and $\alpha_i = 0,\ i = \kappa + 1, \ldots, n$. Since $\sum_{i=1}^{n} \alpha_i = \kappa$, a consequence of Theorems 6 and 7 is the following results which was given by Azé and Hiriart-Urruty [1] for the case $c = 0$.

Theorem 8. *Let $S_4 := \{X : E_\kappa(X) \leq c\}$. Then the set S_4 is nonempty and*

$$d(X, S_4) \leq \frac{\sqrt{n}}{\kappa} \left(E_\kappa(X) - c \right), \quad \forall X \notin S_4.$$

Moreover, if either $c = 0$ or $\kappa = n$, then the constant $\frac{\sqrt{n}}{\kappa}$ is the smallest one satisfying the last inequality.

Theorem 9. *Suppose that there exist $\sigma > 0$ and $d \in \mathbb{R}^m$ such that*

$$\sigma I_n - L(d) \in \mathcal{S}^n_-.$$

Then the set $S_5 := \{x : E_\kappa(\mathcal{A}(x)) \leq c\}$ is nonempty and

$$d(x, S_5) \leq \frac{\|d\|}{\sigma\kappa} (E_\kappa(\mathcal{A}(x)) - c), \quad \forall x \notin S_5.$$

The concept of weak sharp minima in mathematical programming introduced by Burke and Ferris [4] is connected with error bounds. Taking $c := \inf_x E_\kappa(\mathcal{A}(x))$ in the above theorem we have the following result about weak sharp minima of minimizing sum of the κ th largest eigenvalue.

Corollary 2. *Suppose there exist $\sigma > 0$ and $d \in \mathbb{R}^m$ such that*

$$\sigma I_n - L(d) \in \mathcal{S}^n_-,$$

and $\inf\limits_{x \in R^n} E_\kappa(\mathcal{A}(x)) > -\infty$. Then the set of global solutions of the problem of minimizing the function $E_\kappa(\mathcal{A}(x))$ has a weak sharp minima, i.e. $\arg\min E_\kappa(\mathcal{A}(x))$ *is nonempty and*

$$d(x, \arg\min E_\kappa(\mathcal{A}(x))) \leq \frac{\|d\|}{\sigma\kappa} ((E_\kappa(\mathcal{A}(x)) - \inf_x E_\kappa(\mathcal{A}(x))) \qquad \forall x \in \mathbb{R}^m.$$

For integers k, l between 1 and n, with $k \leq l$, consider the function

$$KL(X) := \text{sum of the } k\text{th and } l\text{th largest eigenvalues of } X.$$

Then it is clear that

$$KL(X) = \lambda_k(X) + \lambda_l(X) = \sum_{i=1}^{n} \alpha_i \lambda_i(X) \quad \forall X \in \mathcal{S}^n,$$

with $\alpha_i = 1,\ i = k, l$ and $\alpha_i = 0,\ i \neq k$ or $i \neq l$.
From Theorems 6 and 7 we have the following results.

Theorem 10. *Let $S_6 := \{X : KM(X) \le c\}$. Then S_6 is nonempty and*

$$d(X, S_6) \le \frac{\sqrt{n}}{s(k,l)}(KM(X) - c), \quad \forall X \notin S_6,$$

where $s(k,l) = 1$ if $k = l$ and $s(k,l) = 2$ if $k \ne l$.

Theorem 11. *Suppose that there exist $\sigma > 0$ and $d \in \mathbb{R}^m$ such that*

$$\sigma I_n - L(d) \in \mathcal{S}^n_-.$$

Then the set $S_7 := \{x : KM(\mathcal{A}(x)) \le c\}$ is nonempty and

$$d(x, S_7) \le \frac{\|d\|}{\sigma s(k,m)}(KM(\mathcal{A}(x)) - c), \quad \forall x \notin S_7.$$

5. Semidefinite matrix inequality systems

Let S_{III} and S_{IV} denote the solution sets of the matrix systems (III) and (IV), respectively. Our aim in this section is to give sufficient conditions ensuring error bounds for the matrix systems (III) and (IV).

Theorem 12. *Suppose that there exists $X_0 \in S_{III}$ such that one of the following conditions is satisfied:*

(i) X_0 is positive definite and

$$\left.\begin{array}{l} \lambda_i g_i(X_0) = 0, \quad \lambda_i \ge 0 \ i = 1, \dots, p, \\ \sum_{i=1}^I \lambda_i \nabla g_i(X_0) = 0, \end{array}\right\} \Rightarrow \lambda = 0.$$

(ii) X_0 is positive semidefinite, singular and

$$\left.\begin{array}{l} \sum_{i=1}^I \lambda_i \nabla g_i(X_0) \in \mathcal{S}^n_+, \quad \sum_{i=1}^I \lambda_i tr(\nabla g_i(X_0) X_0) = 0 \\ \lambda_i g_i(X_0) = 0, \ \lambda_i \ge 0 \ i = 1, \dots, p \end{array}\right\} \Rightarrow \lambda = 0.$$

Then S_{III} is metrically regular at X_0, i.e., there exists $\delta > 0, a > 0$ such that

$$d(X, S_{III}) \le a \max\{0, g_1(X), \dots, g_p(X), |g_{p+1}(X)|, \dots, |g_I(X)|\}$$
$$\text{for all } X \in \mathcal{S}^n_+ \cap B(X_0, \delta),$$

where $\mathcal{S}^n_+$ denotes the cone of positive semidefinite matrices of order n.

Proof. Since it is known that the normal cone of $\mathcal{S}^n_+$ is given by

$$N_{\mathcal{S}^n_+}(X_0) = \{B \in \mathcal{S}^n_- : tr(BX_0) = 0\}$$

(see e.g. [9, Theorem 2.1]) and any positive definite matrix lies in the interior of $\mathcal{S}^n_+$, the proof follows from applying Theorem 2 with $C = \mathcal{S}^n_+$. ∎

Specializing the above theorem to the constraint region of the linear semidefinite program:

$$S_D := \{X \in \mathcal{S}^n : tr(A_i X) = b_i \; i = 1, \ldots, m, X \in \mathcal{S}^n_+\},$$

where $A_i \in \mathcal{S}^n, b_i \in \mathbb{R}$ we have the following error bound results for linear semidefinite programs.

Corollary 3. *Suppose that there exists $X_0 \in S_D$ such that one of the following conditions is satisfied*

(i) X_0 is positive definite and $A_1, \ldots, A_m$ are linearly independent;

(ii) X_0 is singular and

$$\left.\begin{array}{l} \sum_{i=1}^m \lambda_i A_i \in \mathcal{S}^n_+, \quad \sum_{i=1}^m \lambda_i tr(A_i X_0) = 0 \\ \lambda \in R^m \end{array}\right\} \Rightarrow \lambda = 0.$$

Then S_D is metrically regular at X_0, i.e., there exists $\delta > 0, a > 0$ such that

$$d(X, S_D) \leq a \max\{|tr(A_1 X) - b_1|, \ldots, |tr(A_m X) - b_m|\}$$
$$\text{for all } X \in \mathcal{S}^n_+ \cap B(X_0, \delta).$$

Corollary 4. *Suppose that S_D is nonempty and compact and $A_1, \ldots, A_m$ are linearly independent. Moreover assume that for each $X_0 \in S_D$ which is singular,*

$$\left.\begin{array}{l} \sum_{i=1}^m \lambda_i A_i \in \mathcal{S}^n_+, \quad \sum_{i=1}^m \lambda_i tr(A_i X_0) = 0 \\ \lambda \in R^m \end{array}\right\} \Rightarrow \lambda = 0.$$

Then there exists $a > 0$ such that

$$d(X, S_D) \leq a \max\{|tr(A_1 X) - b_1|, \ldots, |tr(A_m X) - b_m|\} \quad \text{for all } X \in \mathcal{S}^n_+.$$

Proof. By virtue of Corollary 3, S_D is metrically regular at each point of S_D. Since S_D is convex and compact, by [24, Theorem 5.2], S_D has a global error bound and hence the proof is complete.

∎

Theorem 13. *Suppose that there exist $a > 0$ and $U \in \mathcal{S}^n_+$ with $\|U\| = 1$ such that*

$$tr(A_i U) \le -a^{-1}, \qquad \forall i = 1, \dots, m.$$

Then $S_d := \{X \in \mathcal{S}^n_+ : tr(A_i X) \le b_i \ i = 1, \dots, m\}$ is nonempty and

$$d(X, S_d) \le a \max\{0, tr(A_1 X) - b_1, \dots, tr(A_m X) - b_m\} \quad \text{for all } X \in \mathcal{S}^n_+.$$

Proof. Since the contingent cone to $\mathcal{S}^n_+$ at any $X \in \mathcal{S}^n_+$ contains $\mathcal{S}^n_+$, the results then follow from Theorem 3.

■

Since

$$tr(A_i U) \le \lambda(A_i)^\top \lambda(U)$$

where $a^\top$ denotes the transpose of vector a, the above theorem has the following corollary.

Corollary 5. *Suppose that there exist $a > 0$ and $U \in \mathcal{S}^n_+$ with $\|U\| = 1$ such that*

$$\lambda(A_i)^\top \lambda(U) \le -a^{-1} \qquad \forall i = 1, \dots, m.$$

Then $S_d := \{X \in \mathcal{S}^n_+ : tr(A_i X) \le b_i \ i = 1, \dots, m\}$ is nonempty and

$$d(X, S_d) \le a \max\{0, tr(A_1 X) - b_1, \dots, tr(A_m X) - b_m\} \quad \text{for all } X \in \mathcal{S}^n_+.$$

In particular, it is easy to see that the above corollary has the following consequence.

Corollary 6. *Suppose that $A_i, \ i = 1, \dots, m$ are negative definite. Then $S_d := \{X \in \mathcal{S}^n_+ : tr(A_i X) \le b_i \ i = 1, \dots, m\}$ is nonempty and there exists $a > 0$ such that*

$$d(X, S_d) \le a \max\{0, tr(A_1 X) - b_1, \dots, tr(A_m X) - b_m\} \quad \text{for all } X \in \mathcal{S}^n_+.$$

Taking $U = I$ in Theorem 13, one has the following corollary.

Corollary 7. *Suppose that there exists $a > 0$ such that*

$$tr(A_i) \le -a^{-1}, \qquad \forall i = 1, \dots, m.$$

Then $S_d := \{X \in \mathcal{S}^n_+ : tr(A_i X) \le b_i \ i = 1, \dots, m\}$ is nonempty and

$$d(X, S_d) \le a \max\{0, tr(A_1 X) - b_1, \dots, tr(A_m X) - b_m\} \quad \text{for all } X \in \mathcal{S}^n_+.$$

Theorem 14. *Suppose that there exists $x_0 \in S_{IV}$ such that one of the following conditions is satisfied*

(i) $G(x_0)$ *is negative definite,*

(ii) $G(x_0)$ *is singular, negative semidefinite and*

$$\left.\begin{array}{l} tr(\Omega G_i(x_0)) = 0 \quad i = 1, 2, \ldots, n \\ \Omega \succeq 0 \end{array}\right\} \Rightarrow \Omega = 0.$$

Then S_{IV} is metrically regular at x_0, i.e., there exists $\delta > 0, a > 0$ such that

$$d(x, S_{IV}) \leq a\lambda_1(G(x))_+ \qquad \textit{for all } x \in B(x_0, \delta),$$

where $G_i(x) := \partial G(x)/\partial x_i$ are $n \times n$ partial derivative matrices.

Proof. Denote by $g(x) := \lambda_1(G(x))$. By Theorem 2, it suffices to prove that

$$\left.\begin{array}{l} \gamma \geq 0, \gamma g(x_0) = 0 \\ 0 \in \gamma \partial g(x_0) \end{array}\right\} \Rightarrow \gamma = 0.$$

The above condition is easily seen to be satisfied if $G(x_0)$ is negative definite. Hence one only needs to prove that (ii) implies that $0 \notin \partial g(x_0)$. It is easy to see that

$$\lambda_1(G(x)) = \max_{w \in R^n} \frac{w^\top G(x) w}{w^\top w}. \tag{11}$$

By Danskin's theorem (see e.g. [5, p.99]) the function g is Lipschitz near x_0, regular at x_0 and one has

$$\begin{aligned} \partial g(x_0) &= co\{\nabla_x[\frac{w^\top G(x) w}{w^\top w}] : w \text{ is an eigenvector corresponding to } \lambda_1(G(x_0))\} \\ &= co\{(uu^\top G_1(x), \ldots, uu^\top G_m(x)) : \\ &\qquad u \text{ is an unit eigenvector corresponding to } \lambda_1(G(x_0))\}, \end{aligned}$$

where $co\Omega$ denotes the convex hull of set Ω. Since for any vector u the matrix $uu^\top$ is positive semidefinite, assumption (ii) implies that $0 \notin \partial g(x_0)$.

■

Remark: Note that the above condition is equivalent to the MF condition ([22]): there exists a vector $h \in R^m$ such that the matrix

$$G(x_0) + \sum_{i=1}^{m} h_i G_i(x_0)$$

is negative definite. It is known that the Slater condition implies the MF condition.

According to Shapiro [22], we say that the mapping $G(x)$ is positive semidefinite convex (psd-convex) if it is convex with respect to the order relation imposed by the cone $\mathcal{S}^n_+$. That is the inequality

$$tG(x) + (1-t)G(y) \succeq G(tx + (1-t)y)$$

holds for any $x, y \in \mathbb{R}^m$ and all $t \in [0,1]$. By [22, Proposition 1], the mapping $G(x)$ is psd-convex if and only if for any $v \in \mathbb{R}^n$, the function $\varphi(x) = v^\top G(x)v$ is convex. It was observed in [22] that due to the expression (11), if G is psd-convex then $\lambda_1(G(x))$ is a convex function and the solution set S_{IV} is convex hence the following corollary is immediate.

Corollary 8. *Suppose that $G(x)$ is psd-convex, the set S_{IV} is bounded and the Slater condition is satisfied, i.e., there exists $x_0 \in R^m$ such that $G(x_0)$ is negative definite. Then there exists $a \geq 0$ such that*

$$d(x, S_{IV}) \leq a[\lambda_1(G(x))]_+ \quad \textit{for all } x \in \mathbb{R}^m.$$

Proof. By virtue of Theorem 14, S_{IV} is metrically regular at each point of S_{IV}. Since S_{IV} is convex and compact, by [24, Theorem 5.2], S_{IV} has a global error bound and hence the proof is complete.

■

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