Characterizations of the free disposal condition for nonconvex economies on infinite-dimensional commodity spaces

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Abstract: Our aim in this paper is to prove geometric characterizations of the free disposal condition for nonconvex economies on infinite-dimensional commodity spaces even if the cone and the production set involved in the condition have empty interior such as in L^1 with the positive cone L^1_+. We then use this characterization to prove existence of Pareto and weak Pareto optimal points. We also explore a notion of extremal systems à la Kruger-Mordukhovich. We show that the free disposal hypothesis alone assures extremality of the production set with respect to some set.
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1 Introduction

The importance of the free-disposal condition in producer theory and the corresponding version of the non-satiation assumption in consumer theory is well-known and explained in classical references such as Arrow[2], Debreu[14, 15, 16], Lange [34], Mas-Colell [36, 38] and more recents references Benoist [4], Borwein

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and Jofre [8], Flåm and Jourani [17], Kahn and Vohra [28, 29, 30], Kreps [32], Mordukhovich (Chapter 8)[41, 40], Quinzii [42], Villar[46]. These assumptions are extensively used for obtaining equilibrium or Pareto efficiency in convex economies and also after the seminal works by Guesnerie [19] and Kahn and Vohra [28] for nonconvex economies either finite or infinite dimensional commodity spaces. This assumption is part of the argument for proving the non negativity of equilibrium prices and without it the existence and properties of equilibria are no longer valid. To overcome this difficulty in economies with nonconvex production sets, several notions of equilibria have been proposed. However, the free disposal and nonsatiation conditions are most of the time part of the assumptions. Such existence issues are largely explored in the literature in the past decades either in finite or infinite dimensional spaces.

Economic models with intertemporal decisions and infinite horizon with or without uncertainty such as optimal portfolios in finance and optimal paths in macroeconomics growth models, or models of commodity differentiation in which the decisions space is the Borel measure on a compact metric space. All these cases are good examples of infinite dimensional economic model. In these cases, the free disposability assumption is assumed typically taking the natural positive cone L_+ of the commodity space L or a transformation of this cone, which is often also asked to have nonempty interior. Unfortunately, this is not satisfied for example for commodity space L^p, with $1 \leq p < \infty$ or more general reflexive Banach lattice. Concrete examples of positive cones with empty interior are L^2_+ and L^1_+, very often used in financial markets and equilibrium theory.

In infinite dimensional spaces a common condition for overcome empty interior of the positive cone is the properness property introduced by Mas-Colell (see Mas-Colell [36, 37, 38]) and its extension to nonconvex sets (Florenzano, Gourdel and Jofré [18]). However, the main Theorem 1 below shows that no properness condition is required for the characterization of the free disposal condition.

Our aim in this paper is to prove primal and dual geometric characterizations of the free disposal hypothesis in nonconvex economies with infinite-dimensional commodity spaces even if the cone and the production set involved in this condition have empty interior or the corresponding property for consumers. The proof of these results are based on Danes's Drop Theorem. We then use the characterization to obtain existence of Pareto and weak Pareto optimal (see [1] for distinctions between these two notions), extending in this way characterizations of efficient production vectors in finite dimensional spaces proved by Bonnisseau and Crettez [7]. We also explore the concept of extremal systems à la Kruger-Mordukovich

[41], and show that the free disposal hypothesis alone assures extremality of the production set with respect to a specific set.

An important feature of our characterization is that first it extends the result by Jofré and Rivera [25] from finite dimensional spaces to general Banach spaces. Second, our characterization does not require any interiority (or epi-Lipschitzian) condition on the production or respectively consumption set nor on the positive cone as was used in [25].

2 Formulation of the free disposal condition

We are given

- two (real) Banach spaces U and V which represent *inputs* and *outputs* vectors respectively.
- two closed sets $U_+ \subset U$, $V_+ \subset V$
- two closed sets $U_- = -U_+$ and $V_- = -V_+$
- and the production set which may be defined as

$$\mathcal{P} := \{(u, v) \in U \times V | u \text{ can produce } v\}$$

The production set $\mathcal{P}$ is sometimes described in terms of its sections

$$\mathcal{P}(u) := \{v | (u, v) \in \mathcal{P}\}$$

and

$$\mathcal{P}^{-1}(v) := \{u | (u, v) \in \mathcal{P}\}$$

which form the output feasibility and input requirement sets, respectively.
We say that both inputs and outputs are **free disposable** [2, 14] if

$$\begin{array}{l}\text{For } u' - u \in U_+ \text{ and } v' - v \in V_+ \text{ if } (u, v) \in \mathcal{P} \text{ then} \\ (u', v) \in \mathcal{P} \text{ and } (u, v') \in \mathcal{P}\end{array} \tag{1}$$

Free disposability of inputs or positive monotonicity, guarantees that an increase in inputs cannot result in a decrease in outputs
These free disposability conditions can be characterized as follow:

$$\text{For } \alpha \in U_+ \text{ and } \beta \in V_+ \text{ if } (u, v) \in \mathcal{P} \text{ then } (u + \alpha, v) \in \mathcal{P} \text{ and } (u, v + \beta) \in \mathcal{P}$$

$$\Updownarrow$$

$\forall u \in U$ and $v \in V$;

$$\mathcal{P}(u) - V_- \subset \mathcal{P}(u) \text{ and } \mathcal{P}^{-1}(v) - U_- \subset \mathcal{P}^{-1}(v) \tag{2}$$

$$\Updownarrow \text{ if } \mathcal{P} \text{ is convex and } U_+ \text{ and } V_+ \text{ are convex cones}$$

$$\mathcal{P} - (U_- \times V_-) \subset \mathcal{P} \tag{3}$$

In Economic Theory very often production sets are determined by a set of finite inequalities, linear or nonlinear:

$$\mathcal{P} := \{(u, v) \in U \times V : \, g_i(u, v) \leq 0 \forall i \in \{1, \cdots, m\}\}$$

where $g_1, \cdots, g_m : U \times V \mapsto \mathbb{R}$ are $\mathcal{C}^1$-mappings.
The following questions arise:

How to verify the algebraic inclusions (2)?
How to express the algebraic inclusion (2) in terms of the data $\mathbf{g_1}, \cdots, \mathbf{g_m}$?

These two questions constitute the challenge of our paper. To fix ideas and to avoid overloading the notation, we reduce our study to the case where $\mathcal{P}(u)$ and $\mathcal{P}^{-1}(v)$ are fixed sets and we did our study in either of the two spaces. This means that our conditions can be reduced to the following one:

$$Y - Z \subset Y$$

where Y and Z are closed sets of some Banach space X.

3 Background

Throughout we shall assume that X is a Banach space, X^* its topological dual and $\langle \cdot, \cdot \rangle$ is the pairing between X and X^*. We denote by $\mathbb{B}$ and $B(x, r)$ (resp. $\mathbb{B}^*$ and $B^*(x^*, r)$) the closed unit ball and the closed ball centered at x (resp. x^*) with radius r of X (resp. X^*). By $d(\cdot, S)$ we denote the usual distance function to the set S

$$d(x, S) = \inf_{u \in S} \|x - u\|$$

where $\|\cdot\|$ is a norm on X. We write $x \xrightarrow{f} x_0$ and $x \xrightarrow{S} x_0$ to express $x \to x_0$ with $f(x) \to f(x_0)$ and $x \to x_0$ with $x \in S$, respectively. The closed convex hull of a set A is denoted by $\bar{co}A$.

The Clarke's tangent cone to S at $x_0 \in S$ is defined by

$$T(S, x_0) = \{h \in X : \forall x_n \xrightarrow{S} x_0 \, \forall t_n \to 0^+, \exists h_n \to h; \, x_n + t_n h_n \in S \, \forall n\}$$

and the Clarke's normal cone to S at x_0 is given by

$$N(S, x_0) = [T(S, x_0)]^0$$

where H^0 denotes the negative polar of the cone H, that is

$$H^0 = \{x^* \in X^* : \langle x^*, h \rangle \leq 0 \, \forall h \in H\}.$$

The contingent cone to S at x_0 is defined by

$$K(S, x_0) = \{h \in X : \exists t_n \to 0^+, \exists h_n \to h; \, x_0 + t_n h_n \in S \, \forall n\}$$

while the tangent cone to S at x_0 is given by

$$T_0(S, x_0) = \{h \in X : \forall t_n \to 0^+, \exists h_n \to h; \, x_0 + t_n h_n \in S \, \forall n\}.$$

We always have the following inclusions:

$$T(S, x_0) \subset T_0(S, x_0) \subset K(S, x_0).$$

Whenever S is convex then

$$T(S, x_0) = T_0(S, x_0) = K(S, x_0).$$

4 Primal and dual characterizations of the free disposal assumption for general sets

The following result gives characterizations of the free disposal hypothesis in terms of the Clarke's normal cone without the epi-Lipschitzness assumption on the production set nor the interiority or epi-Lipschitzness property on the set occurring in this hypothesis.
We say that a closed production set $Y \subset X$ satisfies the free disposal hypothesis with respect to a closed set $Z \subset X$, with $0 \in Z$, if

$$Y - Z \subset Y. \tag{4}$$

It satisfies the Clarke's normal geometrical condition with respect to Z if

$$N(Y, y) \subset -[K(Z, 0)]^0, \quad \forall y \in Y. \tag{5}$$

We also consider the following free disposal condition involving the closed convex hull of the contingent cone

$$Y - Z \cap \bar{co} K(Z, 0) \subset Y. \tag{6}$$

Theorem 1 *We have the following implications*

$$(4) \Longrightarrow (5) \Longrightarrow (6).$$

The formal proof of this main Theorem is long and involved; we defer it to the last section. At this point, however, it is appropriate to give the proof of the implication (4) $\Longrightarrow$ (5) in the simplest case where Z is convex or more generally when in relation (5), we consider $T_0(Z,0)$ instead of $K(Z,0)$. Indeed it suffices to establish the following inclusion

$$-T_0(Z,0) \subset T(Y,y) \quad \forall y \in Y.$$

Let $h \in T_0(Z,0)$. Then

$$\forall t_n \to 0^+,\, \exists h_n \to h;\, t_n h_n \in Z\, \forall n.$$

Now let $y \in Y$ and (y_n) be an arbitrary sequence of Y converging to y. Then relation (4) implies that

$$y_n - t_n h_n \in Y, \quad \forall n$$

and hence $-h \in T(Y,y)$.

We have the following characterization in special cases.

Corollary 1 *Suppose that $Z \subset \bar{co}K(Z,0)$. Then*

$$(4) \Longleftrightarrow (5).$$

The inclusion $Z \subset \bar{co}K(Z,0)$ is satisfied for convex sets or more generally for the large class of starshaped sets. A set Z is said to be starshaped at $0 \in Z$ if

$$t \in [0,1],\, z \in Z \Longrightarrow tz \in Z.$$

Thus we have:

Corollary 2 *Suppose that Z is starshaped at 0. Then*

$$(4) \Longleftrightarrow (5) \Longleftrightarrow K(Z,0) \subset -T(Y,y) \quad \forall y \in Y.$$

Remark 1 *Corollary 2 extends the result by Jofré-Rivera [25] from finite dimensional spaces to Banach spaces. Furthermore, contrary to what established in [25], our characterization needs not any interiority (or epi-Lipschitzian) condition on the production set Y nor convexity on Z.*

An important situation in economy is when Z is a closed convex cone. In this case we obtain the following characterization of the free-disposal assumption.

Corollary 3 *Suppose that Z is a closed convex cone. Then*

$$(4) \iff N(Y,y) \subset -Z^0 \quad \forall y \in Y \iff Z \subset -T(Y,y) \quad \forall y \in Y.$$

The important issue in this corollary is the following characterization which tells us that *we may always assume that Z is a closed convex cone.* This is due to the fact that the equivalence $2 \iff 3$ in the corollary below holds without the starshapeness condition of Z.

Corollary 4 *Suppose that Z is a starshaped set containing* 0*. Then the following are equivalent:*

1. *The free disposal condition (4) holds.*
2. *The Clarke's normal geometrical condition (5) holds.*
3. *The free disposal condition holds with respect to $\bar{co}K(Z,0)$, that is*

$$Y - \bar{co}K(Z,0) \subset Y.$$

Proof. It suffices to establish the equivalence $2 \iff 3$. To do this, put $\tilde{Z} = \bar{co}K(Z,0)$. Then $\tilde{Z} = K(\tilde{Z},0) = \bar{co}K(Z,0)$ and hence $\tilde{Z}$ is starshaped at 0 (since it is a convex cone). Now, it suffices to apply Corollary 3 with $\tilde{Z}$ instead of Z and the proof is completed. □

4.1 Verification of the free disposal assumption

In Economic Theory very often production sets are determined by a set of finite inequalities, linear or nonlinear. In what follows, we give examples characterizing the free disposal condition for this family of production sets. We start with a set defined by a finite linear inequalities and then we explore the case of nonlinear inequalities. All these examples are developed in infinite dimensional Banach spaces.

$$\ell^p := \{(x_n)_n : x_n \in \mathbb{R}, \forall n \in \mathbb{N} \text{ and } \sum_n |x_n|^p < \infty\},$$

where $1 \leq p < \infty$. This characterization is based on computing the Clarke's normal cone to the production set Y in terms of the data. In both examples the negative cone $Z := \ell^p_+$ has empty interior.

Remark 2 *Note that the following examples can also be stated in functional spaces, i.e., $L^p(\Omega)$.*

Example 1 *Let the space of goods be equal to ℓ^1 and the production set be determined by finite set of linear inequalities*

$$Y := \{x \in \ell^1 : \langle a_i, x\rangle \leq b_i,\ \forall i = 1, \cdots, m\}$$

where $a_1, \cdots, a_m \in \ell^\infty$ and $b_1, \cdots, b_m \in \mathbb{R}$. For each $y \in \ell^1$, put $I(y) := \{i \in \{1, \cdots, m\} : \langle a_i, y\rangle = b_i\}$. Then

$$co\{a_i : i \in I(y)\} \subset \ell^\infty_+ \,\forall y \in Y \Longleftrightarrow Y - \ell^1_+ \subset Y$$

Example 2 *Let $1 \leq p < \infty$ be an integer. Let the space of goods be equal to ℓ^p and let $g_1, \cdots, g_m : \ell^p \mapsto \mathbb{R}$ be $\mathcal{C}^1$-mappings defining the production set*

$$Y := \{x \in \ell^p : g_i(x) \leq 0 \forall i \in \{1, \cdots, m\}\}.$$

For each $y \in \ell^p$, put $I(y) := \{i \in \{1, \cdots, m\} : g_i(y) = 0\}$. Suppose that for all $y \in Y$, the vectors $\{\nabla g_i(y) : i \in I(y)\}$ are positively linearly independent, that is,

$$\sum_{i \in I(y)} \lambda_i \nabla g_i(y) = 0, \quad \lambda_i \geq 0 \,\forall i \in I(y) \Longrightarrow \lambda_i = 0 \,\forall i \in I(y).$$

Then

$$co\{\nabla g_i(y) : i \in I(y)\} \subset \ell^q_+ \,\forall y \in Y \Longleftrightarrow Y - \ell^p_+ \subset Y$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

An other example similar to the last one and containing Example 1 is the following.

Example 3 *Let $1 \leq p < \infty$ be an integer and $q \in \mathbb{R} \cup \{+\infty\}$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let Y be the production set like in Example 2. Suppose that*

$$\nabla g_i(y) \in \ell^q_+ \backslash \{0\} \quad \textit{if} \quad g_i(y) = 0.$$

This condition implies that for all $y \in Y$, the vectors $\{\nabla g_i(y) : i \in I(y)\}$ are positively linearly independent. Thus, Example 2 ensures that the production set Y satisfies the free disposal hypothesis with respect to ℓ^p_+, that is,

$$Y - \ell^p_+ \subset Y$$

5 Applications to Economic Theory

We will consider a production set Y in X-commodity economy, that is Y is a subset of a Banach space X. The purpose of this section is to show that the free disposal hypothesis guarantees the existence of Pareto optimality, with and without interiority condition of the production set. There are two types of Pareto optimal, the so called the Pareto optimal and the weak Pareto optimal. A feasible allocation [1] is Pareto optimal (or Pareto efficient) if there is no other feasible allocation that makes at least one of the agents in an economy strictly better off without making someone else worse off. A feasible allocation is weakly Pareto optimal if there is no other feasible allocation that makes all the agents in an economy strictly better off. Clearly, if an allocation is Pareto optimal, then it is weakly Pareto optimal as well, for if there is no allocation that can make at least one person better off without making someone else worse off, then there should be no allocation that can make everybody better off. The reverse does not hold: a weak Pareto allocation won't necessarily qualify as a Pareto one. In mathematical point of view, this can be expressed in a compact way as follows: Suppose that the preference is determined by some set Z containing 0.

Definition 1 *An allocation $\bar{y} \in Y$ is a Pareto optimal with respect to the set Z if*

$$Y \cap (\bar{y} + Z) = \{\bar{y}\}.$$

We denote the set of Pareto optimal point of Y with respect to Z by Pareto(Y, Z).

Definition 2 *An allocation $\bar{y} \in Y$ is a weak Pareto optimal with respect to the set Z if*

$$Y \cap (\bar{y} + intZ) = \emptyset.$$

We denote the set of weak Pareto optimal point of Y with respect to Z by W-Pareto(Y, Z).

When the set Z has non interior, the concept of weak Pareto optimal has non sense. In this case we may consider either Pareto optimality or an alternative concept to weak Pareto optimality, called extremality. The concept of extremal points for general set systems appeared in Kruger and Mordukhovich (1980), where some approximate and limiting versions for necessary conditions of extremality were obtained in terms of ε-normals and their sequential limits in Banach spaces admitting Fréchet smooth renorms. Other necessary conditions for extremality were obtained by Mordukhovich [40], Flåm and Jourani [17], Bellaassali and Jourani [3] and others.

Definition 3 *(Extreme systems, Kruger and Mordukhovich [33]) An allocation $\bar{y} \in Y$ is extremal with respect to Z if there exists a sequence $z_k \to 0$ such that*

$$(Y - \bar{y}) \cap (Z + z_k) = \varnothing, \forall k.$$

We denote the set of extremal point of Y with respect to Z by Ext(Y, Z).

[1] An allocation is a specification of how much of each good each agent will receive.

The definition of extremality implies at once the following equality

$$(Y - \bar{y}) \cap Z = \mathrm{bd}(Y - \bar{y}) \cap \mathrm{bd}Z.$$

Remark 3 *It is not difficult to show that, when Z has an interior then the concepts of weak Pareto optimality and extremality coincide. But generally the three concepts can be very different.*

Theorem 2 *(***A characterization of the boundary of the production set***)* *Let the free disposal assumption (4) be satisfied. Then*

$$bdY = Ext(Y, Z).$$

If, in addition $intZ \neq \emptyset$, then

$$bdY = W\text{-}Pareto(Y, Z).$$

Proof. Let $\bar{y} \in \mathrm{bd}Y$. We claim that there exists a sequence $z_k \to 0$ such that

$$(Y - \bar{y}) \cap (Z + z_k) = \varnothing, \; \forall k.$$

Otherwise, for all sequence $z_k \to 0$ there exists a subsequence $(z_{\varphi(k)})$ of (z_k) such that

$$(Y - \bar{y}) \cap (Z + z_{\varphi(k)}) \neq \varnothing, \; \forall k.$$

Let $v_k \in Z$ be such that $v_k + \bar{y} + z_{\varphi(k)} \in Y$. Using relation (4), we get

$$\bar{y} + z_{\varphi(k)} \in Y, \; \forall k.$$

This implies the existence of $r > 0$ such that

$$B(\bar{y}, r) \subset Y$$

and contradicts the fact that $\bar{y}$ is a boundary point of Y. $\square$

Does the free disposal assumption guarantee alone the Pareto optimality of boundary points ? Unfortunately, a sample example shows that this condition is not sufficient to get Pareto optimality. To see this, take $Y = \mathbb{R}_- \times \mathbb{R}$ and $Z = \mathbb{R}_+ \times \mathbb{R}_+$. Then 0 is a boundary point of Y, but $0 \notin \mathrm{Pareto}(Y, Z)$.
In the following, we will give several conditions ensuring Pareto optimality.

Theorem 3 *(***Existence of Pareto optimal under a tangential condition***)* *Suppose, in addition to the free disposal assumption (4), that Z is starshaped at 0 and either the following tangential relation holds at $\bar{y} \in Y$:*

$$T_0(Y, \bar{y}) \cap T_0(Z, 0) = \{0\} \tag{7}$$

or $T_0(Y, \bar{y})$ does not contain any line. Then $\bar{y}$ is a Pareto optimal to Y with respect to Z.

Proof. Let $z \in (Y - \bar{y}) \cap Z$. Suppose that $z \neq 0$. Then, relation (4) ensures that for all $t \in [0,1]$

$$\bar{y} + z - tz \in Y$$

and this is equivalent to

$$\bar{y} + tz \in Y, \forall t \in [0,1].$$

Consequently,

$$tz \in (Y - \bar{y}) \cap Z, \forall t \in [0,1]$$

and hence

$$z \in T_0((Y - \bar{y}) \cap Z, 0) \subset T_0(Y, \bar{y}) \cap T_0(Z, 0).$$

If relation (7) holds then $z = 0$ and this is a contradiction with $z \neq 0$. On the other hand, Corollary 2 ensures that

$$-z \in T_0(Y, \bar{y})$$

and as $T_0(Y, \bar{y})$ does not contain any line, we obtain that $z = 0$. Again we obtain a contradiction with $z \neq 0$. So the proof is completed. $\square$

Now, we are going to establish Pareto optimality under a normality condition. This condition will be constructed from Theorem 1 and its corollaries. Indeed, under the starshapeness assumption of Z at 0, the free disposal hypothesis (4) is equivalent to the Clarke's normal geometrical condition (5). We know that this condition alone is not sufficient for guaranteeing Pareto optimality. To get this later one we introduce the following interiority normal condition at $\bar{y} \in Y$:

$$N(Y, \bar{y}) \backslash \{0\} \subset -\mathrm{int}[K(Z, 0)]^0. \tag{8}$$

But in infinite dimensional case, the existence of nonzero vector in the Clarke normal cone can be problematic, even if $\bar{y}$ is a boundary point of Y, and then relation (8) does not make sense. So, we need additional assumption on the production set Y. It is shown in [[27], Theorem 8.1], that when Y is compactly epi-Lipschitzian at $\bar{y}$ in the sense of Borwein-Strojwas [9], that is, there exist a norm-compact set K and $r > 0$ such that

$$Y \cap B(\bar{y}, r) + t\mathbb{B} \subset Y - tK \quad \forall t \in [0, r], \tag{9}$$

then

$$\bar{y} \in \mathrm{bd}Y \iff \exists \;\; p^* \in N(Y, \bar{y}), \text{ with } p^* \neq 0.$$

As in [26], we will easily show that when Y is epi-Lipschitzian at $\bar{y}$ in the sense of Rockafellar [43]

$$\bar{u} \in \mathrm{bd}T(Y, \bar{y}) \iff T(Y, \bar{y}) \text{ has a supporting hyperplane at } \bar{u}. \tag{10}$$

The following theorem is a generalization of the result by Bonnisseau and Crettez [7] from the finite dimensional spaces to the infinite dimensional ones.

Theorem 4 *(***Existence of Pareto optimal under a normal condition***) Let $\bar{y}$ be a boundary point of Y. Suppose that Z is convex with nonempty interior. Then $\bar{y}$ is a Pareto optimal of Y with respect to Z, provided that relations (4) and (8) hold.*

Proof. The free disposal hypothesis together with $\text{int}Z \neq \emptyset$ implies that the set Y is epi-Lipschitzian at $\bar{y}$ in the sense of Rockafellar [43] and hence compactly epi-Lipschitzian at $\bar{y}$. It results that relation (8) makes sense and [43]

$$\text{int}T(Y,\bar{y}) = \{h \in X : \exists \varepsilon > 0,\, Y \cap B(\bar{y},\varepsilon) +]0,\varepsilon]B(h,\varepsilon) \subset Y\}. \tag{11}$$

Let $z \in (Y - \bar{y}) \cap Z$. As in the proof of Theorem 3, relation (4) assures that

$$tz \in (Y - \bar{y}) \cap Z, \forall t \in [0,1]$$

and by Corollary 2 we obtain

$$-z \in T(Y,\bar{y}).$$

By the interiority normal condition (8), we have for all $p^* \in N(Y,\bar{y})$, with $\|p^*\| = 1$, there exists $\delta > 0$ such that

$$\delta\|h\| \leq \langle p^*, h\rangle \, \forall h \in K(Z,0)$$

and hence

$$\delta\|z\| \leq \langle p^*, z\rangle.$$

If $z \neq 0$ then

$$\langle p^*, -z\rangle < 0 \, \forall p^* \in N(Y,\bar{y}) \backslash \{0\}. \tag{12}$$

We claim that $-z \in \text{int}T(Y,\bar{y})$. Otherwise, relation (10) implies that $T(Y,\bar{y})$ has a supporting hyperplane at $-z$, that is there exists $p^* \in N(Y,\bar{y})$, with $p^* \neq 0$, such that

$$\langle p^*, h\rangle \leq 0,\, \forall h \in T(Y,\bar{y}) \text{ and } \langle p^*, -z\rangle = 0$$

and this is in contradiction with (12). By invoking (11), we obtain

$$\exists\, \varepsilon > 0,\, Y \cap B(\bar{y},\varepsilon) +]0,\varepsilon]B(-z,\varepsilon) \subset Y.$$

Then, for a small $t \in]0,\varepsilon[$, we get

$$\bar{y} + tz \in Y \cap B(\bar{y},\varepsilon)$$

and hence

$$\bar{y} + tz + tB(-z,\varepsilon) \subset Y.$$

This yields $\bar{y} \in \text{int}Y$ and contradicts our assumption $\bar{y} \in \text{bd}Y$. $\square$

6 Proof of Theorem 1

Let us start with some background needed in the proof of our main theorem. If f is an extended-real-valued function on X, we write for any subset S of X

$$f_S(x) = \begin{cases} f(x) & \text{if } x \in S, \\ +\infty & \text{otherwise.} \end{cases}$$

The function

$$d^- f(x,h) = \liminf_{\substack{u \to h \\ t \downarrow 0}} t^{-1}(f(x+tu) - f(x))$$

is the lower Dini directional derivative of f at x in the direction h, and the Dini ε-subdifferential of f at x is the set

$$\partial_\varepsilon^- f(x) = \{x^* \in X^* : \langle x^*, h\rangle \leq d^- f(x;h) + \varepsilon\|h\|, \forall h \in X\}$$

for $x \in Domf$ and $\partial_\varepsilon^- f(x) = \emptyset$ if $x \notin Domf$, where $Domf$ denotes the effective domain of f. For $\varepsilon = 0$ we write $\partial^- f(x)$.
By $\mathcal{F}(X)$ we denote the collection of finite dimensional subspaces of X. The approximate subdifferential of f at $x_0 \in Domf$ is defined by the following expression (see Ioffe [21]-[23])

$$\partial_A f(x_0) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{x \xrightarrow{f} x_0} \partial^- f_{x+L}(x) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{\substack{x \xrightarrow{f} x_0 \\ \varepsilon \downarrow 0}} \partial_\varepsilon^- f_{x+L}(x)$$

where

$$\limsup_{\substack{x \xrightarrow{f} x_0 \\ \varepsilon \downarrow 0}} \partial_\varepsilon^- f_{x+L}(x) = \{x^* \in X^* : x^* = w^*\text{–}\lim x_i^*,\ x_i^* \in \partial_{\varepsilon_i}^- f_{x_i+L}(x_i), x_i \xrightarrow{f} x_0,\ \varepsilon_i \downarrow 0\},$$

that is, the set of w^*-limits of all such nets.
It is easily seen that the set-valued mapping $x \to \partial_A f(x)$ is upper semicontinuous in the following sense

$$\partial_A f(x_0) = \limsup_{x \xrightarrow{f} x_0} \partial_A f(x)$$

and in [22] and [23] Ioffe has shown that when S is a closed subset of X and $x_0 \in S$

$$\partial_A d(x_0, S) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{\substack{x \xrightarrow{S} x_0 \\ \varepsilon \downarrow 0}} \partial_\varepsilon^- d_{x+L}(x, S) \cap (1+\varepsilon)\mathbb{B}^*. \tag{13}$$

The following lemma will be used later. The first part was established by Ioffe [21, Lemma 1], while the second part uses a penalization property by Clarke [12].

Lemma 1 *Let $L \in \mathcal{F}(X)$ and $x^* \in \partial_\varepsilon^- f_{x+L}(x)$. Then the function*

$$u \mapsto f(u) - \langle x^*, u - x\rangle + 2\varepsilon\|u - x\|$$

attains a local minimum at x on $x + L$.
If additionally f is locally Lipschitz near x with constant K, then the function

$$u \mapsto f(u) - \langle x^*, u - x\rangle + 2\varepsilon\|u - x\| + (K + \|x^*\| + 2\varepsilon)d(u, x + L)$$

attains a local minimum at x.

Proof. (4) $\Longrightarrow$ (5): We use the following relation ([22]-[23]) between the the Clarke's normal cone and the Ioffe's approximate subdifferential $\partial_A d(Y, y)$ of the distance function of Y

$$N(Y, y) = w^* - cl[\mathbb{R}_+ co\partial_A d(Y, y)].$$

Hence it suffices to show that

$$\partial_A d(Y, y) \subset -[K(Z, 0)]^0. \tag{14}$$

So let $p^* \in \partial_A d(Y, y)$. Then, by (13) we have for all finite dimensional space $L \subset X$ the existence of nets $y_i \xrightarrow{Y} y$, $p_i^* \xrightarrow{w^*} p^*$ and $\varepsilon_i \to 0^+$ such that

$$p_i^* \in \partial_{\varepsilon_i} d_{y_i+L}(y_i, Y) \cap (1 + \varepsilon_i)B^*.$$

Now Lemma 1 implies that for each i there exists $\delta_i > 0$ such that,

$$(2 + 3\varepsilon_i)d(y, y_i + L) - \langle p_i^*, y - y_i\rangle + 2\varepsilon_i\|y - y_i\| \geq 0 \,\forall y \in Y \cap B(y_i, \delta_i).$$

Now let $h \in K(Z, 0)$. Then there exist sequences $t_n \to 0^+$ and $h_n \to h$ such that

$$t_n h_n \in Z \cap B(0, \delta_i) \,\forall n.$$

Thus, by relation (4), $y_i - t_n h_n \in Y \cap B(y_i, \delta_i)$, we obtain

$$(2 + 3\varepsilon_i)d(h_n, L) + \langle p_i^*, h_n\rangle + \varepsilon_i\|h_n\| \geq 0$$

and hence

$$(2 + 3\varepsilon_i)d(h, L) + \langle p_i^*, h\rangle + \varepsilon_i\|h\| \geq 0 \,\forall h \in K(Z, 0).$$

By passing to the limit on i, this yields

$$2d(h, L) + \langle p^*, h\rangle \geq 0 \,\forall h \in K(Z, 0)$$

and this asserts that for all finite dimensional space $L \subset X$

$$-p^* \in [K(Z, 0)]^0 \cap B(0, 3) + L^\perp$$

where $L^\perp := \{x^* \in X^* : \langle x^*, x\rangle = 0 \,\forall x \in L\}$. Thus as $[K(Z, 0)]^0 \cap B(0, 3)$ is weak-star closed, we have

$$-p^* \in [K(Z, 0)]^0.$$

(5) $\Longrightarrow$ (6): The proof of this implication is based on the following drop theorem.

Theorem 5 *(Daneš's drop theorem) Let $A \subset X$ be a closed set and $B \subset X$ be a closed convex and bounded set be such that*

$$\inf_{(a,b)\in A\times B} \|a-b\| > 0.$$

Then for each $a \in A$ there exists $a_0 \in X$ such that

$$a_0 \in A \cap Drop[a, B] \text{ and } A \cap Drop[a_0, B] = \{a_0\}$$

where $Drop[a, B]$ denotes the drop generated by a and B, that is,

$$Drop[a, B] = \{a + t(b-a) : t \in [0,1], b \in B\}.$$

Proof of Theorem 1 (be continued). Now, let us come back to the proof of our implication (5) $\Longrightarrow$ (6). Pick $y \in Y$ and $z \in Z \cap \bar{co}K(Z, 0)$. Suppose that $y - z \notin Y$. Then there exists $\delta > 0$ such that

$$B(-z, \delta) \cap (Y - y) = \emptyset. \tag{15}$$

Theorem 5, applied with $A = Y - y$, $B = B(-z, \delta)$ and $a = 0$, yields the existence of $y_0 \in X$ such that

$$y_0 \in (Y - y) \cap Drop[0, B(-z, \delta)] \tag{16}$$

$$(Y - y) \cap Drop(y_0, B(-z, \delta)) = \{y_0\}. \tag{17}$$

Relation (16) implies the existence of $b \in B(0, \delta)$ and $t_0 \in [0, 1[$ such that $y_0 = t_0(-z + b)$. Let $\varepsilon > 0$ be such that $t_0 + \varepsilon < 1$. Using relation (17), we obtain that for all $w \in B(-z - y_0, \delta(t_0 + \varepsilon))$

$$(Y - y) \cap (y_0 +]0, 1]B(w, \delta(1 - t_0 - \varepsilon))) = \emptyset$$

and this implies that $w \notin T(Y, y + y_0)$. Consequently,

$$B(-z - y_0, \delta(t_0 + \varepsilon)) \cap T(Y, y + y_0) = \emptyset.$$

By separation theorem, there exists $p^* \in N(Y, y + y_0)$, with $\|p^*\| = 1$, such that

$$0 \leq \langle p^*, v\rangle \, \forall v \in B(-z - y_0, \delta(t_0 + \varepsilon))$$

or equivalently

$$\delta(t_0 + \varepsilon) \leq \langle p^*, -z - y_0\rangle.$$

As $y_0 = t_0(-z + b)$, we have

$$\delta(t_0 + \varepsilon) \leq \langle -p^*, (1 - t_0)z\rangle + \langle p^*, -t_0 b\rangle. \tag{18}$$

Relation (5) implies that $-p^* \in [K(Z, 0)]^0$. Then as $z \in Z \cap \bar{co}K(Z, 0)$, we have $\langle -p^*, z\rangle \leq 0$. So, relation (18) and the fact that $b \in B(0, \delta)$ ensure the inequality

$$\delta(t_0 + \varepsilon) \leq t_0\|b\| \leq t_0\delta$$

and this contradiction allows us to say that $y - z \in Y$. Thus $Y - Z \cap \bar{co}K(Z, 0) \subset Y$. $\square$

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