Controllability and strong controllability of differential inclusions

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Abstract. In this paper, we prove sufficient conditions for controllability and strong controllability in terms of the Mordukhovich's subdifferential for two classes of differential inclusions. The first one is the class of sub-Lipschitz multivalued functions introduced by Loewen-Rockafellar [11]. The second one, introduced recently by Clarke [3], is the class of multivalued functions which are pseudo-Lipschitz and satisfy the so-called tempered growth condition. To do this, we establish an error bound result in terms of Mordukhovich's subdifferential outside Asplund spaces.

Key Words. Mordukhovich's subdifferential, differential inclusion, controllability, strong controllability, sub-Lipschitz and pseudo-Lipschitz multivalued mapping, error bound.

AMS subject classifications. 49J52, 90C30, 49K15, 49K05.

1 Introduction

In this paper, we deal with controllability and strong controllabilty of systems governed by differential inclusions of the form

$$\dot{x}(t) \in F(t, x(t))\, a.e.\, t \in [a, b], \quad (x(a), x(b)) \in S \tag{1}$$

where $F : [a, b] \times H \mapsto H$ is a multivalued mapping and $S \subset H \times H$ is a nonempty closed set and H is a real Hilbert space. The domain over which the study of system (1) occurs is typically the Sobolev space $W^{1,p}([a, b], H)$ (abbreviated $W^{1,p}$) consisting of all absolutely continuous functions $x\colon [a, b] \mapsto H$ for which $|\dot{x}|^p$ is integrable on $[a, b]$. The space $W^{1,p}$ is endowed with the norm

$$\|x\| = |x(a)| + [\int_a^b |\dot{x}(t)|^p dt]^{\frac{1}{p}}$$

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where $|\cdot|$ denotes a norm on H .
Let z be a solution of the system (1).

- The system (1) is said to be *locally controllable* at z if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $v \in B(0,\delta)$ there exists a trajectory x for F, with $\|x-z\| \leq \varepsilon$, satisfying $(x(0), x(1)+v) \in S$.

- It is said to be *strongly locally controllable* ([8]) at z if there exist $a > 0$ and $\delta > 0$ such that for all u and v in $B(0,\delta)$ there exists a trajectory x for F, with $\|x - z\| \leq a(|u| + |v|)$, satisfying $(x(0) + u, x(1) + v) \in S$.

Here $B(0,\delta)$ denotes the closed ball in H centered at 0 and of radius δ. It is clear that each strongly locally controllable system is also locally controllable. But, the converse does not hold. Indeed, the following system is locally controllable at $z = 0$, but it is not strongly locally controllable at this point :

$$(\dot{x}_1, \dot{x}_2)(t) \in \{x_2^3(t)\} \times \mathbb{R}, \qquad x(0) = x(1) = (0,0).$$

The notion of controllability is often expressed with the norm $\|\cdot\|_\infty$ ($\|x\|_\infty = \max_{t\in[a,b]} |x(t)|$) instead of $\|\cdot\|$. But each system controllable in terms of the norm $\|\cdot\|$ is also controllable in terms of the norm $\|\cdot\|_\infty$.
In recent years, the concept of controllability has been studied repeatedly in the literature (see [4], [5], [10], [21], [22]-[23] and references therein). However most of these authors assume that either F is convex-valued and Lipschitz or bounded and Lipschitz or admits a kind of linearization. Note that these assumptions ensure the existence of Lipschitz selection of F, which is usefull in the major proofs in these papers. With the very recent developments on necessary optimality conditions in nonsmooth analysis ([6], [7], [11]-[13], [16], [18]-[20]) the convexity is dropped but the Lipschitz condition remains in force. Moreover only few papers have attacked the problem of strong controllability ([2], [6]).
The aim of this paper is to establish sufficient conditions for controllability and strong controllability without convexity nor boundedness using two general classes of multivalued functions. The first one is the class of sub-Lipschitz multivalued functions introduced by Loewen-Rockafellar [11]. The second one, introduced recently by Clarke [3], is the class of multivalued functions which are pseudo-Lipschitz and satisfy the so-called tempered growth condition. Both classes encompass that of Lipschitz multivalued mappings.
Our study relies on an error bound result established in Section 3 and consequently on a general strong controllability result deduced in Section 4. These results are given in terms of the Mordukhovich's or limiting

subdifferential. It is well-known that this subdifferential possesses exact chain rule only in Asplund spaces. But the domain over which a part of the study of system (1) occurs is $W^{1,1}$ which is not Asplund space. Nevertheless our result on error bound in Section 3 will be established in a general Banach spaces and this allows us to encompass the space $W^{1,1}$. Sufficient conditions for strong controllability are formulated with the Mordukhovich's subdifferential of the distance function of the set C of solutions of the differential inclusion

$$\dot{x}(t) \in F(t, x(t)) \, a.e. \, t \in [a, b].$$

This and the Clarke's necessary optimality conditions [3] allow us to obtain sufficient conditions for controllability and strong controllability of the system (1) in terms of the Mordukhovich's coderivative of F in the case where $H = \mathbb{R}^m$.

Here and throughout the paper we assume the following basic hypothesis:

($\mathbf{H}_1$) *for each x, the multivalued mapping $t \mapsto F(t, x)$ is measurable and the values of F are closed.*

2 Background

In order to make the paper as short as possible, some definitions and the complete wording of the results will not be repeated here, and the reader is referred to [14]-[18] if required.
Let X be a Banach space endowed with the norm denoted by $\| \cdot \|$ with which we associate the distance function $d(S, \cdot)$ to a set S. By $B(x, r)$ we denote the open ball centered at x and of radius r. The topological dual space of X will be denoted by X^* and the pairing between X and X^* by $\langle \cdot, \cdot \rangle$.
For a function f and a set S, we write $x \xrightarrow{f} x_o$ and $x \xrightarrow{S} x_o$ to express $x \to x_0$ with $f(x) \to f(x_0)$ and $x \to x_0$ with $x \in S$, respectively.
Let f be an extended real-valued function on X. The Morddukhovich's or limiting subdifferential of f at x_0 is the set

$$\partial_L f(x_0) = {}^{\text{seq}} \operatorname*{Lim\,sup}_{x \xrightarrow{f} x_0, \, \varepsilon \downarrow 0} \partial_\varepsilon^F f(x),$$

where ${}^{\text{seq}}\text{Lim sup}$ stands for the weak-star (w^*) *sequential* upper limit of subsets in X^*, and where for $\varepsilon \geq 0$

$$\partial_\varepsilon^F f(x) = \{x^* \in X^* : \liminf_{h \to 0} \frac{f(x+h) - f(x) - \langle x^*, h \rangle}{\| h \|} \geq -\varepsilon\}$$

is the Fréchet ε-subdifferential of f at any x where f is finite. We adopt the convention $\partial_\varepsilon^F f(x) = \emptyset$ when $|f(x)| = +\infty$. We also put $\partial^F f(x) = \partial_0^F f(x)$ for $\varepsilon = 0$.
The $\varepsilon-$Fréchet and limiting normal cones to a closed set $S \subset X$ at a point $x_0 \in S$ are given by

$$N_\varepsilon^F(S, x_0) = \partial_\varepsilon^F \delta_S(x_0) \quad \text{and} \quad N_L(S, x_0) = \partial_L \delta_S(x_0),$$

where δ_S denotes the indicator function of S, i.e., $\delta_S(x) = 0$ if $x \in S$ and $\delta_S(x) = +\infty$ otherwise. The theory of Fréchet and limiting subdifferentials are developed, with fairly comprehensive references and remarks, in the paper by Mordukhovich and Shao [17] and in Mordukhovich's books [18, 19].
Next we consider a multivalued mapping G from X into a Banach space Y of graph

$$GrG := \{(x, y) : \ y \in G(x)\}.$$

The multivalued mapping $D^*G(x, y) : Y^* \mapsto X^*$ defined by

$$D_L^*G(x, y)(y^*) := \{x^* \in X^* : \ (x^*, -y^*) \in N_L(GrG; (x, y))\}$$

is called the Mordukhovich's coderivative of G at the point $(x, y) \in GrG$. Here Y^* denotes the dual space of Y.

3 Error bound

It is well-known that some Banach spaces may be characterized in terms of some subdifferentials. For example the Dini subdifferential characterizes the Weak Trustworthy spaces. The $\varepsilon-$Fréchet (and limiting Fréchet) subdifferential gives a characterization of Asplund spaces. To give sufficient conditions for error bounds for systems in terms of the limiting Fréchet subdifferential, the previous works assume that the space is Asplund. Our aim here is to obtain these results in general Banach spaces. Here we consider the following system:

$$x \in C \text{ and } g(x, u) \in D \tag{\textbf{S}}$$

where C and D are closed sets in X and H, and $g : X \times U \mapsto H$ is a mapping. Here X is a Banach space, U is a metric space called parameters set and H is a real Hilbert space
The corresponding parametric solution set of (**S**) is defined by the multivalued mapping

$$S(u) = \{x \in C : \quad g(x, u) \in D\}.$$

Our results are in the line of those obtained in [1] in the case where H is a finite dimensional space. Here we use the following Mordukhovich's condition, called sequential normal compactness (SNC): A set A is said to be sequentially normally compact at $a \in A$ if for any sequences $\varepsilon_n \to 0^+$, $a_n \xrightarrow{A} a$ and $a_n^* \in N^F_{\varepsilon_n}(A, a_n)$ one has

$$a_n^* \xrightarrow{w^*} 0 \Longrightarrow \|a_n^*\| \to 0.$$

Theorem 3.1 *Suppose that*

i) $(\bar{x}, \bar{u})$ *is a solution of the system (***S***).*

ii) g *is of class* $\mathcal{C}^1$ *at* $(\bar{x}, \bar{u})$ *in* x *with respect to* u *,i.e.* g *and its partial derivative* $D_x g(x, u)$ *are continuous at* $(\bar{x}, \bar{u})$*.*

iii) Either H *is of finite dimension or* D *is convex and sequentially normally compact at* $g(\bar{x}, \bar{u})$*.*

Then $\beta) \Longrightarrow \alpha)$*, where*
$\alpha)$ *there exist* $a > 0$ *and* $r > 0$ *such that*

$$d(x, S(u)) \leq a d(g(x, u), D)$$

for all $x \in C \cap B(\bar{x}, r)$ *and all* $u \in B(\bar{u}, r)$*;*
and
$\beta)$ *there is no* $y^* \in N_L(D, g(\bar{x}, \bar{u}))$*,* $y^* \neq 0$*, satisfying* $0 \in y^* \circ D_x g(\bar{x}, \bar{u}) + N_L(C, \bar{x})$*.*

Proof. The case where H is a finite dimensional space was obtained in [1]. Let assume that D is convex and sequentially normally compact at $g(\bar{x}, \bar{u})$. Consider the function $f : X \times U \mapsto \mathbb{R} \cup \{+\infty\}$ defined by

$$f(x, u) = \begin{cases} d(g(x, u), D) & \text{if } x \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

Then

$$S(u) = \{x \in X : \quad f(x, u) \leq 0\}.$$

With these definitions and taking into account the continuity of g in both variables x and u, assertion $\alpha)$ is equivalent to the conclusion of Theorem 2 in [1]. Suppose that $\alpha)$ is false. Then, by Theorem 2 in [1], there are sequences $x_n \to \bar{x}$, with $x_n \in C$, $u_n \to \bar{u}$ and $\varepsilon_n \to 0^+$ such that

$$x_n \notin S(u_n) \text{ and } 0 \in \partial_x^{\varepsilon_n} f(x_n, u_n). \tag{2}$$

So there exists $r_n \to 0^+$ such that

$$f(x_n, u_n) \leq f(x, u_n) + 2\varepsilon_n \|x_n - x\| \quad \forall x \in B(x_n, r_n)$$

or equivalently

$$d(g(x_n,u_n),D) \le d(g(x,u_n),D) + 2\varepsilon_n\|x_n - x\| \quad \forall x \in B(x_n,r_n)\cap C. \quad (3)$$

Let $d_n \in D$ be the projection of $g(x_n,u_n)$ over D, that is,

$$d(g(x_n,u_n),D) = \|g(x_n,u_n) - d_n\|.$$

Then $d_n \to g(\bar{x},\bar{u})$ and by (3) we obtain

$$\|g(x_n,u_n) - d_n\| \le \|g(x,u_n) - d_n\| + 2\varepsilon_n\|x - x_n\| \quad \forall x \in B(x_n,r_n)\cap C$$

and

$$\|g(x_n,u_n) - d_n\| \le \|g(x_n,u_n) - y\| \quad \forall y \in D.$$

Set $y_n^* = \frac{g(x_n,u_n)-d_n}{\|g(x_n,u_n)-d_n\|}$. Using the fact that g is of class $\mathcal{C}^1$ at $(\bar{x},\bar{u})$ in x with respect to u we get a sequence $s_n \to 0^+$ such that

$$-y_n^* \circ D_x g(x_n,u_n) \in N_{s_n}^F(C,x_n)$$

and

$$y_n^* \in N_{s_n}^F(D,d_n).$$

Extracting a subsequence if necessary we may assume that $y_n^* \to y^*$ and since D is SNC at $g(\bar{x},\bar{u})$ and $\|y_n^*\| = 1$, we get $y^* \neq 0$. Thus there exists $y^* \in N_L(D,g(\bar{x},\bar{u}))$, $y^* \neq 0$, such that $0 \in y^* \circ D_x g(\bar{x},\bar{u}) + N_L(C,\bar{x})$. But this inclusion contradicts the assertion β). $\diamondsuit$

4 A general result on seminormality and strong local controllability

Let H be a real Hilbert space and $F : [a,b]\times H \mapsto H$ be a multivalued mapping. We consider the set C of solutions of the differential inclusion

$$\dot{x}(t) \in F(t,x(t)) \quad a.e.\, t \in [a,b]. \quad (4)$$

We introduce the multivalued mapping $G\colon H\times H \mapsto W^{1,p}$ defined by

$$G(u,v) = \{x \in W^{1,p}\colon\ \dot{x}(t) \in F(t,x(t))\, a.e.,\ (x(a)+u, x(b)+v) \in S\} \quad (5)$$

Let z be a solution of system (1). This system is said to be *semi-normal* at z if there exist $\alpha > 0$ and $r > 0$ such that

$$d(x,G(u,v)) \le \alpha d((x(a)+u, x(b)+v);S) \quad (6)$$

for all $x \in B(z,r)\cap C$ and $u,v \in r\mathbb{B}$. Here $\mathbb{B}$ stands for the closed unit ball of H and

$$B(z,r) = \{x \in W^{1,p}\colon\ \|x - z\| \le r\}.$$

It is clear that seminormality implies strong local controllability. Hereafter we give a sufficient condition ensuring seminormality of the system (1). This condition uses the linear continuous mapping $w : W^{1,p} \mapsto H \times H$ defined by

$$w(x) = (x(a), x(b)).$$

Theorem 4.1 *Let $1 \leq p \leq \infty$. Suppose that either H is of finite dimension or S is convex and sequentially normally compact at $(z(a), z(b))$. Then the system is semi-normal at z (in $W^{1,p}$) provided that C is closed (which is the case when the multivalued mapping $x \mapsto F(t, x)$ has closed graph for almost all t) and*

$$w^*(\partial_L d(S, (z(a), z(b))) \cap -\partial_L d(C, z) = \{0\} \tag{7}$$

where w^* denotes the adjoint mapping of w.

Proof : Let us consider the following linear continuous mapping $g : W^{1,p} \times H \times H \mapsto H \times H$ defined by $g(x, u, v) = (x(a)+u, x(b)+v)$. Then

$$G(u, v) = \left\{x \in W^{1,p} : x \in C, g(x, u, v) \in S\right\}$$

and then $(z, 0, 0)$ is a solution of the system

$$x \in C, \quad g(x, u, v) \in S.$$

It suffices to show that part β) of Theorem 3.1 holds for this system. On the contrary, suppose that there exists $(a^*, b^*) \neq 0$ such that:

$$(a^*, b^*) \in N_L(S, g(z, 0, 0))$$

and

$$0 \in D_x^* g(z, 0, 0)(a^*, b^*) + N_L(C, z)$$

However, we have for all $x \in W^{1,p}$

$$D_x g(z, 0, 0)(x) = (x(a), x(b)) = w(x)$$

and then

$$\begin{aligned} \langle D_x^* g(z, 0, 0)(a^*, b^*), x\rangle &= \langle D_x g(z, 0, 0)(x), (a^*, b^*)\rangle \\ &= \langle w(x), (a^*, b^*)\rangle \\ &= \langle w^*(a^*, b^*), x\rangle \end{aligned}$$

and hence $D_x^* g(z, 0, 0)(a^*, b^*) = w^*(a^*, b^*)$, but we have supposed that $(a^*, b^*) \in N_L(S, (z(a), z(b))$ and then

$$w^*(N_L(S, (z(a), z(b))) \cap -N_L(C, z) \neq \{0\}$$

or equivalently

$$w^*(\partial_L d(S,(z(a),z(b))) \cap -\partial_L d(C,z) \neq \{0\}$$

which leads to a contradiction with the hypothesis of the proposition. So part β) of Theorem 3.1 holds and then there exist $a > 0$ and $r > 0$ such that

$$d(x, G(u,v)) \leq ad(g(x,u,v), S)$$

for all $x \in C \cap B(z,r)$ and all $u, v \in B(0,r)$. Therefore

$$d(x, G(u,v)) \leq ad((x(a)+u, x(b)+v), S)$$

and this completes the proof. $\diamondsuit$

5 Seminormality and strong controllability of sub-Lipschitz differential inclusions

In this section we will give sufficient conditions for seminormality and strong controllability of system (1) using the sub-Lipschitz property on the differential inclusion in the case where $H = \mathbb{R}^m$.

Definition 5.1 *F is said to be sub-Lipschitzian in the sense of Loewen-Rockafellar [11] at z if there exist $\beta > 0$, $\varepsilon > 0$ and a summable function $k : [a,b] \mapsto \mathbb{R}$ such that for almost all $t \in [a,b]$, for all $N > 0$, for all $x, x' \in z(t) + \varepsilon\mathbb{B}$ and $y \in \dot{z}(t) + N\mathbb{B}$ one has*

$$d(y, F(t,x)) - d(y, F(t,x')) \leq (k(t) + \beta N)|x - x'|.$$

This is not the original definition by Loewen-Rockafellar [11], but both concepts are equivalent.

Before giving the results, we make a significant focus on the Sobolev space $W^{1,1}$. Using the Hahn-Banach theorem and the Riesz representation theorem, we obtain that for each element x^* of the dual space $(W^{1,1})^*$ of $W^{1,1}$ there exist two elements v_0 and v_1 in $L^\infty := L^\infty([a,b], \mathbb{R}^n)$ such that

$$\langle x^*, x\rangle = \langle v_0, x\rangle_{L^\infty \times L^1} + \langle v_1, x\rangle_{L^\infty \times L^1} \, \forall x \in W^{1,1}.$$

Set $V_0(t) = -\int_t^b v_0(s)ds$. Then $V_0(b) = 0$ and $\dot{V}_0(t) = v_0(t)$. Integrating by part, we get

$$\langle x^*, x\rangle = \langle x(a), -V_0(a)\rangle_{\mathbb{R}^n \times \mathbb{R}^n} + \langle v_1 - V_0, \dot{x}\rangle_{L^\infty \times L^1}, \ \forall x \in W^{1,1}.$$

Now if we put $w(t) = \int_a^t (v_1 - V_0)(s)ds - V_0(a)$, then $w \in W^{1,\infty}$ and

$$\langle x^*, x\rangle = \langle x(a), w(a)\rangle_{\mathbb{R}^n \times \mathbb{R}^n} + \langle \dot{w}, \dot{x}\rangle_{L^\infty \times L^1}, \ \forall x \in W^{1,1}.$$

Thus for each $x^* \in (W^{1,1})^*$ there exists $w \in W^{1,\infty}$ such that

$$\langle x^*, x\rangle = \langle x(a), w(a)\rangle_{\mathbb{R}^n\times\mathbb{R}^n} + \langle \dot{w}, \dot{x}\rangle_{L^\infty\times L^1},\ \forall x \in W^{1,1}.$$

To avoid burdening the notation, we will put

$$w = x^* \quad \langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\mathbb{R}^n\times\mathbb{R}^n} \quad \text{and} \quad \langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{L^\infty\times L^1}.$$

Thus the pairing between $W^{1,1}$ and $(W^{1,1})^*$ may be defined by

$$\langle x^*, x\rangle = \langle x^*(a), x(a)\rangle + \langle \dot{x}^*, \dot{x}\rangle_{L^\infty\times L^1},\ \forall x \in W^{1,1}\ \forall x^* \in (W^{1,1})^*. \qquad (8)$$

With the help of Theorem 2.1 in [8], Corollary 4.1 in [9] and Theorem 3.1.7 in [2], we obtain the following result.

Theorem 5.1 *Let $C \subset W^{1,1}$ be the set of solutions of the system (1), with $S = \mathbb{R}^m \times \mathbb{R}^m$. Suppose that C is closed and F is sub-Lipschitzian at $z \in C$. Then for each $x^* \in \partial_L d(C,z)$ there exists an arc $p \in W^{1,1}$ such that*

$$\dot{p}(t) \in coD_L^*F(t,\cdot)(z(t),\dot{z})(-p(t)-\dot{x}^*(t))\ a.e.\,t, \quad p(a) = -x^*(a),\ p(b) = 0$$

and

$$\langle p(t) + \dot{x}^*(t), \dot{z}(t)\rangle = \max_{y\in F(t,z(t))} \langle p(t) + \dot{x}^*(t), y\rangle\ a.e.\,t.$$

We will show that the following sufficient conditions ensure strong controllability of system (1) : there is nontrivial arc p satisfying

$$\dot{p}(t) \in \mathrm{co}D_L^*F(t, z(t), \dot{z}(t))(-p(t))\ a.e. \qquad (9)$$

$$(p(a), -p(b)) \in N_L(S, (z(a), z(b)) \qquad (10)$$

and

$$\langle p(t), \dot{z}(t)\rangle = \max_{v\in F(t,z(t))} \langle p(t), v\rangle \quad a.e. \qquad (11)$$

In fact, we will show that these relations imply (7) and then, by Theorem 4.1, the seminormality as well as the strong local controllability.

Using the definition of the pairing between $W^{1,1}$ and its topological dual, we obtain

Lemma 5.1 *For each $a^*, b^* \in \mathbb{R}^m$ satisfying $u^* := -w^*(a^*, b^*) \in \partial_L d(C,z)$, we have*

$$u^*(t) = -a^* - b^* - b^*(t-a) \quad \forall t \in [a,b].$$

Proof. For all $x \in W^{1,1}$ we have

$$\langle u^*, x\rangle = \langle -w^*(a^*, b^*), x\rangle = -\langle (a^*, b^*), w(x)\rangle = -\langle a^*, x(a)\rangle - \langle b^*, x(b)\rangle$$

and hence

$$\langle u^*(a), x(a)\rangle + \int_a^b \langle \dot{u}^*(t), \dot{x}(t)\rangle dt = -\langle a^*, x(a)\rangle - \langle b^*, x(b)\rangle. \tag{12}$$

Now pick an arbitrary $v \in \mathbb{R}^m$ and put $x(t) = v$ for all $t \in [a, b]$. Then

$$\langle u^*(a), v\rangle = -\langle a^*, v\rangle - \langle b^*, v\rangle.$$

Thus

$$u^*(a) = -a^* - b^*.$$

Now replace $u^*(a)$ in relation (12) we obtain

$$\langle b^*, x(b) - x(a)\rangle + \int_a^b \langle \dot{u}^*(t), \dot{x}(t)\rangle dt = 0 \quad \forall x \in W^{1,1}$$

or equivalently, for $B^*(t) = b^*$ for all $t \in [a, b]$,

$$\int_a^b \langle B^*(t) + \dot{u}^*(t), \dot{x}(t)\rangle dt = 0 \quad \forall x \in W^{1,1}.$$

This last one gives the desired equality. $\diamondsuit$

Theorem 5.1 and Lemma 5.1 guarantee the following result.

Theorem 5.2 *The assumptions in Theorem 5.1 ensure the following implication*

$$(9) + (10) + (11) \Longrightarrow (7).$$

Proof. Let $(a^*, b^*) \in \partial_L d(S, (z(a), z(b))$ be such that $u^* := -w^*(a^*, b^*) \in \partial_L d(C, z)$. By Lemma 5.1, we have

$$u^*(t) = -a^* - b^* - b^*(t - a) \quad \forall t \in [a, b].$$

Theorem 5.1 ensures the existence of $p \in W^{1,1}$ such that

$$\dot{p}(t) \in \mathrm{co} D_L^* F(t, \cdot)(z(t), \dot{z})(-p(t) - \dot{u}^*(t)) \ a.e.\, t, \quad p(a) = -u^*(a), \ p(b) = 0$$

and

$$\langle p(t) + \dot{u}^*(t), \dot{z}(t)\rangle = \max_{y \in F(t, z(t))} \langle p(t) + \dot{u}^*(t), y\rangle \ a.e.\, t.$$

Put $p_0(t) = p(t) + \dot{u}^*(t)$ for all $t \in [a, b]$. Then $p_0 \in W^{1,1}$ and

$$\dot{p}_0(t) = \dot{p}(t), \ a.e. t \in [a, b], \quad p_0(a) = a^*, \quad p_0(b) = -b^*.$$

Thus p_0 satisfies relations (9), (10) and (11) and hence $p_0 = 0$. Consequently, we obtain that $a^* = b^* = 0$ and $u^* = 0$. This asserts that relation (7) holds true. $\diamondsuit$

6 Controllability and strong controllability of pseudo-Lipschitz differential inclusions

Let $F : [a,b] \times \mathbb{R}^m \mapsto \mathbb{R}^m$ be a multivalued mapping and $R : [a,b] \times [0,+\infty[\mapsto]0,+\infty]$ be a measurable function, in the first variable, referred as the radius function.
Following Clarke [3], F is said to satisfy

1) a *pseudo-Lipschitz property for radius* R near z if for each $s \geq 0$ there exist $\varepsilon_s > 0$ and a summable function k_s such that for almost all $t \in [a,b]$, for every x and x' in $B(z(t), \varepsilon_s)$

$$F(t,x) \cap B(\dot{z}(t), R(t,s)) \subset F(t,x') + k_s(t)|x - x'|B \tag{13}$$

2) the *tempered growth condition for radius* R near z if for each $s \geq 0$ there exist $\varepsilon_s > 0$, $\lambda_s \in]0,1[$ and a summable function r_s such that for almost every $t \in [a,b]$ we have $0 < r_s(t) \leq \lambda_s R(t,s)$ and

$$|x - z(t)| \leq \varepsilon_s \Longrightarrow F(t,x) \cap B(\dot{z}(t), r_s(t)) \neq \emptyset \tag{14}$$

Note that it is not the original Clarke's definition, but its parametrized version.

For the class of pseudo-Lipschitzian differential inclusions, the definitions of controllability and strong controllability will be modified as follows:

1) The system (1) is said to be *locally controllable* at z if there exists $s \geq 0$ such that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $v \in B(0,\delta)$ there exists a trajectory x for F, with $\|x - z\| \leq \varepsilon$, satisfying $(x(0), x(1) + v) \in S$ and for almost all $t \in [0,1]$, $\dot{x}(t) \in B(\dot{z}(t), R(t,s))$.

2) It is said to be *strongly locally controllable* at z if there exist $a > 0$ and $\delta > 0$ such that for all u and v in $B(0,\delta)$ there exists a trajectory x for F, with $\|x - z\| \leq a(|u| + |v|)$, satisfying $(x(0) + u, x(1) + v) \in S$ and for almost all $t \in [0,1]$, $\dot{x}(t) \in B(\dot{z}(t), \mathcal{R}(t))$, where

$$\mathcal{R}(t) = \liminf_{s \to +\infty} R(t,s).$$

6.1 Strong controllability under Mordukhovich's regularity

In this subsection the Lipschitz constant as well as the radius R *for* F *near* z *are not depending on the parameter* s.

In this section we are interested in the strong controllability of the system (1) under Mordukhovich regularity of the solution set C to the following system :

$$\dot{x}(t) \in F(t, x(t)) \cap B(\dot{z}(t), R(t)) \quad a.e.\, t \in [0, 1] \tag{15}$$

We say that C is *Mordukhovich regular* at z if

$$\partial_L d(C, z) = \partial^F d(C, z).$$

This happens, for example, when the set-valued mapping $x \mapsto F(t, x)$ is convex, in the sense that its graph is.

Theorem 6.1 *Let z be a solution of the system (1). Suppose that C is closed and Mordukhovich regular at z and that F is Clarke's pseudo-Lipschitz and satisfies the tempered growth condition for radius R near z. Then the system (1) is strongly locally controllable at z, provided that there is nontrivial arc p satisfying the inclusions (9) and (10) as well as the following maximum condition*

$$\langle p(t), \dot{z}(t)\rangle = \max_{v \in F(t,z(t)) \cap B(\dot{z}(t),R(t))} \langle p(t), v\rangle \quad a.e. \tag{16}$$

The proof of the theorem uses the following lemma whose proof's is a consequence of Theorem 4.1.1 in [3].

Lemma 6.1 *Let $a^*, b^* \in \mathbb{R}^m$ be such that $-w^*(a^*, b^*) \in \partial_L d(C, z)$. Then, under the assumptions of the theorem, there exists un arc p which satisfies, in addition to relations (9) and (16), the following ones : $p(a) = a^*$ and $p(b) = -b^*$.*

Proof. Let $-w^*(a^*, b^*) \in \partial_L d(C, z)$. Since C is Mordukhovich regular at z, we have for all $\gamma > 0$ there exists $\delta > 0$ such that

$$\langle w^*(a^*, b^*), x - z\rangle + \gamma\|x - z\| \geq 0\, \forall x \in B(z, \delta) \cap C$$

or equivalently

$$\langle a^*, x(a) - z(a)\rangle + \langle b^*, x(b) - z(b)\rangle + \gamma|x(a) - z(a)| + \gamma \int_a^b |\dot{x}(t) - \dot{z}(t)|dt \geq 0$$

for all $x \in B(z, \delta) \cap C$. Since F is pseudo-Lipschitz for radius R near z, with a summable function k, there exists $\varepsilon > 0$ such that for almost all $t \in [a, b]$, for every x and x' in $B(z(t), \varepsilon)$

$$F(t, x) \cap B(\dot{z}(t), R(t)) \subset F(t, x') + k(t)|x - x'|B \tag{17}$$

The tempered growth condition for radius R near z implies the existence of $\varepsilon_s > 0$, $\lambda_s \in]0,1[$ and a summable function r_s such that for almost every $t \in [a,b]$ we have $0 < r_s(t) \leq \lambda_s R(t,s)$ and

$$|x - z(t)| \leq \varepsilon_s \Longrightarrow F(t,x) \cap B(\dot{z}(t), r_s(t)) \neq \emptyset$$

We may assume that $\varepsilon = \varepsilon_0 = \delta$. Consider the function $L : [a,b] \times \mathbb{R}^m \times \mathbb{R}^m \mapsto \mathbb{R} \cup \{+\infty\}$ and $\ell : \mathbb{R}^m \times \mathbb{R}^m \mapsto \mathbb{R}$ defined by

$$L(t,x,y) = \begin{cases} \gamma|y - \dot{z}(t)| & \text{if } y \in F(t,x) \\ +\infty & \text{otherwise} \end{cases}$$

and

$$\ell(u,v) = \langle a^*, u - z(a)\rangle + \langle b^*, v - z(b)\rangle + \gamma|u - z(a)|.$$

So z is a local solution of radius R (in the Clarke sense [3]) of the problem

$$\min \ell(x(a), x(b)) + \int_a^b L(t, x(t), \dot{x}(t))dt.$$

It is not difficult to see that all the assumptions of Theorem 4.1.1 in [3] are satisfied and to apply this theorem to complete the proof.$\diamondsuit$

Proof of Theorem 6.1. We are ready to show that relation (7) is satisfied and to apply Theorem 4.1. Let $(a^*, b^*) \in \partial_L d(S, (z(a), z(b)))$ be such that $w^*(a^*, b^*) \in -\partial_L d(C, z)$. Then, Lemma 6.1 ensures the existence of un arc p satisfying relations (9) and (16) and $p(a) = a^*$ and $p(b) = -b^*$. So that $(p(a), -p(b)) \in \partial_L d(S, (z(a), z(b)))$, and the assumption of the theorem implies that $p = 0$. Thus $a^* = b^* = 0$. The proof is then completed. $\diamondsuit$

6.2 Controllability under the closedness of the attainable set

In this section, the set S in (1) takes the following form

$$S = S_0 \times S_1$$

where S_0 and S_1 are closed sets in $\mathbb{R}^m$.
We are concerned with local controllability of the system (1) under the closedness of the attainable set $\mathcal{A}(F, S_0)$ with a pseudo-Lipschitz differential inclusion. For a set $S_0 \subset \mathbb{R}^m$ and a parameter $s \geq 0$, the *attainable set* for trajectories of F emanating from S_0 is defined by

$$\mathcal{A}_s(F, S_0) := \{x(b) : \quad \dot{x}(t) \in F(t, x(t)) \cap B(\dot{z}(t), R(t,s)) \; a.e., \; x(a) \in S_0\}.$$

For $s \geq 0$, we say that $\mathcal{A}_s(F, S_0)$ is *locally closed* near z if for each $\varepsilon > 0$ sufficiently small the following set is closed in $\mathbb{R}^m$:

$$\mathcal{A}_{s,\varepsilon}(F, S_0) := \{x(b) : \|x - z\| \leq \varepsilon, \; \dot{x}(t) \in F(t, x(t)) \cap B(\dot{z}(t), R(t,s)) \; a.e., \; x(a) \in S_0\}.$$

Theorem 6.2 *Let z be a solution of the system (1). Suppose that, for each $s \geq 0$, the attainable set $\mathcal{A}_s(F, S_0)$ is locally closed near z and that F is Clarke's pseudo-Lipschitz and satisfies the tempered growth condition for radius R near z. Then the system (1) is locally controllable at z, provided that there is nontrivial arc p satisfying relations (9) as well as the following ones*

$$p(a) \in N_L(S_0, z(a)), \quad -p(b)) \in N_L(S_1, z(b)) \tag{18}$$

$$\langle p(t), \dot{z}(t) \rangle = \max_{v \in F(t,z(t)) \cap B(\dot{z}(t), \mathcal{R}(t))} \langle p(t), v \rangle \quad a.e. \tag{19}$$

Where for almost all $t \in [a, b]$, we set

$$\mathcal{R}(t) = \liminf_{s \to +\infty} R(t, s) \tag{20}$$

Proof. Note that we may put $a = 0$ and $b = 1$ and assume that S is a compact set and so there exists $r > 0$ such that $S \subset 2rB$. We introduce the following transformation of system (1), which is in the line of Theorem 3.5.3 in [2] :

$$(\dot{x}(t), \dot{y}(t)) \in \tilde{F}(t, x(t), y(t)) \, a.e.t \in [0, 2]; \tag{21}$$

$$(x(0), y(0)) \in C_0, \; (x(2), y(2)) = (0, 0) \tag{22}$$

where $\tilde{F} : [0, 2] \times \mathbb{R}^m \times \mathbb{R}^m \mapsto \mathbb{R}^m \times \mathbb{R}^m$ is the multivalued mapping defined by

$$\tilde{F}(t, x, y) = \begin{cases} F(t, x + y) \times \{0\} & \text{if } 0 \leq t \leq 1 \\ \{0\} \times 2rB & \text{if } 1 \leq t \leq 2 \end{cases}$$

and $C_0 = \{(c_0 - c_1, c_1) : \, (c_0, c_1) \in S\}$. Define the arc $\tilde{z}$ by

$$\tilde{z}(t) = \begin{cases} (z(t) - z(1), z(1)) & \text{if } 0 \leq t \leq 1 \\ (0, (2 - t)z(1)) & \text{if } 1 \leq t \leq 2 \end{cases}$$

It is not difficult to see that $\tilde{z}$ is a solution of system (23) and $\tilde{F}$ is Clarke's pseudo-Lipschitz and satisfies the tempered growth condition for radius $\tilde{R}$ near $\tilde{z}$, where

$$\tilde{R}(t, s) = \begin{cases} R(t, s) & \text{if } 0 \leq t \leq 1 \\ 2r & \text{if } 1 < t \leq 2 \end{cases}$$

Then, we have the following properties.

Lemma 6.2

1. *Define two functions* f_t *and* g_t *by*

$$f_t(y,z,v) = d(v, F(t, y+z)) \quad \text{and} \quad g_t(x,v) = d(v, F(t,x)).$$

   Let $(y^*, z^*, v^*) \in \partial_L f_t(\bar{y}, \bar{z}, \bar{v})$. *Then* $y^* = z^*$ *and* $(y^*, v^*) \in \partial_L g_t(\bar{y}+\bar{z}, \bar{v})$.

2. *If* $0 \le t \le 1$, $\bar{v} \in F(t, \bar{y}+\bar{z})$ *and* $(y^*, z^*) \in D_L^* \tilde{F}(t, (\bar{y}, \bar{z}), (\bar{v}, 0))(v^*, w^*)$ *then* $y^* = z^*$, $y^* \in D_L^* F(t, \bar{y}+\bar{z}, \bar{v})(v^*)$.

3. *If* $1 \le t \le 2$ *and* $(y^*, z^*) \in D_L^* \tilde{F}(t, (0, (2-t)z(1)), (0, -z(1)))(v^*, w^*)$, *then* $y^* = z^* = w^* = 0$.

4. *Let* $(x_0, y_0) \in C_0$ *and* $(x^*, y^*) \in N_L(C_0, (x_0, y_0))$. *Then* $x^* \in N_L(S_0, x_0 + y_0)$ *and* $y^* - x^* \in N_L(S_1, y_0)$.

Proof. We establish only the item 1. The other ones are easy to obtain. Let $(y^*, z^*, v^*) \in \partial_L f_t(\bar{y}, \bar{z}, \bar{v})$. Then, by the definition of the Mordukhovich's subdifferential, there are sequences $(y_n^*, z_n^*, v_n^*) \to (y^*, z^*, v^*)$, $(y_n, z_n, v_n) \to (\bar{y}, \bar{z}, \bar{v})$, $\delta_n \to 0^+$ and $\varepsilon_n \to 0^+$ such that

$$d(v, F(t, y+z) - d(v_n, F(t, y_n + z_n) - \langle y_n^*, y - y_n \rangle - \langle z_n^*, z - z_n \rangle - \\ \langle v_n^*, v - v_n \rangle + \varepsilon_n [\|y - y_n\| + \|z - z_n\| + \|v - v_n\|] \ge 0$$

for all $y \in B(y_n, \delta_n)$, $z \in B(z_n, \delta_n)$ and $v \in B(v_n, \delta_n)$. Putting $x_n = y_n + z_n$ and $x = y + z$, the last inequality implies that

$$d(v, F(t, x) - d(v_n, F(t, x_n) - \langle y_n^*, x - x_n \rangle - \langle z_n^* - y_n^*, z - z_n \rangle - \\ \langle v_n^*, v - v_n \rangle + 2\varepsilon_n [\|x - x_n\| + \|z - z_n\| + \|v - v_n\|] \ge 0$$

for all $x \in B(x_n, \frac{\delta_n)}{2}$, $z \in B(z_n, \frac{\delta_n)}{2})$ and $v \in B(v_n, \delta_n)$. Take $z = z_n$ (resp. $x = x_n$ and $v = v_n$) we get

$$d(v, F(t, x) - d(v_n, F(t, x_n) - \langle y_n^*, x - x_n \rangle - \langle v_n^*, v - v_n \rangle + \\ 2\varepsilon_n [\|x - x_n\| + \|v - v_n\|] \ge 0$$

for all $x \in B(x_n, \frac{\delta_n)}{2}$ and $v \in B(v_n, \delta_n)$ (resp. $\|y_n^* - z_n^*\| \le 2\varepsilon_n$). Thus $(y^*, v^*) \in \partial_L g_t(\bar{y} + \bar{z}, \bar{v})$ and $y^* = z^*$.$\diamondsuit$

The following lemma shows that the local closedness of the attainable set for F implies that for $\tilde{F}$.

Lemma 6.3 *For each* $s \ge 0$, *the attainable set* $\mathcal{A}_s(\tilde{F}, C_0)$ *for trajectories of* $\tilde{F}$ *emanating from* C_0 *is locally closed near* $\tilde{z}$.

Proof. Let $((u_n, v_n))$ be a sequence of $\mathcal{A}_s(\tilde{F}, C_0)$ converging to some (u_x, v_y). We will show that $(u_x, v_y) \in \mathcal{A}_s(\tilde{F}, C_0)$. Indeed, let (x_n, y_n) be a trajectory for $\tilde{F}$ be such that

$$(\dot{x}_n(t), \dot{y}_n(t)) \in B(\dot{\tilde{z}}(t), \tilde{R}(t,s)), \quad (x_n(0), y_n(0)) \in C_0, \quad (x_n(2), y_n(2)) = (u_n, v_n).$$

Then $x_n(0) + y_n(0) \in S_0$, $y_n(0) \in S_1$ and

$$\begin{cases} \dot{y}_n(t) = 0, \quad \dot{x}_n(t) \in F(t, x_n(t) + y_n(t)) \cap B(\dot{z}(t), R(t,s)) \ a.e.\, t \in [0,1] \\ \dot{x}_n(t) = 0, \quad \dot{y}_n(t) \in 2rB \ a.e.\, t \in [1,2] \end{cases}$$

It follows that $x_n(1) = x_n(2) = u_n$ and $y_n(1) = y_n(0) \in S_1$. Set $w_n(t) = x_n(t) + y_n(t)$ for all $t \in [0,2]$. Then $w_n(0) \in S_0$ and

$$\begin{cases} \dot{w}_n(t) \in F(t, w_n(t)) \cap B(\dot{z}(t), R(t,s)) \ a.e.\, t \in [0,1] \\ \dot{w}_n(t) \in 2rB \ a.e.\, t \in [1,2] \end{cases}$$

First, we consider the case $\dot{w}_n(t) \in 2rB$ $a.e.t \in [1,2]$, with $w_n(1) - u_n = y_n(1) \in S_1$. Set $W_n(t) = w_n(t) - u_n$. Since S_1 is compact, Theorem 3.1.7 in [2] asserts that there exists a subsequence of (W_n) which converges uniformly to an arc W satisfying

$$\dot{W}(t) \in 2rB \, a.e.\, t \in [1,2] \quad \text{and} \quad W(1) \in S_1$$

and hence (w_n) converges uniformly to an arc w which satisfies

$$\dot{w}(t) \in 2rB \, a.e.t \in [1,2] \quad \text{and} \quad w(1) - u_x \in S_1$$

Second, $(w_n(1))$ is a sequence of the set $\mathcal{A}_s(F, S_0)$ which is closed. Then, since $w_n(1) \to w(1) \in \mathcal{A}_s(F, S_0)$, there exists an arc v such that

$$\dot{v}(t) \in F(t, v(t)) \cap B(\dot{z}(t), R(t,s)) \ a.e.t \in [0,1], \quad v(0) \in S_0, \quad v(1) = w(1).$$

Now since $y_n(1) = w_n(1) - u_n$, then $y_n(1) \to w(1) - u_x$. Set

$$x(t) = \begin{cases} v(t) - v(1) + u_x & \text{if } 0 \le t \le 1 \\ u_x & \text{if } 1 \le t \le 2 \end{cases} \quad \text{and } y(t) = \begin{cases} w(1) - u_x & \text{if } 0 \le t \le 1 \\ w(t) - u_x & \text{if } 1 \le t \le 2 \end{cases}$$

with $w(2) = u_x + v_x$. Then (x, y) is a trajectory for $\tilde{F}$ which satisfies

$$(\dot{x}(t), \dot{y}(t)) \in B(\dot{\tilde{z}}(t), \tilde{R}(t,s)), \quad (x(0), y(0)) \in C_0, \quad (x(2), y(2)) = (u_x, v_y)$$

and the proof is completed.$\diamondsuit$

The following result is a consequence of Theorem 2.3.3 in [3].

Lemma 6.4 *Let z be a solution to the system*

$$\dot{x}(t) \in F(t, x(t))\, a.e.,\ x(0) \in S_0 \tag{23}$$

where $S_0 \subset \mathbb{R}^m$ is closed and F is Clarke's pseudo-Lipschitz and satisfies the tempered growth condition for radius R near z, and that for each $s \geq 0$ there exists $\varepsilon_s > 0$ such that $z(1)$ is a boundary point of the set

$$\mathcal{A}_s(F, S_0) := \{x(1) :\ \|x - z\| \leq \varepsilon_s,\ \dot{x}(t) \in F(t, x(t)) \cap B(\dot{z}(t), R(t, s))\, a.e.,\ x(0) \in S_0\}.$$

Then there exists a nontrivial arc p such that

$$\dot{p}(t) \in coD_L^* F(t, z(t), \dot{z}(t))(-p(t))\ a.e.$$

$$p(0) \in N_L(S_0, z(0))$$

$$\langle p(t), \dot{z}(t) \rangle = \max_{v \in F(t, z(t)) \cap B(\dot{z}(t), \mathcal{R}(t))} \langle p(t), v \rangle \quad a.e.$$

Proof of Theorem 6.2 (be continued) : It is easy to see that $\tilde{F}$ satisfies the Clarke's pseudo-Lipschitz property near $\tilde{z}$ as well as the tempered growth condition of radius $\tilde{R}$ near $\tilde{z}$. We will show that the conlusion of Lemma 6.4 does not hold for $\tilde{F}$ at $\tilde{z}$. Let p and q be two arcs satisfying:

the Euler-Lagrange inclusion

$$(\dot{p}(t), \dot{q}(t)) \in \text{co} D^* \tilde{F}(t, \tilde{z}(t), \dot{\tilde{z}}(t))(-p(t), -q(t)) \quad a.e. \tag{24}$$

the transversality condition

$$(p(0), q(0)) \in N_L(C_0, \tilde{z}(0)) \tag{25}$$

as well as the maximum condition

$$\langle (p(t), q(t)), \dot{\tilde{z}}(t) \rangle = \max_{(w_1, w_2) \in \tilde{F}(t, \tilde{z}(t) \cap B(\tilde{z}(t), \tilde{R}(t))} \langle p(t), w_1 \rangle + \langle q(t), w_2 \rangle \tag{26}$$

By Lemma 6.2, the Euler-Lagrange inclusion (24) implies that

$$\dot{p}(t) = \dot{q}(t) \quad \text{and} \quad \dot{p}(t) \in \text{co} D_L^* F(t, z(t), \dot{z}(t))(-p(t))\, a.e. t \in [0, 1] \tag{27}$$

This implies, in particular, that $p(1) - p(0) = q(1) - q(0)$. The transversality condition (25) and Lemma 6.2 ensure that

$$p(0) \in N_L(S_0, z(0)) \quad \text{and} \quad q(1) - p(1) = q(0) - p(0) \in N_L(S_1, z(1)) \tag{28}$$

It follows from the maximum condition (26) that

$$\langle p(t), \dot{z}(t)\rangle = \max_{w\in F(t,z(t))\cap B(\dot{z}(t),R(t))} \langle p(t), w\rangle \; a.e. t \in [0, 1] \tag{29}$$

and $q(t) = 0$ for all $t \in [1, 2]$. So that $q(1) = 0$ and hence $-p(1) \in N_L(S_1, z(1))$. Now, it is time to use relations (9), (18) and (19) to get $p = q = 0$. This allows us to apply Lemma 6.4 to get the existence of $s \geq 0$ such that for all $\varepsilon > 0$, $\tilde{z}(2)$ is not a boundary point of the set

$$\mathcal{A}_{s,\varepsilon}(\tilde{F}, C_0).$$

But as this set is closed and $\tilde{z}(2) \in \mathcal{A}_{s,\varepsilon}(\tilde{F}, C_0)$, it follows that $\tilde{z}(2)$ is an internal point of $\mathcal{A}_{s,\varepsilon}(\tilde{F}, C_0)$. A simple computation leads to the conclusion of our theorem.$\diamondsuit$

The following example shows the necessity of the tempered growth condition.

Example 6.1 *Let $k : [0, 1] \mapsto [0, +\infty[$ be a summable function, which is unbounded on any open interval. Define ([3]) the multivalued mapping $F : [0, 1] \times \mathbb{R} \mapsto \mathbb{R}$ by $F(t, x) =]-\infty, -k(t)|x|]$. Then F is pseudo-Lipschitz for any choice of radius function R near $z = 0$. Then for any essentially bounded function R, the attainable set $\mathcal{A}(F, \{0\}) = \{0\}$ (since the only trajectory x for F emanating from 0 and satisfying $\dot{x}(t) \in B(0, R(t))$ is $x = 0$) and hence the following system is not locally controllable at $z = 0$:*

$$\dot{x}(t) \in F(t, x(t)) \; a.e. t \in [0, 1] \quad x(0) = 0.$$

However the only adjoint arc p satisfying relations (9), (18) and (19) is $p = 0$.

References

[1] P. Bosch, R. Henrion, A. Jourani, Error bounds and applications, Applied Math. Optim., 50 (2004) 161-181.

[2] F.H. Clarke, Optimisation and nonsmooth analysis, Wiley, New-York (1983).

[3] F.H. Clarke, Necessary conditions in dynamic programming, *Memoirs AMS* **73** (2005).

[4] H. Frankowska, Local controllability and infinitesimal generators of semigroups of set-valued maps, SIAM J. Cont. Optim., 25 (1987) 412–432.

[5] H. Frankowska, B. Kaskosz, A Maximum principle for differential inclusion problems with state constraints, Systems Control Lett., 11 (1988) 189-194.

[6] A. D. Ioffe (1997) Euler-Lagrange and Hamiltonian formalisms in dynamic optimization, Trans. Amer. Math. Soc. 349: 2871–2900.

[7] A. D. Ioffe and R. T. Rockafellar, The Euler and Weierstrass conditions for nonsmooth variational problems, Calculus of Variations and PDE, 4 (1996) 59–87.

[8] A. Jourani, Normality, local controllability and NOC in multiobjective optimal control problems, Contemporary Trends in Nonlinear Geometric Control Theory and its Applications, Eds A. Anzaldo-Meneses, B. Bonnard, J.-P. Gauthier and F. Monroy, World Scientific Publishers, Singapore, 2002.

[9] A. Jourani, Lagrangian and Hamiltonian necessary conditions for the generalized Bolza problem and applications, J. Nonlin. Convex Anal., 10 (2009), 437-454

[10] B. Kaskosz, S. Lojasiewicz Jr, Lagrange-type extremal trajectories in differential inclusions, Systems Control Lett., 19 (1992), 241-247.

[11] P.D. Loewen and R.T. Rockafellar, Optimal control of unbounded differential inclusions, SIAM J. Control Optim. 32 (1994) 442–470.

[12] P.D. Loewen and R.T. Rockafellar, New necessary conditions for the generalized problem of Bolza, SIAM J. Control Optim., 34 (1996) 1496–1511.

[13] P.D. Loewen and R.T. Rockafellar, Bolza problems with general time constraints, SIAM J. Control Optim., 35 (1997) 2050–2069.

[14] B.S. Mordukhovich, Maximum principle in problems of time optimal control with nonsmooth constraints, J. Appl. Math. Mech., 40 (1976) 960–969.

[15] B.S. Mordukhovich, Metric approximations and necessary optimality conditions for general classes of nonsmooth extremal problems, Soviet Math. Dokl., 22 (1980) 526–530.

[16] B.S. Mordukhovich, Discrete approximations and refined Euler-Lagrange conditions for nonconvex differential inclusions, SIAM J. Control Optim., 33 (1995) 882–915.

[17] B. S. Mordukhovich, Y. Shao, Nonsmooth sequential analysis in Asplund spaces, Trans. Amer. Math. Soc., 348 (1996), 1235-1280.

[18] B. S. Mordukhovich, Variational Analysis and Generalized Differentiation. I: Basic Theory, Grundlehren Series (Fundamental Principles of Mathematical Sciences), Vol. 330, 584 pp., Springer-Verlag, Berlin, 2006.

[19] B. S. Mordukhovich, Variational Analysis and Generalized Differentiation. II: Applications, Grundlehren Series (Fundamental Principles of Mathematical Sciences), Vol. 331, 592 pp., Springer-Verlag, Berlin, 2006.

[20] B.S. Mordukhovich, Variational analysis of evolution inclusions, SIAM J. Control Optim., 18 (2007), 752-777.

[21] H.D. Tuan, Controllability and extremality in nonconvex differential inclusions, J. Optim. Th. Appl., 85 (1995) 435-472.

[22] J. Warga, Controllability, extremality and abnormality in nonsmooth optimal control, J. Optim. Th. Appl. 41 (1983) 239260.

[23] J. Warga, Optimization and controllability without differentiability assumptions, SIAM J. Control Optim., 21 (1983) 837855.