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Conditioning and Upper-Lipschitz Inverse Subdifferentials in Nonsmooth Optimization Problems[1,2,3]

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Abstract. In this paper, we study conditioning problems for convex and nonconvex functions defined on normed linear spaces. We extend the notion of upper Lipschitzness for multivalued functions introduced by Robinson, and show that this concept ensures local conditioning in the nonconvex case via an abstract subdifferential; in the convex case, we obtain complete characterizations of global conditioning in terms of an extension of the upper-Lipschitz property.

Key Words. Subdifferentials, upper-Lipschitz property, conditioning, Ekeland variational principle.

1. Introduction

To solve the optimization problem

$$\min f(x), \qquad x \in X,$$

or the inclusion

$$0 \in T(x),$$

one may consider some iterative methods. In 1970,. Martinet (Refs. 1-2) introduced the proximal method as a regularization in the context of convex

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optimization in Hilbert spaces. Since then, it has been studied by many people for the general maximal monotone inclusion problem under various instances (see Ref. 3 for survey). The proximal method consists in finding a zero of a maximal monotone operator T from a Hilbert space H into itself. This method generates iteratively a sequence as follows:

$$x_{k+1}=(I+\lambda_k T)^{-1}(x_k), \qquad k=0, 1, 2, 3 \ldots, \tag{1}$$

with x_0 known, where I denotes the identity operator and $\{\lambda_k\}$ is a given sequence of positive reals. Note that this algorithm can be considered as a discretization of the following differential inclusion:

$$-du/dt \in Tu, \qquad t>0,$$

$$u(0)=x_0.$$

The finite convergence of the algorithm (1) was first proved by Rockafellar (Ref. 4, Proposition 8) for T the subdifferential in the sense of convex analysis of a proper closed polyhedral convex function defined on a finite-dimensional space. To ensure the finite convergence of the algorithm in the general convex case, several authors imposed the assumption

$$(\partial f)^{-1}(z) \subset (\partial f)^{-1}(0), \tag{2}$$

for all z near zero, or

$$f(x) \geq \min f + cd(x, S), \qquad \forall x, \tag{3}$$

for some $c>0$, where $S=\arg\min f$ is assumed to be a nonempty set. In his paper (Ref. 4), Rockafellar studied this algorithm in the case of an arbitrary maximal monotone operator T. He showed that the algorithm (1) converges superlinearly if the sequence $\{\lambda_k\}$ is such that $\lambda_k \to +\infty$, and if the operator T is such that T^{-1} satisfies the following condition: there exist $\alpha \geq 0$ and $r>0$ such that

$$d(x, T^{-1}(0)) \leq \alpha \|z\|, \tag{4}$$

for $\|z\| \leq r$ and $x \in T^{-1}(z)$, when $T^{-1}(0)$ is a singleton. Note that this result has been extended by Cornejo and Jourani (Ref. 5) to the case where $T^{-1}(0)$ is a nonempty set. In the case where $T=\partial f$, with f a convex function, Lemaire (Refs. 6–7) used the so-called ψ-conditioning to get the finite convergence of Algorithm (1) when ψ is a linear function. A real-valued function f is said to be ψ-conditioned if

$$f(x) \geq \min f + \psi(d(x, S)), \qquad \forall x, \tag{5}$$

where $\psi: R_+ \to R_+$. In finite-dimensional spaces, Zhang and Treiman (Ref. 8) showed that (4) implies (5) whenever f is a lower-semicontinuous function (for short, l.s.c. function) and $\psi(t) = ct^2$, $c > 0$, by using the Mordukhovich subdifferential (Ref. 9).

In this paper, we study the conditioning problems for convex and nonconvex functions defined on normed linear spaces. For other related results, see the recent paper of Penot (Ref. 10). We extend the notion of upper Lipschitzness for multivalued functions introduced by Robinson (Ref. 11), and we show that this concept ensures the local conditioning in the nonconvex case via an abstract subdifferential. In the convex case, we obtain complete characterizations of global conditioning in terms of an extension of the upper-Lipschitz property. Other characterizations are also investigated in the paper.

2. Background

Throughout this paper, X denotes a normed vector space and X^* its topological dual endowed with its dual norm. If $x \in X$ and $x^* \in X^*$, the element $x^*(x)$ is denoted by $\langle x, x^* \rangle$ or $\langle x^*, x \rangle$. If $A \subset X$, $\operatorname{int} A$ denotes its interior, while $\overline{\operatorname{conv}}\, A$ denotes the closed convex hull of A. The set $R \cup \{-\infty, +\infty\}$ is denoted by $\bar{R}$, while R_+ and $\bar{R}_+$ denote the sets of nonnegative reals and $R_+ \cup \{\infty\}$, respectively. Consider a function $f: X \to R \cup \{\infty\}$. The domain of f is the set

$$\operatorname{dom} f = \{x \in X \mid f(x) < \infty\};$$

f is proper if $\operatorname{dom} f \neq \varnothing$. The function f is convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y), \qquad \forall x, y \in \operatorname{dom} f,\ \forall \lambda \in [0, 1].$$

The subdifferential of f at $\bar{x} \in \operatorname{dom} f$ (in the sense of convex analysis) is the set

$$\partial f(\bar{x}) = \{x^* \in X^* \mid \langle x - \bar{x}, x^* \rangle \leq f(x) - f(\bar{x}), \forall x \in X\}.$$

If $\bar{x} \notin \operatorname{dom} f$, then $\partial f(\bar{x}) = \varnothing$. If the function is convex, we consider always this subdifferential. The conjugate of f is the w^*-l.s.c. convex function $f^*: X^* \to \bar{R}$, defined by

$$f^*(x^*) = \sup_{x \in X} (\langle x, x^* \rangle - f(x)).$$

The indicator function of the set $A \subset X$ is the function $I_A : X \to R \cup \{\infty\}$, defined by

$$I_A(x) = \begin{cases} 0, & x \in A, \\ \infty, & x \notin A. \end{cases}$$

The convolution of the functions f and g is denoted by $f \triangle g$ and is defined by

$$(f \triangle g)(x) = \inf_{z \in X} (f(x-z) + g(z)).$$

For a subset A of X, we denote the distance from the point $x \in X$ to A by

$$d(x, A) = \inf_{y \in A} \|x - y\|,$$

with the convention that $d(x, \varnothing) = \infty$. The closed ball centered at $x \in X$ and having radius $\rho > 0$ will be denoted by $B_X(x, \rho)$; we also denote by B_X the ball $B(0, 1)$ and by B_{X^*} the ball $B_{X^*}(0, 1)$. For C and D subsets of X, the Hausdorff excess of C over D is defined by

$$e(C, D) = \sup_{x \in C} d(x, D),$$

with the convention $e(\varnothing, D) = 0$; the ρ-Hausdorff excess of C over D, for $\rho > 0$, is defined by

$$e_\rho(C, D) = e(C \cap \rho B, D).$$

Let X, Y be normed linear spaces. We endow $X \times Y$ with the box norm

$$\|(x, y)\| = \max\{\|x\|, \|y\|\}, \qquad (x, y) \in X \times Y.$$

Generally, a multivalued mapping $T: X \rightrightarrows Y$ will be identified with its graph

$$\{(x, y) \in X \times Y \mid y \in F(x)\};$$

the inverse of F will be denoted by F^{-1}; of course,

$$F^{-1}(y) = \{x \in X \mid y \in F(x)\}.$$

3. Upper φ-Lipschitz Multivalued Mappings: Error Estimates

We recall first the notion of diagonally stationary sequences introduced by Lemaire in Ref. 12.

Definition 3.1. Let $\{T^n\}$ be a sequence of multivalued mappings from X into X^*. A sequence $\{x_n\}$ in X is diagonally stationary for $\{T^n\}$ [for short,

$\{T^n\}$-DS] if

$$\lim_{n\to\infty} d(0, T^n(x_n)) = 0.$$

Definition 3.2. Let $T: X \rightrightarrows X^*$ be a multivalued mapping with $T^{-1}(0) \neq \emptyset$, and let $\varphi: R_+ \to R_+$ be a function. We say that T^{-1} is upper φ-Lipschitz at 0 on the nonempty set $C \subset X^*$ containing 0 if

$$\varphi(d(x, T^{-1}(0))) \leq \|x^*\|, \qquad \forall x^* \in C, \forall x \in T^{-1}(x^*).$$

Note that, for $\varphi(t) = t$, we recover the definition of upper Lipschitzness given by Robinson (Ref. 11). See Refs. 9, 13–15 for more details on various Lipschitzian behaviors, other than the upper-Lipschitz continuity of multivalued mappings.

Example 3.1. Let the operator T be strongly monotone. Then, T^{-1} is upper φ-Lipschitz at zero with $\varphi(t) = \alpha t$ for some $\alpha > 0$.

Example 3.2. See Zhang–Treiman (Ref. 8). Let $f: R^n \to R \cup \{\infty\}$ and $\bar{z} \in \operatorname{dom} f$. The subdifferential of f at $\bar{z}$, in the sense of Mordukhovich (Ref. 9), is the set

$$\partial^- f(\bar{z}) = \limsup_{z \xrightarrow{f} \bar{z}} \{v \mid \langle v, w\rangle \leq f^-(\bar{z}; w), \forall w \in R^n\},$$

where

$$f^-(\bar{z}; w) = \liminf_{\substack{u \to w \\ t \to 0^+}} [f(\bar{z} + tu) - f(\bar{z})]/t$$

is the contingent directional derivative of f at $\bar{z}$ in the direction w, and where $z \xrightarrow{f} \bar{z}$ means that $z \to \bar{z}$ and $f(z) \to f(\bar{z})$. Let C be a closed subset of $\mathbb{R}^n$.

(i) If $f(x) = d(x, C)$, there exists $r > 0$ such that, for all $z \in rB_{\mathbb{R}}{}^n$ and $x \in (\partial^- f)^{-1}(z)$, one has

$$d(x, (\partial^- f)^{-1}(0)) = 0,$$

i.e.,

$$(\partial^- f)^{-1}(z) \subset (\partial^- f)^{-1}(0), \qquad \forall z \in rB_{\mathbb{R}^n}.$$

(ii) If $f(x) = (d(x, C))^2$, there exists $r > 0$ such that, for all $z \in rB_{\mathbb{R}^n}$ and $x \in (\partial^- f)^{-1}(z)$, one has

$$d(x, (\partial^- f)^{-1}(0)) = (1/2)\|z\|.$$

The next result was established by Lemaire for subdifferentials (Ref. 12).

Proposition 3.1. Let $\{x_n\}$ be a $\{T^n\}$-DS sequence with $\limsup \|x_n\| < \rho$ for some $\rho > 0$. Assume that $\lim_{n\to\infty} e_\rho(T^n, T) = 0$ and that T^{-1} is upper φ-Lipschitz at 0 on rB_{X^*}, $r > 0$, where φ: $[0, \infty[\to [0, \infty]$ is a nondecreasing function with $\varphi(0) = 0$ and such that φ is increasing, finite, and continuous on $[0, t_0]$ for some $t_0 > 0$. Then, there exists a sequence $\{x_n^*\} \subset X^*$ converging to 0 such that, for n sufficiently large, $x_n^* \in T^n(x_n)$ and

$$d(x_n, T^{-1}(0)) \leq e_\rho(T^n, T) + \varphi^{-1}(\|x_n^*\| + e_\rho(T^n, T)). \tag{6}$$

Proof. By hypothesis, there exists a sequence $\{x_n^*\} \subset X^*$ such that $x_n^* \in T^n(x_n)$ and $\|x_n^*\| \to 0$. Also, from the hypothesis, there exists $\delta \in]0, \rho[$, $\delta < \varphi(t_0)$, and $n_0 \in \mathbb{N}$ such that T^{-1} is upper φ-Lipschitz on $B_{X^*}(0, \delta)$, $\|x_n^*\| < \delta/3$, $\|x_n\| < \rho$, and $e_\rho(T^n, T) < \delta/3$, for $n \geq n_0$. It follows that, for $n \geq n_0$,

$$d((x_n, x_n^*), T) \leq e_\rho(T^n, T).$$

Let us fix $\epsilon \in]0, \delta/3[$. Then, there is a sequence $\{(u_n, u_n^*)\} \subset T$ such that

$$\max\{\|x_n - u_n\|, \|x_n^* - u_n^*\|\} \leq e_\rho(T^n, T) + \epsilon, \qquad n \geq n_0.$$

It follows that

$$\|u_n^*\| \leq \|u_n^* - x_n^*\| + \|x_n^*\| \leq e_\rho(T^n, T) + \epsilon + \|x_n^*\| < \delta, \qquad n \geq n_0. \tag{7}$$

By the upper φ-Lipschitz property of T^{-1} on $B_{X^*}(0, \delta)$, and using the properties of φ and (7), we get

$$\varphi(d(u_n, T^{-1}(0))) \leq \|u_n^*\|, \qquad n \geqslant n_0.$$

So, for $n \geqslant n_0$, we obtain

$$\begin{aligned} d(x_n, T^{-1}(0)) &\leq \|x_n - u_n\| + d(u_n, T^{-1}(0)) \\ &\leq e_\rho(T^n, T) + \epsilon + \varphi^{-1}(\|x_n^*\| + e_\rho(T^n, T) + \epsilon). \end{aligned}$$

Letting $\epsilon \to 0$, we obtain that (6) holds for $n \geq n_0$. □

In the next proposition, we obtain the following best approximation.

Proposition 3.2. Let $\{x_n\}$ be a $\{T^n\}$-DS sequence with $\limsup \|x_n\| < \rho$ for some $\rho > 0$. Assume that $\lim_{n\to\infty} e_\rho(T^n, T) = 0$ and that there exists $r > 0$ such that

$$T^{-1}(x^*) \subset T^{-1}(0), \qquad \forall x^* \in rB_{X^*}. \tag{8}$$

$\{T^n\}$-DS] if

$$\lim_{n\to\infty} d(0, T^n(x_n))=0.$$

Definition 3.2. Let $T: X \rightrightarrows X^*$ be a multivalued mapping with $T^{-1}(0)\neq\varnothing$, and let $\varphi: R_+ \to R_+$ be a function. We say that T^{-1} is upper φ-Lipschitz at 0 on the nonempty set $C\subset X^*$ containing 0 if

$$\varphi(d(x, T^{-1}(0)))\leq \|x^*\|, \qquad \forall x^*\in C, \forall x\in T^{-1}(x^*).$$

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$$\partial^- f(\bar{z})=\limsup_{z \xrightarrow{f} \bar{z}}\{v\,|\,\langle v, w\rangle \leq f^-(\bar{z}; w), \forall w\in R^n\},$$

where

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is the contingent directional derivative of f at $\bar{z}$ in the direction w, and where $z \xrightarrow{f} \bar{z}$ means that $z\to\bar{z}$ and $f(z)\to f(\bar{z})$. Let C be a closed subset of $\mathbb{R}^n$.

(i) If $f(x)=d(x, C)$, there exists $r>0$ such that, for all $z\in rB_{\mathbb{R}}{}^n$ and $x\in(\partial^- f)^{-1}(z)$, one has

$$d(x, (\partial^- f)^{-1}(0))=0,$$

i.e.,

$$(\partial^- f)^{-1}(z)\subset(\partial^- f)^{-1}(0), \qquad \forall z\in rB_{\mathbb{R}^n}.$$

(ii) If $f(x)=(d(x, C))^2$, there exists $r>0$ such that, for all $z\in rB_{\mathbb{R}^n}$ and $x\in(\partial^- f)^{-1}(z)$, one has

$$d(x, (\partial^- f)^{-1}(0))=(1/2)\|z\|.$$

(P3) $\partial f(x)$ is equal to the subdifferential of f at x in the sense of convex analysis whenever f is convex and l.s.c.;

(P4) $0 \in \partial f(x)$, whenever $x \in \operatorname{dom} f$ is a local minimum point of f;

(P5) if f is l.s.c. on a neighborhood of x, and if g is convex and continuous, then

$$\partial(f+g)(x) \subset \partial f(x) + \partial g(x).$$

Definition 4.1. Let $\psi: R_+ \to \bar{R}_+$. Consider a function $f: X \to R \cup \{+\infty\}$ whose set of critical points, defined by $S = (\partial f)^{-1}(0)$, is nonempty. We say that f is ψ-conditioned on the subset A of X with $A \cap S \neq \varnothing$, if $\inf f \in R$ and

$$f(x) \geq \inf f + \psi(d(x, S)), \qquad \forall x \in A.$$

If $A = X$, we simply say that f is ψ-conditioned.

We give now some examples of ψ-conditioning, putting in evidence the function ψ.

Example 4.1. Consider the quadratic programming problem

$$(\mathscr{P}) \quad \min f(x) = (1/2)\langle Ax, x\rangle - \langle c, x\rangle,$$

where A is a positive-definite symmetric $n \times n$ matrix and $c \in R^n$. It is known that f is convex and that the unique optimal solution of $\mathscr{P}$ is

$$x_0 = A^{-1}c;$$

therefore,

$$S = \{x_0\},$$

and so,

$$d(x, S) = \|x - x_0\|.$$

One has

$$\begin{aligned} f(x) - f(x_0) &= (1/2)\langle Ax, x\rangle - \langle c, x\rangle + (1/2)\langle c, A^{-1}c\rangle \\ &= (1/2)\langle A(x - A^{-1}c), x - A^{-1}c\rangle, \end{aligned}$$

whence

$$f(x) \geq \inf f + K\|x - x_0\|^2, \qquad \forall x \in R^n,$$

where

$$K = \inf\{\langle Ax, x\rangle \mid \|x\| = 1\} > 0.$$

Therefore, f is ψ-conditioned with $\psi(t) = Kt^2$.

Another example is given by Bahraoui and Lemaire in Ref. 19, where the solution set S is not a singleton.

Example 4.2. See Lemaire (Ref. 12) and Tiba (Ref. 20). Let $f: X \to \bar{R}$ be a convex proper function satisfying the Slater condition [i.e., there exists $x_0 \in X$ such that $f(x_0) < 0$], and consider

$$C = \{x \in X \mid f(x) \leq 0\}.$$

Then, for all $r > 0$, there exists $\alpha_r > 0$ such that

$$f(x) \geq \alpha_r d(x, C), \qquad \forall x \in (x_0 + rB_X) \setminus C;$$

i.e., $g = \max(f, 0)$ is ψ-conditioned on $x_0 + rB_X$ with $\psi(t) = \alpha_r t$.

Indeed, for $x \in (x_0 + rB_X) \setminus C$, we have

$$f(x) > 0 > f(x_0).$$

We can assume that $f(x) < \infty$. Let us take

$$\lambda = -f(x_0)/[f(x) - f(x_0)] \in]0, 1[.$$

By the convexity of f, we get

$$f(\lambda x + (1 - \lambda)x_0) \leq \lambda f(x) + (1 - \lambda) f(x_0) = 0,$$

whence

$$\lambda x + (1 - \lambda)x_0 \in C;$$

this implies that

$$\begin{aligned} d(x, C) &\leq \|x - (\lambda x + (1 - \lambda)x_0)\| \\ &= (1 - \lambda)\|x - x_0\| \\ &= [f(x)/(f(x) - f(x_0))]\|x - x_0\| \\ &\leq [r/-f(x_0)] f(x). \end{aligned}$$

Example 4.2 can be stated in a more general situation.

Example 4.3. See Robinson (Ref. 11) and Ursescu (Ref. 21). Let X and Y be Banach spaces, and let $F: X \rightrightarrows Y$ be a multivalued mapping with closed and convex graph. Suppose that 0 is an internal point of the range of F. Then, for every $x_0 \in F^{-1}(0)$, there exist $c, r > 0$ such that

$$d(x, F^{-1}(0)) \leq c(r + 1) d(0, F(x)), \qquad \forall x \in (x_0 + rB_X);$$

i.e., $g(x)=d(0, F(x))$ is ψ-conditioned on x_0+rB_X with $\psi(t)=(1/c(1+r))t$. So, Example 4.2 is a direct consequence of this example. Indeed, it suffices to consider $F(x)=-f(x)+R_-$ as a multivalued mapping.

Note that this result is extended in Jourani (Ref. 22) to γ-paraconvex multivalued mappings with $\gamma>1$; F is γ-paraconvex, $\gamma>0$, if there exists $c>0$ such that, for all $x, y\in X$, $\lambda\in[0, 1]$,

$$\lambda F(x)+(1-\lambda)F(y)\subset F(\lambda x+(1-\lambda)y)+c\|x-y\|^\gamma B_Y.$$

We establish now a relation between the upper-Lipschitz property of the inverse of the subdifferential and the conditioning of a function f. Our proof is in line with that given by Zhang and Treiman (Ref. 8). It uses the Ekeland varational principle (Ref. 23).

Theorem 4.1. Let φ: $R_+\to R_+$ be an increasing function with $\varphi(0)=0$, and suppose that there exists $r>0$ and $t_0>0$ such that

$$0<\varphi(t)\le r, \qquad \forall t\in\,]0, t_0].$$

Let $f: X\to R\cup\{+\infty\}$ be a l.s.c. proper function, such that the set of its critical points $S=(\partial f)^{-1}(0)$ is nonempty. Suppose that $(\partial f)^{-1}$ is upper φ-Lipschitz on rB_{X^*}. Then, f is ψ-conditioned on $S+rB_X$, where $\psi(t)=(1-\gamma)t\varphi(\gamma t)$ for $t\ge 0$, with $\gamma\in\,]0, 1[$ a fixed number satisfying $\gamma r\le t_0$.

Proof. Let $\gamma\in\,]0, 1[$ be such that $\gamma r\le t_0$. Suppose that f is not ψ-conditioned on $S+rB_X$, where ψ is defined above. Then, there exists $x_0\in S+rB_X$ such that

$$f(x_0)<\inf f+\psi(d(x_0, S)).$$

It follows that

$$\psi(d(x_0, S))>0,$$

whence there exists $c\in\,]0, 1[$ such that

$$f(x_0)<\inf f+c\psi(d(x_0, S)). \tag{10}$$

Take

$$\epsilon=c\varphi(\gamma d(x_0, S))>0.$$

By the Ekeland variational principle (Ref. 23), there exists $u\in X$ such that

$$f(u)+\epsilon\|u-x_0\|\le f(x_0), \tag{11}$$

$$f(u)\le f(x)+\epsilon\|x-u\|, \qquad \forall x\in X. \tag{12}$$

From (12), taking into account (P5), we obtain the existence of $u^* \in \partial f(u)$ such that $\|u^*\| \leq \epsilon$, while from (10) and (11) we get

$$\|x_0 - u\| \leq (1-\gamma) d(x_0, S).$$

By the assumption and the choice of ϵ, we have

$$\|u^*\| \leq r \quad \text{and} \quad \varphi(d(u, S)) \leq \|u^*\|.$$

But

$$\begin{aligned} d(u, S) &\geq d(x_0, S) - \|u - x_0\| \\ &\geq d(x_0, S) - (1-\gamma) d(x_0, S) \\ &= \gamma d(x_0, S), \end{aligned}$$

and thus we obtain the contradiction

$$\epsilon \geq \|u^*\| \geq \varphi(d(u, S)) \geq \varphi(\gamma d(x_0, S)) > c\varphi(\gamma d(x_0, S)) = \epsilon.$$

Therefore, f is ψ-conditioned on $S + rB_X$. □

Theorem 4.2. Suppose that $T = \partial f$ satisfies relation (8) of Proposition 3.2. Then, f is ψ-conditioned with ψ a linear function from R_+ into R_+.

Proof. The proof is analogous to that of Theorem 4.1. □

5. Conditioning and Upper φ-Lipschitz Inverse Subdifferentials: Convex Case

In this section, we give necessary and sufficient conditions to get the upper φ-Lipschitzness and linear conditioning (i.e., ψ is a linear function) in the convex case.

Theorem 5.1. Let X be a normed vector space, $f: X \to \bar{R}$ a convex proper l.s.c. function, $S \subset X$ a nonempty closed convex set, and $c \in]0, \infty[$. Consider the following statements:

(i) $S = \arg\min f$ and $f(x) \geq \inf f + cd(x, S)$, $\forall x \in X$;
(ii) $f^*(0) \in R$ and $f^*(x^*) = f^*(0) + I_S^*(x^*)$, $\forall x^* \in X^*$, $\|x^*\| \leq c$;
(iii) $\partial f^*(x^*) \subset \partial f^*(0) = S$, $\forall x^* \in X^*$, $\|x^*\| < c$.

Then, (i)$\Leftrightarrow$(ii)$\Rightarrow$(iii). Moreover, if X is a Banach space, then (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii).

Proof.

(i) $\Rightarrow$ (ii). Clearly,

$$-\inf f = f^*(0) \in \mathbb{R}.$$

Since

$$d(\cdot, S) = I_S \triangle \|\cdot\|,$$

and taking into account that

$$(g \triangle h)^* = g^* + h^*,$$

from (i) we have that

$$\begin{aligned} f^*(x^*) &\leq -\inf f + (cd(\cdot, S))^*(x^*) \\ &= f^*(0) + c(I_S^*(x^*/c) + \|\cdot\|^*(x^*/c)) \\ &= f^*(0) + I_S^*(x^*) + I_{cB^*}(x^*). \end{aligned}$$

Therefore,

$$f^*(x^*) \leq f^*(0) + I_S^*(x^*), \qquad \forall x^*, \ \|x^*\| \leq c.$$

Moreover, if $a \in S$, then

$$\langle a, x^* \rangle \leq f(a) + f^*(x^*) = -f^*(0) + f^*(x^*),$$

whence

$$f^*(0) + I_S^*(x^*) \leq f^*(x^*).$$

Thus,

$$f^*(x^*) = f^*(0) + I_S^*(x^*), \qquad \forall x^* \in X^*, \ \|x^*\| \leq c.$$

(ii) $\Rightarrow$ (i). As shown above, one has that

$$(cd(\cdot, S))^* = I_S^* + I_{cB^*}.$$

Since $cd(\cdot, S)$ is a continuous convex function, we have that

$$(I_S^* + I_{cB^*})^* = cd(\cdot, S).$$

Condition (ii) implies that

$$f^* \leq f^*(0) + I_S^* + I_{cB^*};$$

passing to conjugates, we obtain the inequality in (i). As

$$f^*(x^*) = f^*(0) + I_S^*(x^*),$$

for x^* in a neighborhood of 0, one has that

$$\partial f^*(0) = \partial I_S^*(0) = S;$$

thus,

$$S = \arg\min f.$$

(ii) $\Rightarrow$ (iii). This is obvious because

$$\partial f^*(x^*) = \partial I_S^*(x^*) \subset S = \partial f^*(0), \qquad \|x^*\| < c.$$

(iii) $\Rightarrow$ (i). Suppose that X is a Banach space. Since $S = \partial f^*(0)$, we have that

$$S = \arg\min f.$$

By contradiction, suppose that there exists $\bar{x} \in X$ such that

$$f(\bar{x}) < \inf f + cd(\bar{x}, S).$$

It follows that $\bar{x} \notin S$. Therefore, there exists $0 < c' < c$ such that

$$f(\bar{x}) < \inf f + c'd(\bar{x}, S). \tag{13}$$

Fix $c' < \epsilon < c$. Using the Ekeland variational principle (Ref. 23), one obtains $u \in X$ such that

$$f(u) + \epsilon\|u - \bar{x}\| \leq f(\bar{x}), \tag{14}$$

$$f(u) \leq f(x) + \epsilon\|x - u\|, \qquad \forall x \in X. \tag{15}$$

The relation (15) is equivalent to

$$\partial f(u) \cap \epsilon B_{X^*} \neq \varnothing.$$

Therefore, there exists u^* with

$$u^* \in \partial f(u) \quad \text{and} \quad \|u^*\| \leq \epsilon < c.$$

From (iii), it follows that $\partial f^*(u^*) \subset S$, whence $u \in S$. Therefore, using also (13) and (14), we have

$$\epsilon d(\bar{x}, S) \leq \epsilon\|u - \bar{x}\| < c'd(\bar{x}, S).$$

As $d(\bar{x}, S) > 0$, we get the contradiction $\epsilon < c'$. □

Remark 5.1. Note that (i) $\Rightarrow$ (ii) in Theorem 5.1 is true for f and S not necessarily convex, since the formula

$$(g \triangle h)^* = g^* + h^*$$

is valid for nonconvex functions. If (ii) of Theorem 5.1 holds, taking into account that

$$I_S^* = (I_{\overline{\text{conv}}\, S})^*,$$

from (ii) ⇒ (i) of Theorem 5.1 we get that

$$\overline{\text{conv}}\, S = \arg\min f^{**}$$

and

$$f^{**}(x) \geq \inf f^{**} + cd(x, \overline{\text{conv}}\, S), \qquad \forall x \in X.$$

But

$$\inf f^{**} = \inf f \quad \text{and} \quad f \geq f^{**},$$

so that

$$f(x) \geq \inf f + cd(x, \overline{\text{conv}}\, S), \qquad \forall x \in X.$$

If we already know that $S = \arg\min f$ and S is (closed) convex, we obtain that the implication (ii) ⇒ (i) from Theorem 5.1 is true for f not convex. So, we obtain Proposition 3.3 of Lemaire (Ref. 6), stated for f not convex, but with $S = \arg\min f$ a nonempty closed convex set. Note that, without assuming that S is convex, (ii) ⇒ (i) in Theorem 5.1 can be false as shown by the following example: take $f(x) = ||x| - 1|$; then, f^* verifies (ii) for $S = [-1, 1]$ and $c = 1$, but $\arg\min f = \{-1, 1\}$.

Corollary 5.1. Let X be a Banach space and $f: X \to \bar{\mathbb{R}}$ be a proper l.s.c. convex function, $\bar{x} \in X$, and $c \in]0, \infty[$. The following statements are equivalent:

(i) $f(x) \geq f(\bar{x}) + c\|x - \bar{x}\|$, $\forall x \in X$;
(ii) $f^*(0) \in \mathbb{R}$ and $f^*(x^*) = f^*(0) + \langle \bar{x}, x^* \rangle$, $\forall x^* \in X^*, \|x^*\| \leq c$;
(iii) $\partial f^*(x^*) = \{\bar{x}\}$, $\forall x^* \in X^*$, $\|x^*\| < c$;
(iv) $\partial f^*(x^*) \subset \{\bar{x}\} = \partial f^*(0)$, $\forall x^* \in X^*$, $\|x^*\| < c$;
(v) $cB_{X^*} \subset \partial f(\bar{x})$.

Proof. The equivalences (i) ⇔ (ii) ⇔ (iv) are stated in Theorem 5.1. The implications (ii) ⇒ (iii) ⇒ (iv) are obvious. Also, from (iii) it follows easily that

$$\text{int } cB_{X^*} \subset \partial f(\bar{x}).$$

As $\partial f(\bar{x})$ is closed (even w^*-closed),

$$cB_{X^*} \subset \partial f(\bar{x}).$$

Thus, we have (iii) $\Rightarrow$ (v).

(v) $\Rightarrow$ (i). Let $x \in X$ and $x^* \in X^*$, $\|x^*\| \leq c$. Then, $x^* \in \partial f(\bar{x})$, whence

$$\langle x-\bar{x}, x^*\rangle \leq f(x)-f(\bar{x}).$$

It follows that

$$c\|x-\bar{x}\| = \sup_{\|x^*\| \leq c} \langle x-\bar{x}, x^*\rangle \leq f(x)-f(\bar{x}).$$

The implication is proved. □

When X is not complete and f is not convex, we still have some of the equivalences stated above.

Corollary 5.2. Let X be a normed linear space, $f: X \to \bar{R}$ a proper l.s.c. function, $\bar{x} \in X$, and $c>0$. The following statements are equivalent:

(i) $f(x) \geq f(\bar{x}) + c\|x-\bar{x}\|$, $\forall x \in X$;
(ii) $f^*(0) \in R$ and $f^*(x) = f^*(0) + \langle \bar{x}, x^*\rangle$, $\forall x^* \in X^*$, $\|x^*\| \leq c$;
(iii) $cB_{X^*} \subset \partial f(\bar{x})$, $\partial f(\bar{x})$ being meant in the sense of convex analysis;
(iv) $cB_{X^*} \subset \partial f^{**}(\bar{x})$.

Proof.

(i) $\Rightarrow$ (iii). This is obvious: if $\|x^*\| \leq c$, one has

$$\langle x-\bar{x}, x^*\rangle \leq c\|x-\bar{x}\| \leq f(x)-f(\bar{x}).$$

(iii) $\Rightarrow$ (i). This is the same as (v) $\Rightarrow$ (i) from Corollary 5.1.

(i) $\Rightarrow$ (ii). This follows from Theorem 5.1 and the discussion after its proof.

(ii) $\Rightarrow$ (i). As in Remark 5.1, one obtains that $\arg\min f^{**} = \{\bar{x}\}$ and

$$f(x) \geq \inf f + c\|x-\bar{x}\|, \qquad \forall x \in X.$$

One must show that $f(\bar{x}) = \inf f$. In the contrary case, take μ such that $f(\bar{x}) > \mu > \inf f$. As f is l.s.c. at $\bar{x}$, there exists $r>0$ such that

$$f(x) > \mu, \qquad \|x-\bar{x}\| < r.$$

From the relation stated above, for $\|x-\bar{x}\| \geq r$ we have

$$f(x) \geq \inf f + c\|x-\bar{x}\| \geq \inf f + cr.$$

Therefore

$$f(x) \geq \min\{\mu, \inf f + cr\}, \qquad \forall x \in X,$$

whence $\inf f > \inf f$, a contradiction; the implication holds.

(ii) $\Leftrightarrow$ (iv). This follows from (ii) $\Leftrightarrow$ (iii) applied to f^{**}. □

Remark 5.2. The equivalences (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iv) from Corollary 5.1, when X is a reflexive Banach space, follow from Theorem 2.1 in Ref. 24. The equivalence (i)$\Leftrightarrow$(iv) in a milder form is established in Hilbert spaces by Tossing (Ref. 25, Proposition 4.8). The equivalences (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iv) from Corollary 5.2 are stated by Lemaire in Ref. 6, Proposition 3.1.

Let us consider the function

$$\psi_f\colon [0, [\to [0, \infty], \qquad \psi_f(t) = \inf\{f(x) - \inf f \mid d(x, S) = t\}.$$

Proposition 5.1. Let X be a normed vector space, and let $f\colon X \to \bar{R}$ be a proper convex function. Suppose that $S = \arg\min f$ is nonempty. Then,

$$\psi_f(ct) \geq c\psi_f(t), \qquad \forall t \geq 0, \forall c \geq 1,$$

or equivalently $t \to \psi_f(t)/t$ is nondecreasing on $]0, \infty[$.

Proof. Penot (Ref. 10, Proposition 2.2) obtained the same conclusion, even for f starshaped at every $u \in S$, so we omit our original proof. □

The aim of the following corollary is to show that the local conditioning implies the global conditioning but with a different function ψ (more exactly, with a prolongation of ψ).

Corollary 5.3. Let X and f be as in the preceding proposition. If $\psi\colon [0, \alpha] \to [0, \infty[$, where $\alpha > 0$, is such that

$$f(x) \geq \inf f + \psi(d(x, S)), \qquad \forall x \in X, d(x, S) \leq \alpha,$$

then

$$f(x) \geq \inf f + \psi_\alpha(d(x, S)), \qquad \forall x \in X,$$

where

$$\psi_\alpha(t) = \begin{cases} \psi(t), & t \in [0, \alpha], \\ [\psi(\alpha)/\alpha]t, & t \in [\alpha, \infty[. \end{cases}$$

Proof. We have that $\psi_f \geq \psi$ on $[0, \alpha]$. For $d(x, S) = \gamma > \alpha$, we have

$$\begin{aligned} f(x) &\geq \inf f + \psi_f(\gamma) \geq \inf f + (\gamma/\alpha)\psi_f(\alpha) \\ &\geq \inf f + (\gamma/\alpha)\psi(\alpha) \\ &= \inf f + [\psi(\alpha)/\alpha] d(x, S) \\ &= \inf f + \psi_\alpha(d(x, S)). \end{aligned}$$

Thus, the corollary is proved. □

Now, under Corollary 5.3, in the convex case we can work with global conditioning; the next theorem gives several characterizations for this. Consider first the following class of functions:

$F = \{\psi \mid \psi$ convex, l.s.c., $\psi(t)=0 \Leftrightarrow t=0$ and $\psi(t_0) < \infty$ for some $t_0 > 0\}$.

Theorem 5.2. Let X be a Banach space, let $f: X \to \bar{\mathbb{R}}$ be a l.s.c. proper convex function with $S = \arg\min f \neq \varnothing$. The following statements are equivalent:

(i) there exists a function ψ: $[0, \infty[\to [0, \infty]$ and $t_0 > 0$ such that $\psi(t) > 0$ for $t \in]0, t_0[$ and f is ψ-conditioned on X;

(ii) there exists $\psi \in F$ such that f is ψ-conditioned on X;

(iii) there exists $\psi \in F$ such that $f^*(x^*) \leq f^*(0) + I_S^*(x^*) + \psi^*(\|x^*\|)$, $\forall x^* \in X^*$;

(iv) there exists $\psi \in F$ such that $\langle x - \bar{x}, x^* \rangle \geq \psi(d(x, S))$, $\forall (x, x^*) \in \partial f$, $\forall \bar{x} \in S$;

(v) there exists φ: $[0, \infty[\to [0, \infty[$ nondecreasing such that $\varphi(t) = 0 \Leftrightarrow t = 0$ and ∂f^* is upper φ-Lipschitz at 0 on X^*;

(vi) there exists θ: $[0, \infty[\to [0, \infty]$ nondecreasing such that $\lim_{t \to 0^+} \theta(t) = \theta(0) = 0$ and

$$d(x, S) \leq \theta(\|x^*\|), \qquad \forall (x, x^*) \in \partial f;$$

(vii) f has good asymptotical behavior; i.e., (see Ref. 26), if $\{x_n\} \subset X$ is such that $d(0, \partial f(x_n)) \to 0$, then $d(x_n, S) \to 0$.

Moreover, in the implications (ii)$\Leftrightarrow$(iii)$\Rightarrow$(iv), one can take the same ψ; in (iv)$\Rightarrow$(v), one can take $\varphi(t) = \psi(t)/t$ for $t > 0$ and $\varphi(0) = 0$; in (v)$\Leftrightarrow$(vi), one can take φ and θ to be quasi-inverse to each other; in (v)$\Rightarrow$(ii), one can take

$$\psi(t) = (1 - \gamma) \int_0^t \varphi(\gamma s)\, ds,$$

with $\gamma \in]0, 1[$ arbitrary, but fixed.

Proof.

(i)$\Rightarrow$(ii). It is obvious that $\psi_f \geq \psi$, so that $\psi_f(t) > 0$ for $t \in]0, t_0]$. As $t \mapsto \psi_f(t)/t$ is nondecreasing on $]0, \infty[$ (see Proposition 5.1), it follows that $\psi_f(t) > 0$ for $t > 0$ and $\liminf_{t \to \infty} \psi_f(t)/t > 0$. Using Ref. 24, Proposition A.5, we obtain that $\psi = \overline{\text{conv}}\, \psi_f \in F$. Of course, f is ψ-conditioned.

(ii)$\Rightarrow$(i). This is obvious.

(ii) $\Rightarrow$ (iii). Let $x^* \in X^*$. Then,

$$
\begin{aligned}
f^*(x^*) &= \sup_{x\in X}(\langle x, x^*\rangle - f(x)) \\
&\leq \sup_{x\in X}(\langle x, x^*\rangle - \inf f - \psi(d(x, S))) \\
&= f^*(0) + (\psi \circ d(\cdot, S))^*(x^*) \\
&= f^*(0) + \min_{\lambda\geq 0}\,[\psi^*(\lambda) + (\lambda d(\cdot, S))^*(x^*)] \\
&= f^*(0) + \min_{\lambda> 0}\,[\psi^*(\lambda) + \lambda d(\cdot, S)^*(x^*/\lambda)] \\
&= f^*(0) + \min_{\lambda> 0}\,[\psi^*(\lambda) + \lambda\,(I_S \triangle \|\cdot\|)^*(x^*/\lambda)] \\
&= f^*(0) + \min_{\lambda> 0}\,[\psi^*(\lambda) + \lambda\,(I_S^*(x^*/\lambda) + I_{B^*}(x^*/\lambda))] \\
&= f^*(0) + \min_{\lambda> 0}\,[\psi^*(\lambda) + I_S^*(x^*) + I_{\lambda B^*}(x^*)] \\
&= f^*(0) + I_S^*(x^*) + \min_{\lambda\geq \|x^*\|} \psi^*(\lambda) \\
&= f^*(0) + I_S^*(x^*) + \psi^*(\|x^*\|),
\end{aligned}
$$

because ψ^* is nondecreasing.

(iii) $\Rightarrow$ (ii). In the first part, we got

$$(\psi(d(\cdot, S))^* = I_S^* + \psi^* \circ \|\cdot\|.$$

Since the function $x \mapsto \psi(d(\cdot, S))$ is convex, proper, and l.s.c., we have that

$$(I_S^* + \psi^* \circ \|\cdot\|)^* = \psi(d(\cdot, S)).$$

Dualizing the inequality in (iii), we get (ii).

(i) $\Rightarrow$ (iv). Let $\bar{x}\in S$ and $(x, x^*)\in \partial f$. Then,

$$
\begin{aligned}
&f(x) \geq f(\bar{x}) + \psi(d(x, S)), \\
&f(\bar{x}) \geq f(x) + \langle \bar{x} - x, x^*\rangle.
\end{aligned}
$$

Adding the above two inequalities, we get the conclusion with the same ψ.

(iv) $\Rightarrow$ (v). From (iv), we obtain

$$\|x - \bar{x}\|\,\|x^*\| \geq \psi(d(x, S)), \qquad \forall \bar{x}\in S,\ \forall (x, x^*)\in \partial f,$$

whence

$$\|x^*\| d(x, S) \geq \psi(d(x, S)), \qquad \forall (x, x^*)\in \partial f.$$

Taking

$$\varphi(t) = \psi(t)/t, \qquad t > 0,$$

and $\varphi(0) = 0$ we have the desired conclusion.

(v) $\Rightarrow$ (vi). Take θ: $[0, \infty[\to [0, \infty]$ defined by

$$\theta(t) = \sup\{s \geq 0 \mid \varphi(s) \leq t\}.$$

The function θ is nondecreasing and

$$\lim_{t \to 0^+} \theta(t) = 0 = \theta(0).$$

Moreover, from its definition,

$$\theta(\|x^*\|) \geq d(x, S), \qquad \forall (x, x^*) \in \partial f.$$

(vi) $\Rightarrow$ (v). Suppose that

$$\theta(\|x^*\|) \geq d(x, S), \qquad \forall (x, x^*) \in \partial f.$$

Consider the function φ: $[0, \infty[\to [0, \infty]$ defined by

$$\varphi(t) = \inf\{s \geq 0 \mid \theta(s) \geq t\}.$$

It is obvious that $\varphi(0) = 0$ and φ is nondecreasing. If $t > 0$, then $\varphi(t) > 0$; otherwise, there exists $s_n \to 0^+$ such that $\theta(s_n) \geq t$, contradicting $\lim_{s \to 0^+} \theta(s) = 0$. From its definition, one obtains that

$$\varphi(\|x^*\|) \leq d(x, S), \qquad \forall (x, x^*) \in \partial f.$$

(v) $\Rightarrow$ (ii). Let us fix $\gamma \in]0, 1[$, and consider $\psi: R_+ \to \bar{R}_+$ defined by

$$\psi(t) = (1 - \gamma) t \varphi(\gamma t).$$

Suppose that (ii) does not hold for this ψ. Then, there exists $\bar{x} \in X$ such that

$$f(\bar{x}) < \inf f + \psi(d(\bar{x}, S)).$$

It follows that $\bar{x} \notin S$ and there exists $0 < c < 1$ such that

$$f(\bar{x}) < \inf f + c\psi(d(\bar{x}, S)).$$

Take

$$\epsilon = c\varphi(\gamma d(\bar{x}, S)) > 0.$$

Using the Ekeland variational principle (Ref. 23), there exists $u \in X$ such that

$$f(u) + \epsilon \|u - \bar{x}\| \leq f(\bar{x}),$$

$$f(u) \leq f(x) + \epsilon \|x - u\|, \qquad \forall x \in X.$$

The last relation being equivalent to $\partial f \cap cB^* \neq \emptyset$, there exists u^* such that

$$(u, u^*) \in \partial f, \qquad \|u^*\| \leq \epsilon.$$

It follows that

$$\|\bar{x} - u\| \leq (1-\gamma) d(\bar{x}, S),$$
$$\varphi(d(u, S)) \leq \|u^*\| \leq \epsilon = c\varphi(\gamma d(\bar{x}, S)).$$

But

$$d(u, S) \geq d(\bar{x}, S) - \|u - \bar{x}\| \geq d(\bar{x}, S) - (1-\gamma) d(\bar{x}, S) = \gamma d(\bar{x}, S),$$

whence

$$\varphi(d(u, S)) \geq \varphi(\gamma(d(\bar{x}, S)) > c\varphi(\gamma d(\bar{x}, S)),$$

a contradiction. Therefore, (ii) holds for this ψ. As

$$t\varphi(\gamma t) \geq \int_0^t \varphi(\gamma s)\, ds,$$

we can replace ψ by $\bar{\psi} \in F$, where

$$\bar{\psi}(t) = (1-\gamma) \int_0^t \varphi(\gamma s)\, ds.$$

(vi) $\Rightarrow$ (viii). This is obvious.

(vii) $\Rightarrow$ (vi). Let us consider φ: $[0, \infty[0, \infty]$ defined by

$$\varphi(t) = \sup\{d(x, S) \mid (x, x^*) \in \partial f, \|x^*\| \leq t\}.$$

It is obvious that φ is nondecreasing and

$$d(x, S) \leq \varphi(\|x^*\|), \qquad \forall (x, x^*) \in \partial f.$$

If $\lim_{t \to 0^+} \varphi(t) > 0$, there exists $(t_n) \downarrow 0$ and $\epsilon_0 > 0$ such that $\varphi(t_n) > \epsilon_0$ for every n. By the definition of φ, we have that, for every n, there exists $(x_n, x_n^*) \in \partial f$ such that

$$\|x_n^*\| \leq t_n \quad \text{and} \quad d(x_n, S) > \epsilon_0. \qquad \square$$

Remark 5.3. We have used that ψ is convex only for the equivalence (i)$\Leftrightarrow$(ii). For the other equivalences, it suffices to consider that $t \to \psi(t)/t$ is nondecreasing. As seen in Proposition 5.1, the function ψ_f satisfies this condition.

Note that Theorem 5.2 covers Proposition 3.1 of Lemaire (Ref. 26). The implication (i)$\Rightarrow$(iv) of Theorem 5.2 is just Ref. 26, Proposition 4.1; a similar result to (iv) $\Rightarrow$ (i), under different conditions, is proved in Ref. 26, Proposition 4.2. Taking in (i) of Theorem 5.2,

$$\psi(t)=ct^2, \qquad \text{for } t\in[0, \alpha], (c, \alpha>0),$$

and taking its prolongation to $[0, \infty[$ as in Corollary 5.3, and $\varphi(t)=ct$ in (vi), one obtains Theorem 4.3 of Zhang and Treiman (Ref. 8). Applying Theorem 5.2 for $\psi(t)=ct$, one obtains a weaker form of Theorem 5.1.

References

1. Martinet, B., *Régularisation d'Inéquations Variationelles par Approximations Sucessives*, Revue Française d'Informatique et de Recherche Opérationnelle, Vol. 2, pp. 154–159, 1970.
2. Martinet, B., *Algorithmes pour la Résolution de Problèmes d'Optimisation et de Minimax*, PhD Thesis, Université de Grenoble, 1972.
3. Lemaire, B., *The Proximal Algorithm*, New Methods in Optimization and Their Industrial Uses, International Series of Numerical Mathematics, Birkhauser Verlag, Basel, Switzerland, Vol. 87, pp. 73–89, 1987.
4. Rockafellar, R., *Monotone Operators and the Proximal Point Algorithm*, SIAM Journal on Control and Optimization, Vol. 14, pp. 877–898, 1976.
5. Cornejo, O., and Jourani, A., *Coupling Proximal Algorithm*, Working Paper, University of Bourgogne, Dijon, France, 1995.
6. Lemaire, B., *Quelques Résultats Récents sur l'Algorithme Proximal*, Technical Report, Séminaire d'Analyse Numérique, Toulouse, France, 1989.
7. Lemaire, B., *About the Convergence of the Proximal Method*, Advances in Optimization, Lecture Notes in Economics and Mathematical Systems, Springer Verlag, Berlin, Germany, Vol. 382, pp. 122–148, 1991.
8. Zhang, R., and Treiman, J., *Upper-Lipschitz Multifunctions and Inverse Subdifferentials*, Nonlinear Analysis: Theory, Methods, and Applications, Vol. 24, pp. 273–286, 1995.
9. Mordukhovich, M., *Sensitivity Analysis in Nonsmooth Optimization*, Theoretical Aspects of Industrial Design, Edited by D. Field and V. Komkov, SIAM, Philadelphia, Pennsylvania, pp. 32–46, 1992.
10. Penot, J., *Conditioning Convex and Nonconvex Problems*, Journal of Optimization Theory and Applications, Vol. 86, pp. 68–123, 1995.
11. Robinson, S., *Regularity and Stability for Convex Multivalued Functions*, Mathematical Operations Research, Vol. 1, pp. 130–143, 1976.
12. Lemaire, B., *Bounded Diagonally Stationary Sequences in Convex Optimization*, Journal of Convex Analysis, Vol. 1, pp. 75–86, 1994.
13. Borwein, J., and Zhuang, D., *Verifiable Necessary and Sufficient Conditions for Regularity of Set-Valued Maps and Single-Valued Maps*, Journal of Mathematical Analysis and Applications, Vol. 134, pp. 441–459, 1988.

14. PENOT, J., *Metric Regularity, Openness, and Lipschitz Behavior of Multifunctions*, Nonlinear Analysis: Theory, Methods, and Applications, Vol. 13, pp. 629–643, 1989.
15. ROCKAFELLAR, R., *Lipschitz Properties of Multifunctions*, Nonlinear Analysis: Theory, Methods, and Applications, Vol. 9, pp. 867–885, 1985.
16. CORREA, R., JOFRÉ, A., and THIBAULT, L., *Characterization of Lower Semicontinuous Convex Functions*, Proceedings of the American Mathematical Society, Vol. 116, pp. 67–72, 1992.
17. THIBAULT, L., *A Note on the Zagrodny Mean-Value Theorem*, Preprint, 1994.
18. THIBAULT, L., and ZAGRODNY, D., *Integration of Subdifferentials of Lower Semicontinuous Functions on Banach Spaces*, Journal of Mathematical Analysis and Applications, Vol. 78, pp. 267–308, 1995.
19. BAHRAOUI, M., and LEMAIRE, B., *Convergence of Diagonally Stationary Sequences in Convex Optimization*, Set-Valued Analysis, Vol. 2, pp. 49–61, 1994.
20. TIBA, D., *On the Convex Programming Problem*, Preprint 2, Institutul de Matematică al Academiei Române, 1995.
21. URSESCU, C., *Multifunction with Convex Closed Graph*, Czechoslovak Mathematical Journal, Vol. 22, pp. 438–441, 1975.
22. JOURANI, A., *Open Mapping Theorem and Inversion Theorem for γ-Paraconvex Multivalued Mappings and Applications*, Studia Mathematica, Vol. 117, pp. 123–136, 1995.
23. EKELAND, I., *On the Variational Principle*, Journal of Mathematical Analysis and Applications, Vol. 134, pp. 441–459, 1974.
24. ZALINESCU, C., *On Uniformly Convex Functions*, Journal of Mathematical Analysis and Applications, Vol. 95, pp. 344–374, 1983.
25. TOSSINGS, P., *The Perturbed Proximal Point Algorithm and Some of Its Applications*, Applied Mathematics and Optimization, Vol. 29, pp. 125–159, 1994.
26. LEMAIRE, B., *Bonne Position, Conditionnement et Bon Comportement Asymptotique*, Technical Report, Séminaire d'Analyse Convexe, Montpellier, France, 1992.

Then for n large enough, we have

$$d\left(x_n, T^{-1}(0)\right) \leq e_\rho(T^n, T).$$

Proof. The proof is analogous to that of Proposition 3.1. ∎

It is known that many problems in mathematical programming can be formulated as inclusion problems for monotone operators. The proximal algorithm is the most general method known for solving such inclusions.

Example 3.3 Let T be a maximal monotone operator from a Hilbert space into itself and let $\{x_n\}$ be generated by the proximal point algorithm, i.e.

$$x_{n+1} = (I + \lambda_n T)^{-1}(x_n), \tag{9}$$

where $\lambda_n > 0$ for every n and $\lim_{n\to\infty} \lambda_n = \infty$. The relation (9) is equivalent to

$$0 \in T(x_{n+1}) + \frac{x_{n+1} - x_n}{\lambda_n}.$$

Let $T^n = T + \frac{\cdot - x_n}{\lambda_n}$ and suppose that $T^{-1}(0) \neq \emptyset$. Then the sequence $\{x_n\}$ is bounded and, obviously,

$$\lim_{n\to\infty} d(0, T^n(x_{n+1})) = 0.$$

Moreover, for every $\rho > 0$, we have

$$e_\rho(T^n, T) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus we get that $\lim_{n\to\infty} d(x_n, T^{-1}(0)) = 0$ provided that T^{-1} is upper-Lipschitz at 0.

4 Conditioning and upper φ-Lipschitzian inverse subdifferentials: the nonconvex case

In this section we assume that X is a Banach space. Following Refs. 16–18, we will call *subdifferential on* X an operator ∂ which satisfies the following properties for functions f, g from X into $\mathbb{R} \cup \{+\infty\}$:

$P_1)$ $\partial f(x) \subset X^*$ and $\partial f(x) = \emptyset$ if $x \notin \operatorname{dom} f$,

$P_2)$ $\partial f(x) = \partial g(x)$ whenever f and g coincide on a neighbourhood of x,