BULL. AUSTRAL. MATH. SOC. 54C60, 46A30
VOL. 44 (1991) [1–9]

ON METRIC REGULARITY OF MULTIFUNCTIONS

A. JOURANI

The aim of this paper is to give a metric regularity theorem for multifunctions between metric spaces involving some known results for multifunctions by using the notion of strict (G, δ)-differentiability of multifunctions and a simple convergence procedure.

0. INTRODUCTION

The notion of metric regularity of multifunctions (see Definition 1.1) plays an important role in optimisation theory, in particular in mathematical programming problems (see for example [**5, 8, 15, 19, 23, 26, 27, 28**]). This concept has been extensively studied in varying degrees of generality by many authors (see [**3, 5, 6, 14, 16, 17, 18, 21, 22, 24**]). Probably the first result goes back to Graves [**14**], stating that a continuously differentiable mapping F between Banach spaces whose derivative $DF(x_0)$ at x_0 is surjective is metrically regular around $(x_0, F(x_0))$. Extending the well-known Banach perturbation lemma [**20**, Theorem 3 (2.V)], Robinson [**24**] has made explicit the metric regularity result for convex multifunctions in Banach spaces. Applying Ekeland's variational principle [**11**], Ioffe [**16**] and Borwein [**5**] proved some metric regularity results for Lipschitz mappings between Banach spaces. We can also consider that the results obtained by the authors [**1, 2, 4, 7, 13, 25**] are metric regularity results since one can establish a relationship between the metric regularity and the pseudo-Lipschitzianity of multifunctions (see [**22**] and [**6**]). Robinson [**24**] established that a multifunction F between Banach spaces with convex closed graph which is open at a point (x_0, y_0) of its graph is regular around (x_0, y_0) and conversely. Without convexity of its graph the openness of F at (x_0, y_0) is not sufficient to recover the metric regularity of F around (x_0, y_0) (see Example 4.6 in [**6**]). Recently Penot [**22**] and Borwein and Zhuang [**6**] showed that the openness of F around (x_0, y_0) is equivalent to the metric regularity of F around (x_0, y_0).

In this paper we study metric regularity properties of multifunctions in metric spaces. Namely, we show that a multifunction F preserves some properties of its derivative, in particular the openness and the metric regularity properties.

Received 28 June 1990
The author is indebted to Professors L. Thibault and D. Azé for valuable remarks helping to improve the paper.

To do this we introduce the notion of strict (G, δ)-differentiability of multifunctions. To this end, let us introduce some notation.

Let X, Y and Z be linear spaces equipped with some distance d which is invariant by translations, that is, $d(b+a, c+a) = d(b, c)$; let $F\colon X \rightrightarrows Y$, $L\colon X \rightrightarrows Y$ and $G\colon Z \rightrightarrows X$ be multifunctions and let $\delta\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly monotone continuous function. We denote by

$$GrF = \{(x, y) \in X \times Y\colon y \in F(x)\}$$

the graph of F and by B_X, B_Y and B_Z the closed unit balls of X, Y and Z respectively. We denote by F^{-1} the multifunction whose graph is deduced from GrF by exchanging x and y and we set

$$L \circ G(z) = \{y \in Y\colon \exists\, x \in G(z); y \in L(x)\}.$$

Observe that, for each $y \in Y$,

$$(L \circ G)^{-1}(y) = G^{-1} \circ L^{-1}(y) = \{z \in Z\colon y \in L \circ G(z)\}.$$

DEFINITION 0.1: F is said to be strictly (G, δ)-differentiable at $(x_0, y_0) \in GrF$ if there exists a multifunction $L\colon X \rightrightarrows Y$ such that for each $\varepsilon > 0$, there exists $r > 0$ such that

$$LoG(z) \cap B_Y \subset F(x + G(z)) - y + \varepsilon\delta(d(0, z))B_Y$$

for all $x \in x_0 + rB_X$, $y \in (y_0 + rB_Y) \cap F(x)$ and $z \in rB_Z$.

REMARK. When $X = Z$, $G(x) = x$ and $\delta(t) = t$, we recover the notion of strict differentiability of a multifunction (see [4] and [**9**]).

1. THE MAIN RESULT

DEFINITION 1.1: A multifunction F is said to be metrically δ-regular around $(x_0, y_0) \in GrF$ if there exist $K > 0$ and $r > 0$ such that

$$d\big(x, F^{-1}(y)\big) \leqslant K\delta^{-1}(d(y, F(x)))$$

for all $x \in x_0 + rB_X$ and $y \in y_0 + rB_Y$ where

$$d(v, D) = \inf\{d(v, v')\colon v' \in D\}.$$

DEFINITION 1.2: F is nicely δ-invertible around $y_0 \in F(x_0)$ if there exist $\alpha > 0$ and $a > 0$ such that for all $y \in y_0 + \alpha B_Y$ there exists $x \in F^{-1}(y)$ such that $\delta(d(x_0, x)) \leqslant a d(y_0, y)$.

REMARK. If F is δ-regular around $(x_0, y_0) \in GrF$ and if δ is continuous and strictly monotone then F is nicely δ-inversible around $y_0 \in F(x_0)$.

Let us assume that G and δ satisfy the following:

$H(G)$: There exist $\alpha > 0$ and $b > 0$ such that

$$G(z) \subset bd(0, z)B_X \text{ for each } z \in \alpha B_Z.$$

$H(\delta)$: $\delta(1) = 1$ and for all t, $s \in [0, +\infty)$,

$$\delta^{-1}(ts) \leqslant \delta^{-1}(t)\delta^{-1}(s).$$

EXAMPLE. $Z = \mathbb{R}_+$, $G(t) = t^s B_X$ and $\delta(t) = t^r$ with $s \geqslant 1$ and $r > 0$.

THEOREM 1.3. *Let L be a strict (G, δ)-derivative of F at $(x_0, y_0) \in GrF$. Suppose that:*

(i) *X is complete and F has a closed graph.*
(ii) *$L \circ G$ is nicely δ-invertible around $0 \in (L \circ G)(0)$.*
(iii) *G and δ satisfy respectively $H(G)$ and $H(\delta)$.*

Then F is δ-regular around (x_0, y_0). More precisely there exist $a > 0$ and $b > 0$ such that for some $\varepsilon > 0$ there is $s > 0$ with:

$$d(x, F^{-1}(y)) \leqslant \frac{b\delta^{-1}(a)}{1 - \delta^{-1}(\varepsilon a)} \delta^{-1}(d(y, F(x)))$$

for all $x \in x_0 + sB_X$ and $y \in y_0 + sB_Y$.

PROOF: We follow the lines of Azé [4]. Since $L \circ G$ is nicely δ-invertible there exist $a > 0$ and $\alpha > 0$ (we can choose α as in $H(G)$) such that for all $y \in \alpha B_Y$ we can exhibit $z \in G^{-1} \circ L^{-1}(y)$ satisfying:

$$\delta(d(0, z)) \leqslant ad(0, y). \tag{1.3.1}$$

From the strict (G, δ)-differentiability of F at (x_0, y_0) it follows that for all $\varepsilon > 0$ (with $\varepsilon a < 1$) there exists $r > 0$ such that

$$L \circ G(z) \cap B_Y \subset F(x + G(z)) - y + \varepsilon\delta(d(0, z))B_Y \tag{1.3.2}$$

for all $x \in x_0 + rB_X$, $y \in (y_0 + rB_Y) \cap F(x)$ and $z \in rB_Z$. Choose $\eta > 0$ such that $\max(\eta, (\delta(\eta))/a) \leqslant \min\big(\alpha, r, r(1 - \delta^{-1}(\varepsilon a))/(b\delta^{-1}(\varepsilon a))\big)$ where b is given by $H(G)$. Let $y \in y_0 + (\delta(\eta)/a)B_y$. We shall construct a sequence $((x_n, y_n)) \subset GrF$ such that $((x_n, y_n))$ converges to (x, y) with $y \in F(x)$ and $x \in \big(x_0 + K\delta^{-1}(d(y, y_0))B_X\big)$, where $K > 0$ does not depend on x and y. Put $x_{-1} = x_0$. Let $n \in \mathbb{N}$ be such that, for each $k \in [0, n]$,

$$x_k \in (x_0 + rB_X),\ y_k \in F(x_k),\ d(y, y_k) \leqslant (\varepsilon a)^k d(y, y_0),$$
$$d(x_k, x_{k-1}) \leqslant b\big(\delta^{-1}(\varepsilon a)\big)^k \delta^{-1}(ad(y, y_0)). \tag{1.3.3}$$

We want to show that (1.3.3) holds for $k = n+1$. Since $y - y_n \in (\delta(\eta)/a)B_Y$, we derive from (1.3.1) the existence of $z_n \in G^{-1} \circ L^{-1}(y - y_n)$ such that $\delta(d(0, z_n)) \leqslant ad(y, y_n)$. It follows from (1.3.2) that there exist $u_n \in G(z_n)$ and $y_{n+1} \in F(x_n + u_n)$ such that

$$\begin{aligned} d(y, y_{n+1}) &\leqslant \varepsilon\delta(d(0, z_n)) \\ &\leqslant \varepsilon ad(y, y_n). \end{aligned} \tag{1.3.4}$$

Let us set $x_{n+1} = x_n + u_n$. Then, by assumption $H(G)$, we have

$$\begin{aligned} d(x_{n+1}, x_n) = d(0, u_n) &\leqslant bd(0, z_n) \\ &\leqslant b\delta^{-1}(ad(y, y_n)). \end{aligned} \tag{1.3.5}$$

On the other hand we obtain, by using the induction assumption (1.3.3), $H(G)$ and by the choice of η,

$$\begin{aligned} d(x_{n+1}, x_0) &\leqslant d(x_{n+1}, x_n) + \sum_{k=1}^{n} d(x_k, x_{k-1}) \\ &\leqslant \frac{b\delta^{-1}(a)\eta}{1 - \delta^{-1}(\varepsilon a)} \\ &\leqslant r. \end{aligned} \tag{1.3.6}$$

In virtue of (1.3.3), (1.3.4), (1.3.5) and (1.3.6) we obtain

$$\begin{aligned} &x_{n+1} \in (x_0 + rB_X),\ y_{n+1} \in F(x_{n+1}), \\ &d(x_{n+1}, x_n) \leqslant b\big(\delta^{-1}(\varepsilon a)\big)^n \delta^{-1}(ad(y, y_0)), \\ &d(y_{n+1}, y) \leqslant (\varepsilon a)^{n+1} d(y, y_0), \end{aligned}$$

so that (1.3.3) holds for $k = n + 1$. Observe that, for each n, $p \in \mathbb{N}$,

$$\begin{aligned} d(x_{n+p}, x_n) &\leqslant \sum_{k=1}^{p} d(x_{n+k}, x_{n+k-1}) \\ &\leqslant \frac{b}{1 - \delta^{-1}(\varepsilon a)} \big(\delta^{-1}(\varepsilon a)\big)^n; \end{aligned}$$

hence (x_n) is a Cauchy sequence and then it converges to some $x \in X$. As (y_n) converges to y, it ensues from the closedness of GrF that (x, y) belongs to GrF which together with (1.3.6) yields

$$y \in F(x) \text{ and } x \in x_0 + K\delta^{-1}(d(y, y_0))B_X \tag{1.3.7}$$

where $K = \big(b\delta^{-1}(a)\big)/\big(1 - \delta^{-1}(\varepsilon a)\big)$. Now pick $y \in y_0 + (\delta(\eta)/2a)B_Y$ and $x_1 \in x_0 + (r/2)B_X$ from (1.3.7); $F^{-1}(y) \neq \emptyset$ and $d\big(x_1, F^{-1}(y)\big) < +\infty$. Let us consider $y_1 \in F(x_1) \cap (y_0 + (\delta(\eta)/2a)B_Y)$; (if this set is empty, there is nothing to prove). The same argument as above applied to (x_1, y_1) instead of (x_0, y_0) provides the existence of $x \in x_1 + rB_X$ such that $y \in F(x)$ and $x \in x_1 + K\delta^{-1}(d(y, y_1))B_X$. Hence,

$$d(x_1, x) \leqslant K\delta^{-1}(d(y, y_1)),$$

and then,

$$d\big(x_1, F^{-1}(y)\big) \leqslant K\delta^{-1}\left(d(y, F(x_1)) \cap \left(y_0 + \frac{\delta(\eta)}{2a}B_Y\right)\right) \tag{1.3.8}$$

for all $x_1 \in x_0 + (r/2)B_X$ and $y \in y_0 + (\delta(\eta)/2a)B_Y$. As in Rockafellar [25] (see also Thibault [27]), one can show that there is $\gamma \in (0, r/2)$ and $\beta \in (0, \delta(\eta)/2a)$ such that

$$d(y, F(x_1) \cap (y_0 + (\delta(\eta)/2a)B_Y)) = d(y, F(x_1)) \tag{1.3.9}$$

for all $x_1 \in x_0 + \gamma B_X$ and $y \in y_0 + \beta B_Y$, which completes the proof of the theorem. ∎

REMARKS. (1) When $X = Z$, $G(x) = x$ and $\delta(t) = t$, we recover the results of [4] and [7].

(2) It follows from the proof of Theorem 1.3 that there is $b > 0$ and $a > 0$ such that for some $\varepsilon > 0$ (with $\varepsilon a < 1$) there is $s > 0$ such that for all $y \in y_0 + sB_Y$ there is $x \in F^{-1}(y)$ satisfying $d(x_0, x) \leqslant \big(b\delta^{-1}(a)\big)\big(\delta^{-1}(d(y_0, y))\big)/\big(1 - \delta^{-1}(\varepsilon a)\big)$.

The proof of the following theorem is exactly the same as that of Theorem 1.3.

THEOREM 1.4. *Theorem 1.3 remains true if we replace the completeness of X by the completeness of GrF.*

COROLLARY 1.5. *If in Theorem 1.4 we assume that $L \circ G$ is δ-regular around $(0, 0) \in Gr(L \circ G)$, then F is δ-regular around (x_0, y_0).*

PROOF: It suffices to apply Theorem 1.4 since the δ-regularity of $L \circ G$ implies the nice δ-inversibility of $L \circ G$. ∎

Borwein and Zhuang [6] have shown that, if for some strictly monotone continuous function δ', F is approximately δ'-open around (x_0, y_0) (that is, there is $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$

with $\limsup_{t\downarrow 0} \left(\delta'^{-1}(\alpha(t))\right)/(t) < 1$, and $r > 0$ and $\eta > 0$ such that for all $x \in x_0 + rB_X$ and all $y \in F(x) \cap (y_0 + \eta B_Y)$ and all $t \in (0, \eta)$,

$$y + \delta'(t)B_Y \subset \mathrm{cl}\,(F(x + tB_X) + \alpha(t)B_Y)$$

where "cl" denotes the closure), then F is δ-regular around (x_0, y_0) for all $\delta \leqslant \delta'$ with $\limsup_{t\downarrow 0} (\delta(t))/t < +\infty$.

In the following corollary we obtain the same result but with the condition $H(\delta')$ instead of $\limsup_{t\downarrow 0} (\delta(t))/t < +\infty$. Let us remark that from this corollary one can get information about the regularity constant K of F. But first we give the following lemma:

Lemma 1.6. *Let $c \in (0, 1)$ and δ satisfy $H(\delta)$. Then for all $b \in [1/(\delta(1/c)), 1)$ and all $t > 0$,*

$$\delta(ct) \leqslant b\delta(t). \tag{1.6}$$

Proof: Consider the failure of (1.6). Then there is $b \in [1/(\delta(1/c)), 1)$ and $t > 0$ such that $\delta(ct) > b\delta(t)$. This latter is equivalent to $1/b > \delta(1/c)$ thanks to $H(\delta)$ and the strict monotonicity of δ which contradicts $b \geqslant 1/(\delta(1/c))$. ∎

Corollary 1.7. *Let δ satisfy $H(\delta)$. If F is approximately δ-open around (x_0, y_0) and GrF is complete then F is δ-regular around (x_0, y_0).*

Proof: The condition $\limsup_{t\downarrow 0} \left(\delta^{-1}(\alpha(t))\right)/t < 1$ implies that there are $0 < c < 1$ and $\eta > 0$ such that $\alpha(t) < \delta(ct)$ for all $t \in [0, \eta]$. Let $b \in [1/(\delta(1/c)), 1)$. Then by Lemma 1.6 and the approximately δ-openness property of F one has the existence of $r > 0$ and $\gamma > 0$ (we can assume that $\gamma = \eta$) such that.

$$\begin{aligned}\delta(t)B_Y &\subset F(x + tB_X) - y + (\delta(ct) - \alpha(t) + \alpha(t))B_Y \\ &\subset F(x + tB_X) - y + \delta(ct)B_Y \\ &\subset F(x + tB_X) - y + b\delta(t)B_Y\end{aligned}$$

for all $x \in x_0 + rB_X$, $y \in (y_0 + \gamma B_Y) \cap F(x)$ and $t \in [0, \gamma]$. Let us set $\varepsilon = b$, $(\varepsilon < 1)$, $Z = \mathbb{R}_+$, $G(t) = tB_X$ and $L(x) = \delta(d(0, x))B_Y$. It is not difficult to see that the assumptions of Theorem 1.4 are satisfied. ∎

Remarks. (1) The result of Corollary 1.7 remains true for every strictly monotone continuous function $\delta' \leqslant \delta$.

(2) It is not difficult to see that for some strictly monotone continuous function δ' the approximately δ'-openness of F around (x_0, y_0) is equivalent to the following: there

is $c \in (0, 1)$, $r > 0$ and $\eta > 0$ such that for all $x \in x_0 + rB_X$, $y \in (y_0 + \eta B_Y) \cap F(x)$ and $t \in (0, \eta)$, $y + \delta'(t)B_Y \subset F(x + tB_X) + \delta'(ct)B_Y$.

Frankowska [12] has introduced the concept of high order variations for multifunctions as follows:

$$F^r(x_0, y_0) = \liminf_{(x,y)\xrightarrow{GrF}(x_0,y_0)} \frac{F(x + tB_X) - y}{t^r}$$

where $r > 0$ and "lim inf" is taken in the sense of Kuratowski.

In the following corollary we obtain Frankowska's result proved in [13] as a consequence of our main Theorem 1.3 since the δ-regularity implies the δ'-openness for some $\delta' \leqslant \delta$ (see [6, 22]).

Corollary 1.8. *Let X and Y be two linear metric spaces, X complete and $r > 0$. Assume that*

$$0 \in \text{int } F^r(x_0, y_0).$$

Then F is δ-regular around (x_0, y_0) with $\delta(t) = t^r$.

Proof: Since $0 \in \text{ int } F^r(x_0, y_0)$ then there is $c > 0$ such that for all $0 < \varepsilon < c$ there exists $r > 0$ satisfying:

$$t^r cB_Y \subset F(x + tB_X) - y + \varepsilon t^r B_Y$$

for all $x \in x_0 + rB_X$, $y \in (y_0 + rB_Y) \cap F(x)$ and $t \in [0, r]$. Put $Z = \mathbb{R}_+$, $G(t) = tB_X$, $\delta(t) = t$ and $L(x) = (d(0, x))^r cB_X$. We easily see that the assumptions of Theorem 1.3 are satisfied. $\square$

The following metric regularity result for closed convex multifunctions is an extension of Robinson [24].

Corollary 1.9. *Let X and Y be two linear metric spaces with X complete and let F be a multifunction from X into Y with closed convex graph. Then the openness of F at $(x_0, y_0) \in GrF$ (that is, there exist $r > 0$ such that $y_0 + rB_Y \subset F(x_0 + B_X)$) is equivalent to the metric δ-regularity of F around $(x_0, y_0) \in GrF$, with $\delta(t) = t$.*

Proof: In the remainder of the proof we assume for notational convenience that $x_0 = 0$ and $y_0 = 0$; this simply translates the origins in X and Y. Suppose that F is open at $(0, 0)$. Then by the convexity of F we have for all $0 < \varepsilon < r$ and $t \in [0, 1]$,

$$trB_Y \subset F(tB_X) + \varepsilon tB_Y.$$

So by the convexity of F we have that for all $x \in (r/2)B_X$, $t \in [0, 1/2]$ and $y \in (1/2)B_Y \cap F(x)$,

$$trB_Y \subset F(x + tB_X) - y + \varepsilon tB_Y.$$

As in Corollary 1.7 we set $Z = \mathbb{R}_+$, $G(t) = tB_X$, $\delta(t) = t$ and $L(x) = d(0, x)rB_Y$. Applying Theorem 1.3 we obtain the δ-regularity of F which completes the proof since the other implication is obvious. □

Relying on the remark following Theorem 1.3 we obtain a substraction result for convex multifunctions.

COROLLARY 1.10. *If the assumptions of Corollary 1.9 are satisified and if for* $0 < \alpha < \beta$ *one has* $y_0 + \beta B_Y \subset F(x_0 + B_X) + \alpha B_Y$, *then there is* $s > 0$ *such that,*

$$y_0 + (\beta - \alpha)sB_Y \subset F(x_0 + sB_X).$$

PROOF: As in the proof of Corollary 1.9 we asusme that $(x_0, y_0) = (0, 0)$ and for r sufficiently small one has

$$t\beta B_Y \subset F(x + tB_X) - y + \alpha tB_Y$$

for all $x \in rB_X$, $t \in [0, r]$ and all $y \in rB_Y \cap F(x)$. If we set $Z = \mathbb{R}_+$, $G(t) = tB_X$ and $L(x) = d(0, x)B_Y$, we have $b = 1$ and $a = 1/\beta$ where b is as in $H(G)$ and a is the invertibility constant of $L \circ G$. Then it ensues from the remarks following Theorem 1.3 that there is $s > 0$ such that for all $y \in y_0 + (\beta - \alpha)sB_Y$ there is $x \in F^{-1}(y)$ satisfying:

$$d(x_0, x) \leqslant (1/(\beta - \alpha))d(y_0, y) \leqslant s$$

which completes the proof. □

REFERENCES

[1] J.P. Aubin, 'Lipschitz behavior of solutions to convex minimization problems', *Math. Oper. Res.* **9** (1984), 87–111.

[2] J.P. Aubin and H. Frankowska, 'On the inverse function theorem', *J. Math. Pures Appl.* **66** (1987), 71–89.

[3] A. Auslender, 'Stability in mathematical programming with nondifferentiable data', *SIAM J. Control Optim.* (1984), 239–254.

[4] D. Azé, 'An inversion theorem for set-valued maps', *Bull. Austral. Math. Soc.* **37** (1988), 411–414.

[5] J.M. Borwein, 'Statility and regular point of inequality systems', *J. Optim. Theory Appl.* **48** (1986), 9–52.

[6] J.M. Borwein and D.M. Zhuang, 'Verifiable necessary and sufficient conditions for openness and regularity for set-valued maps', *J. Math. Anal. Appl.* **134** (1988), 441–459.

[7] C.C. Chou, 'Inverse function theorems for multifunctions in Frechet spaces' (to appear).

[8] F. Clark, *Optimization and nonsmooth analysis* (Wiley Interscience, New York, 1983).

[9] F.S. De Blasi, 'On the differentiability of multifunctions', *Pacific J. Math.* **66** (1976), 67–81.

[10] S. Dolecki, 'A general theory of necessary optimality conditions', *J. Math. Anal. Appl.* **7** (1980), 267–308.

[11] I. Ekeland, 'Nonconvex minimization problems', *Bull. Amer. Math. Soc.* **1** (1979), 443–474.

[12] H. Frankowska, 'An open mapping principle for set-valued maps', *J. Math. Anal. Appl.* **127** (1987), 172–180.

[13] H. Frankowska, 'Some inverse mapping and open mapping theorems' (to appear).

[14] L.M. Graves, 'Some mapping theorems', *Duke Math. J.* **17** (1950), 111–114.

[15] J.B. Hiriart-Uruty, 'Refinement of necessary optimality conditions in nondifferentiable programming', *Appl. Math. Optim.* **5** (1979), 63–82.

[16] A.D. Ioffe, 'Regular points of Lipschitz mapping', *Trans. Amer. Math. Soc.* **251** (1979), 61–69.

[17] A.D. Ioffe, 'On the local surjection property', *Nonlinear Anal.* **11** (1987), 565–592.

[18] A. Jourani and L. Thibault, 'Approximate subdifferential and metric regularity: the finite dimensional case', *Math. Programming* **47** (1990), 203–218.

[19] A. Jourani and L. Thibault, 'The use of metric graphical regularity in approximate subdifferential calculus rules in finite dimensions', *Optimization* **21** (1990), 509–519.

[20] L.V. Kantorovich and G.P. Akilov, *Functional analysis in normed spaces* (Macmillan, New York, 1964).

[21] J.P. Penot, 'On regularity conditions in mathematical programming', *Math. Programming Stud.* **19** (1982), 167–199.

[22] J.P. Penot, 'Metric regularity, openness and Lipschitz behavior of multifunctions', *Nonlinear Anal.* **13** (1989), 629–643.

[23] S.M. Robinson, 'Stability theorems of systems of inequality part II: Differentiable nonlinear systems', *SIAM J. Numer. Anal.* **13** (1976), 497–513.

[24] S.M. Robinson, 'Regularity and stability for convex multivalued functions', *Math. Oper. Res.* **1** (1976), 130–143.

[25] R.T. Rockafellar, 'Lipschitzian properties of multifunctions', *Nonlinear Anal.* **9** (1985), 867–885.

[26] R.T. Rockafellar, 'Extensions of subgradient calculus with application to optimization', *Nonlinear Anal.* **9** (1985), 665–698.

[27] L. Thibault, 'On subdifferential of optimal value function', *SIAM J. Control Optim.* (to appear).

[28] J. Zowe and S. Kurcyusz, 'Regularity and stability for mathematical programming problems in Banach spaces', *Appl. Math. Optim.* **5** (1979), 49–62.

Mathématiques
Université de Perpignan
66025 Perpignan Cedex
France