JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **192**, 867–891 (1995)

Intersection Formulae and the Marginal Function in Banach Spaces

A. JOURANI

Université de Bourgogne, Laboratoire "Analyse Appliquée et Optimisation," B.P. 138, 21004, Dijon Cedex, France

Submitted by Philip D. Loewen

Received July 12, 1993

Intersection formulae are central to the development of subdifferential calculus and the differentiation of marginal functions. In this paper, we reexamine the connection between independence conditions and intersection formulae. Then we apply the formulae to a general parametric mathematical programming problem in which the constraints are defined by multivalued functions. These results allow us to obtain generalized chain rules for composite functions. Corollaries of this work include several well-known intersection formulae and calculus rules.

1. INTRODUCTION

Given two subsets C_1 and C_2 of a Banach space X and a point $x_0 \in C_1 \cap C_2$, our objective in this paper is to study conditions ensuring the inclusion

$$\partial d(x_0, C_1 \cap C_2) \subset k[\partial d(x_0, C_1) + \partial d(x_0, C_2)], \tag{1.1}$$

where k is a positive real number and $\partial d(x_0, C)$ is the subdifferential of the distance function $d(x, C)$ to C at x_0 (see Sections 2 and 3). This formula plays an important role in optimization theory, namely in subdifferential calculus rules (see Rockafellar [24], Mordukhovich [20–22], Ward and Borwein [29], Ioffe [10–11], Jourani and Thibault [14, 18]), in the study of necessary optimality conditions (see Clarke [5], Ioffe [9], Raïssi [23], Journai [12]), and in the differentiability of the marginal function (see Rockafellar [25–26], Mordukhovich [20–22], Borwein [2], Thibault [27]).

0022-247X/95 $6.00

Several conditions have been proposed to ensure formula (1.1). Some are stated in terms of the subdifferential of the distance function, and others in terms of the tangent cone or the normal cone.

In finite dimensions, Aubin and Ekeland [1, pp. 441] (see also Ward and Borwein [29]) establish formula (1.1) under the condition that the difference of the tangent cones to C_1 and C_2 at x_0 is equal to the space X. In Banach spaces, Rockafellar [24] shows that formula (1.1) holds whenever the tangent cone to C_2 at x_0 meets the interior of the tangent cone to C_1 at x_0 and C_1 is epi-Lipschitzian at x_0. Ioffe [11] obtains formula (1.1) under the two assumptions that

$$\partial d(x_0, C_1) \cap (-\partial d(x_0, C_2)) = \{0\} \tag{1.2}$$

and that C_1 is epi-Lipschitzian at x_0. In Hilbert spaces, Clarke and Raïssi [6] establish inclusion (1.1) by assuming that C_1 is compact and requiring that

$$\begin{aligned}
&\text{for all } \forall \varepsilon > 0\ \exists \delta > 0,\ \exists \text{ neighbourhood } V \text{ of } x_0 \text{ such that}\\
&\forall x_i \in C_i \cap V,\ \forall x_i^* \in \partial d(x_i, C_i) \qquad i = 1, 2, \qquad\qquad (1.3)\\
&\text{satisfying } \|x_1^*\| + \|x_2^*\| \geq \varepsilon \text{ then } \|x_1^* + x_2^*\| \geq \delta.
\end{aligned}$$

In [28], Ward has extended the results in [1, 29] to the case in which X is a Banach space and C_1 and C_2 are epi-Lipschitz-like at x_0 (see [3–4]). This class of sets includes epi-Lipschitzian sets and finite dimensional sets.

In infinite dimensional spaces most of the criteria ensuring inclusion (1.1) require that either C_1 or C_2 is epi-Lipschitzian, or C_1 and C_2 are epi-Lipschitz-like, or they require certain conditions to hold at all points in a neighbourhood of x_0. Morever they are stated in a complicated way. Recently, Jourani and Thibault [18] have shown that condition (1.2) implies formula (1.1) when C_1 is compactly epi-Lipschitzian at x_0 in the sense of Borwein and Strojwas [4]. Let us note (following Borwein [3]) that the class of compactly epi-Lipschitzian sets includes epi-Lipschitzian sets, epi-Lipschitz-like sets, and finite dimensional sets.

The purposes of this paper are threefold. First, we show that, when C_1 is compactly epi-Lipschitzian, condition (1.2) is equivalent to condition (1.3) and both are equivalent to a third condition introduced in Section 3 below. Second, we extend the result of Clarke and Raïssi [6] to general Banach spaces without any compactness assumption on C_1. The results obtained generalize those of Ioffe [11] and Rockafellar [24]. Finally we apply our results to estimate the subdifferential of the marginal function m defined by

$$m(x) := \inf \{f(x, y) \colon y \in F(x)\},$$

where f is an extended-real-valued function on $X \times Y$, X and Y are Banach spaces, and F is a multivalued function from X into Y. These results allow us to derive chain rules for nonsmooth functions. Such rules includes sum formulae as well as the calculus rules in [11, 20–22].

Throughout the paper, we will let X and Y be Banach spaces equipped with the norm $\| \|$ and X^* and Y^* their topological dual spaces endowed with the weak-star topology w^*. We denoted by $\langle \cdot, \cdot \rangle$ the pairing between the space and its topological dual and by B_X, B_X^*, ..., the closed unit balls of X, X^*, ... For an extended-real-valued function f on X, we write

$$\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R}: f(x) \leq r\}$$

and

$$\operatorname{dom} f := \{x \in X: |f(x)| < +\infty\}.$$

For a multivalued function F from X into Y, we denote the graph of F by Gr F, that is,

$$\operatorname{Gr} F = \{(x, y) \in X \times Y: y \in F(x)\}.$$

The abbreviations for interior, weak-star closure, and convex hull are "int," "cl*," and "conv." The distance function $d(x, S)$ to a given set S is

$$d(x, S) = \inf_{u \in S} \|x - u\|.$$

We write $x \xrightarrow{f} x_0$ and $x \xrightarrow{S} x_0$ to express $x \to x_0$ with $f(x) \to f(x_0)$ and $x \to x_0$ with $x \in S$, respectively. Finally, if not specified, the norm in a product of two Banach spaces is defined by $\|(u, v)\| = \|u\| + \|v\|$.

2. Preliminaries

In this paper we will use the notations and definitions in Ioffe [10–11] and Clarke [5]. Let f be a function from X into $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ and let $\varepsilon > 0$. We write for any subset S of X

$$f_S(x) = \begin{cases} f(x), & \text{if } x \in S, \\ +\infty, & \text{otherwise.} \end{cases}$$

Recall that for $x \in \operatorname{dom} f$ the lower Dini directional derivative, the Dini subdifferential and the Dini ε-subdifferential of f at x are defined by

$$d^- f(x, h) = \liminf_{\substack{u \to h \\ t \downarrow 0}} t^{-1} (f(x + tu) - f(x))$$

$$\partial^- f(x) = \{x^* \in X^*: \langle x^*, h \rangle \le d^- f(x, h), \forall h \in X\}.$$

and

$$\partial_\varepsilon^- f(x) = \{x^* \in X^*: \langle x^*, h \rangle \le d^- f(x, h) + \varepsilon \|h\|, \forall h \in X\}.$$

If $x \notin \operatorname{dom} f$ we put $\partial^- f(x) = \partial_\varepsilon^- f(x) = \emptyset$.

Let $\mathcal{F}(X)$ be the family of all finite dimensional subspaces of X and let f be a lower semicontinuous function at $x_0 \in \operatorname{dom} f$. The A-subdifferential of f at x_0 (see Ioffe [10–11]) is defined by

$$\partial_A f(x_0) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{x \xrightarrow{f} x_0} \partial^- f_{x+L}(x),$$

where

$$\limsup_{x \xrightarrow{f} x_0} \partial^- f_{x+L}(x) = \{x^* \in X^*: x^* = w^* - \lim x_i^*, x_i^* \in \partial^- f_{x_i+L}(x_i), x_i \xrightarrow{f} x_0\},$$

that is, the set of weak-star limits of all such nets.

The normal cone generated by $\partial_A d(x_0, S)$ is denoted by $\hat{N}_G(S, x_0)$, that is,

$$\hat{N}_G(S, x_0) = \bigcup_{\lambda > 0} \lambda \partial_A d(x_0, S).$$

The $\hat{G}$-subdifferential $\hat{\partial}_G f(x_0)$ of f at $x_0 \in \operatorname{dom} f$ and its singular counterpart $\hat{\partial}_G^\infty f(x_0)$ are given by (see [10–11])

$$\hat{\partial}_G f(x_0) = \{x^* \in X^*: (x^*, -1) \in \hat{N}_G(\operatorname{epi} f; x_0, f(x_0))\},$$
$$\hat{\partial}_G^\infty f(x_0) = \{x^* \in X^*: (x^*, 0) \in \hat{N}_G(\operatorname{epi} f; x_0, f(x_0))\}.$$

Following Clarke [5], a vector $h \in X$ will be in the Clarke tangent cone $T_C(S, x_0)$ to S at x_0 if for any sequence $(x_n) \subset S$ converging to x_0 and any $t_n \to 0^+$ there exists $h_n \to h$ such that for all positive integers n,

$$x_n + t_n h_n \in S.$$

The Clarke normal cone $N_C(S, x_0)$ to S at x_0 is the polar of $T_C(S, x_0)$; i.e.,

$$N_C(S, x_0) = \{x^* \in X^*: \langle x^*, x\rangle \leq 0, \forall x \in T_C(S, x_0)\}.$$

Clarke's subdifferential of f at x_0 is given by

$$\partial_C f(x_0) = \{x^* \in X^*: (x^*, -1) \in N_C(\operatorname{epi} f; x_0, f(x_0))\}.$$

Suppose finally that F is a multivalued function from X into Y. The $\hat{G}$-coderivative and the Clarke coderivative of F at $(x_0, y_0) \in \operatorname{Gr} F$, respectively, are the multivalued functions of $y^* \in Y^*$ defined by

$$\hat{D}_G F(x_0, y_0)(y^*) = \{x^* \in X^*: (x^*, -y^*) \in \hat{N}_G(\operatorname{Gr} F, x_0, y_0)\}$$

and

$$D_C F(x_0, y_0)(y^*) = \{x^* \in X^*: (x^*, -y^*) \in N_C(\operatorname{Gr} F, x_0, y_0)\}.$$

The following theorem lists some of the important properties of the $\hat{G}$-subdifferential.

THEOREM 2.1. [10–11]. *Let f be an extended-real-valued function on X which is lower semicontinuous around $x_0 \in \operatorname{dom} f$ and let S be a closed subset of X containing x_0:*

$$\partial_A d(x_0, S) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{\substack{x \xrightarrow{S} x_0 \\ \varepsilon \downarrow 0}} \partial_\varepsilon^- d_{x+L}(x, S), \tag{2.1}$$

$$\partial_C f(x_0) = \operatorname{cl}^* \operatorname{conv}(\hat{\partial}_G f(x_0) + \hat{\partial}_G^\infty f(x_0)). \tag{2.2}$$

If f is Lipschitz near x_0 with Lipschitz constant k_f,

$$\hat{\partial}_G f(x_0) = \partial_A f(x_0) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{\substack{x \xrightarrow{f} x_0 \\ \varepsilon \downarrow 0}} [\partial_\varepsilon^- f_{x+L}(x) \cap (k_f + \varepsilon) B_X^*], \tag{2.3}$$

$$\hat{\partial}_G f(x_0) = \limsup_{x \xrightarrow{f} x_0} \hat{\partial}_G f(x) \qquad \text{(upper semicontinuity condition)}. \tag{2.4}$$

A thorough discussion of these concepts can be found in [10–11, 5]. We mention here that if f is convex, then the $\hat{G}$-subdifferential coincides with the subdifferential of convex analysis. The singular $\hat{G}$-subdifferential reduces to zero whenever f is Lipschitz near x_0.

Following Borwein and Strojwas [4], we call a set $S \subset X$ compactly epi-Lipschitzian at $x_0 \in S$ if there exist a norm-compact set K and a scalar $r > 0$ such that

$$S \cap (x_0 + rB_X) + \operatorname{tr} B_X \subset S - tK, \qquad \forall t \in]0, r[.$$

The following proposition gives us an important property of such sets.

PROPOSITION 2.2. [17]. *If S is compactly epi-Lipschitzian at $x_0 \in S$, then there exist $k_1, \ldots, k_m \in X$ and $r > 0$ such that*

$$\|x^*\| \leq \max_{i=1,\cdots,m} |\langle x^*, k_i\rangle|$$

for all $x \in x_0 + rB_X$ and $x^* \in \hat{\partial}_G d(x, S)$.

Proof. Let K be a compact subset of X and $r \in]0, 1[$ such that

$$S \cap (x_0 + 3rB_X) + \operatorname{tr} B_X \subset S - tK, \text{ for all } t \in]0, r[.$$

Choose an open neighbourhood V of x_0 with $V \subset x_0 + rB_X$ and $d(x, S) \leq r$ for all $x \in V$. For all $\varepsilon \in]0, \frac{1}{2}[$ there exist $h_1, \ldots, h_m \in K$ such that

$$K \subset \bigcup_{i=1}^{m} (h_i + \varepsilon rB_X).$$

For each $x \in V$ and each $t \in]0, r]$ we may select some $p(x, t) \in S$ such that

$$\|x - p(x, t)\| \leq d(x, S) + t^2. \tag{2.5}$$

By the choice of V we have for each $x \in V$ and each $t \in]0, r]$

$$\begin{aligned}\|x_0 - p(x, t)\| &\leq \|x - p(x, t)\| + \|x - x_0\| \\ &< d(x, S) + t^2 + r < 3r\end{aligned}$$

and, hence, $p(x, t) \in x_0 + 3rB_X$. Fix any $x \in V$ and any $x^* \in \partial_A d(x, S)$. Fix any $b \in B_X$ and any $L \in \mathscr{F}(X)$ with $\{b, h_1, \ldots, h_m\} \subset L$. We may write $x^* = w^* - \lim_{j \in J} x_j^*$ with $x_j^* \in \partial^- d_{x_j + L}(x_j, S)$ and $x_j \to x$. Choose $j_0 \in J$ such that $x_j \in V$ for any $j \in J, j \geq j_0$. Let $(t_n) \subset]0, r]$ converging to zero. For each $n \in \mathbb{N}$ and each $j \in J$, $j \geq j_0$ choose $h_{i(n,j)} \in \{h_1, \ldots, h_m\}$ and $b_{n,j} \in B_X$ with

$$p(x_j, t_n) + t_n(rb + h_{i(n,j)} + \varepsilon rb_{n,j}) \in S. \tag{2.6}$$

For each $j \in J$, $j \geq j_0$, we may suppose that $h_{i(n,j)} = h_{q(j)}$, for all $n \in \mathbb{N}$. Then for each $j \in J$, $j \geq j_0$ (recalling that $d_{x_j+L}(x, S) = d(x, S)$ if $x \in x_j + L$ and $d_{x_j+L}(x, S) = +\infty$ if $x \notin x_j + L$), we have

$$\begin{aligned} t_n^{-1}&[d_{x_j+L}(x_j + t_n(rb + h_{q(j)}), S) - d_{x_j+L}(x_j, S)] \\ &= t_n^{-1}[d(x_j + t_n(rb + h_{q(j)}), S) - d(x_j, S)] \\ &\leq t_n^{-1}[d(p(x_j, t_n) + t_n(rb + h_{q(j)}), S) + \|x_j - p(x_j, t_n)\| - d(x_j, S)] \\ &\leq t_n^{-1}[d(p(x_j, t_n) + t_n(rb + h_{q(j)}, S) + t_n \qquad \text{(by (2.5))} \\ &\leq t_n + \varepsilon r\|b_{n,j}\| + t_n^{-1}d(p(x_j, t_n) + t_n(rb + h_{q(j)} + \varepsilon rb_{n,j}), S) \\ &\leq t_n + \varepsilon r \qquad \text{(by (2.6))}. \end{aligned}$$

So for each $j \geq j_0$ we have

$$d^- d_{x_j+L}(\cdot, S)(x_j; rb + h_{q(j)}) \leq \varepsilon r$$

which implies $\langle x_j^*, rb + h_{q(j)}\rangle \leq \varepsilon r$ and, hence,

$$r\langle x_j^*, b\rangle \leq \varepsilon r + \max_{i=1,\ldots,m} |\langle x_j^*, h_i\rangle|.$$

Therefore we get for each $\varepsilon \in]0, \frac{1}{2}[$ and each $b \in B_X$

$$\begin{aligned} \langle x^*, b\rangle &\leq \varepsilon + r^{-1} \max_{i=1,\ldots,m} |\langle x^*, h_i\rangle| \\ &\leq \varepsilon + r^{-1} \sup_{h\in K} |\langle x^*, h\rangle| \end{aligned}$$

and, hence,

$$\|x^*\| \leq \frac{1}{r(1-\varepsilon)} \max_{i=1,\ldots,m} |\langle x^*, h_i\rangle|$$

and the proposition is proved. ∎

Remarks. (1) It follows from this proposition that in $\hat{\partial}_G d(x, S)$, weak-star and strong convergences of nets to zero are equivalent, that is,

$$x_j^* \xrightarrow{w^*} 0 \qquad \text{iff } \|x_j^*\| \to 0.$$

(2) Loewen [19] has shown that in weakly locally compact cones (of reflexive Banach spaces), weak-star and strong convergences of sequences to zero are equivalent.

3. Independence Conditions

DEFINITION 3.1. Let C_1 and C_2 be two nonempty subsets of X and let $x_0 \in C_1 \cap C_2$. Let $\partial = \hat{\partial}_G$ or $\partial = \partial C$.

(a) C_1 and C_2 and ∂-independent at x_0 [11] if (1.2) holds for ∂.

(b) C_1 and C_2 are strongly ∂-independent at x_0 [6] if (1.3) holds for ∂.

Remark. Since the approximate subdifferential of Lipschitz functions is always a subset of the Clarke subdifferential, ∂_C-indepencence implies $\hat{\partial}_G$-independence, and this implication can be strict (see Example 6.5).

PROPOSITION 3.2. *Let C_1 and C_2 be two nonempty subsets of X, with C_1 compactly epi-Lipschitzian at $x_0 \in C_1 \cap C_2$. Then the following assertions are equivalent:*

(i) *For all $\varepsilon > 0$ there exists $\delta > 0$ and a neighbourhood V of x_0 such that for all $x_i \in V$, $x_i^* \in \hat{\partial}_G d(x_i, C_i)$, $i = 1, 2$, satisfying $\|x_1^*\| + \|x_2^*\| \geq \varepsilon$ then $\|x_1^* + x_2^*\| \geq \delta$.*

(ii) *C_1 and C_2 are strongly $\hat{\partial}_G$-independent at x_0.*

(iii) *C_1 and C_2 are $\hat{\partial}_G$-independent at x_0.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious.

(iii) $\Rightarrow$ (i) Suppose that (i) is false. Then there are $\varepsilon > 0$, $u_n \to x_0$, $v_n \to x_0$, $u_n^* \in \hat{\partial}_G d(u_n, C_1)$, and $v_n^* \in \hat{\partial}_G d(v_n, C_2)$ such that

$$\|u_n^*\| + \|v_n^*\| \geq \varepsilon \qquad \text{and} \qquad \|u_n^* + v_n^*\| \leq 1/n. \tag{3.1}$$

Since the function $d(x, C)$ is Lipschitz with constant 1, the sequences (u_n^*) and (v_n^*) are bounded. So, extracting subnets if necessary, we may suppose that $u_n^* \to u^*$ and $v_n^* \to -u^*$. By (2.4),

$$u^* \in \hat{\partial}_G d(C_1, x_0) \cap (-\hat{\partial}_G d(C_2, x_0)).$$

We shall arrive at a contradiction with (iii) if we show that $u^* \neq 0$. So suppose that $u^* = 0$. Then Proposition 2.2 implies the existence of $k_1, \ldots, k_m$ in X (not depending on n) satisfying

$$\|u_n^*\| \leq \max_{i=1,\ldots,m} |\langle u_n^*, k_i\rangle|.$$

Thus $\|u_n^*\| \to 0$. Now, by (3.1), $\|v_n^*\| \to 0$ (because $-1/n + \|u_n^*\| \leq \|v_n^*\| \leq \|u_n^*\| + 1/n$) and so $\varepsilon \leq 0$ which leads to the contradiction and terminates the proof. ∎

Remark. Assuming that C_1 is epi-Lipschitz-like at x_0 in the sense of Borwein [3] and using the result of Jourani and Thibault [18], instead of Proposition 2.2, we may show that the result of Proposition 3.2 is valid if we replace $\hat{\partial}_G$ by ∂_C.

To close this section let us give an other characterization of the $\hat{\partial}_G$-independence condition.

PROPOSITION 3.3. *Let C_1 and C_2 be two closed subsets of X, with $x_0 \in C_1 \cap C_2$. Then C_1 and C_2 are $\hat{\partial}_G$-independent at x_0 iff*

$$\hat{N}_G(C_1, x_0) \cap (-\hat{N}_G(C_2, x_0)) = \{0\}.$$

Proof. First note that if $t \in [0, 1]$ and $u^* \in \hat{\partial}_G d(x_0, C)$ then $tu^* \in \hat{\partial}_G d(x_0, C)$. Let $x^* \in [\hat{N}_G(C_1, x_0) \cap (-\hat{N}_G(C_2, x_0))]$. Then there are $r > 0$ and $s > 0$ such that $x^*/r \in \hat{\partial}_G d(x_0, C_1)$ and $-x^*/s \in \hat{\partial}_G d(x_0, C_2)$. Suppose, for example, that $r \geq s$. Then $-sx^*/rs \in \hat{\partial}_G d(x_0, C_2)$ and, hence,

$$x^*/r \in \hat{\partial}_G d(x_0, C_1) \cap (-\hat{\partial}_G d(x_0, C_2))$$

which implies that $x^* = 0$.

The converse is immediate, since $\hat{\partial}_G d(x_0, C) \subset \hat{N}_G(C, x_0)$. ∎

4. INTERSECTION FORMULAE

We start this section by establishing the following important result which represents the key to the proof of our intersection theorem.

PROPOSITION 4.1 *Let C_1 and C_2 be two closed subsets of X, with $x_0 \in C_1 \cap C_2$. Suppose that one of the following assumptions is satisfied:*

(i) *C_1 and C_2 are strongly $\hat{\partial}_G$-independent at x_0, or*

(ii) *C_1 is compactly epi-Lipschitzian at x_0 and C_1 and C_2 are $\hat{\partial}_G$-independent at x_0.*

Then there exist $r > 0$ and $a > 0$ such that

$$d(x, C_2 \cap (C_1 - y)) \leq a\, d(x + y, C_1) \tag{4.1}$$

for all $x \in C_2 \cap (x_0 + rB_X)$ and $y \in rB_X$. In particular,

$$d(x, C_2 \cap C_1) \leq (a + 1)(d(x, C_1) + d(x, C_2))$$

for all $x \in x_0 + rB_X$.

Proof. Suppose, for example, that (i) holds. Consider the multivalued function

$$F(x) = \begin{cases} -x + C_1, & \text{if } x \in C_2 \\ \varnothing, & \text{otherwise.} \end{cases}$$

Then (4.1) is equivalent to

$$d(x, F^{-1}(y)) \leq a\, d(y, F(x))$$

for all $x \in x_0 + rB_X$ and $y \in rB_X$, since $F^{-1}(y) = C_2 \cap (C_1 - y)$. Suppose that (4.1) is false. Then, by Lemma 1.2 of Journai [12], there are $s_n \downarrow 0$, with $s_n \leq 1$, $x_n \to x_0$, $z_n \to 0$, $y_n \to 0$, and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$(x_n, z_n) \in \text{Gr } F, \qquad y_n \notin F(x_n), \tag{4.2}$$

and for all $(x, y) \in \text{Gr } F$,

$$\|z_n - y_n\| \leq \|y - y_n\| + s_n(\|x - x_n\| + \|y - z_n\|)$$

and, hence, the function

$$(x, y) \to \|y - y_n\| + 2d(x, y; \text{Gr } F) + s_n(\|x - x_n\| + \|y - z_n\|)$$

attains a local minimum at (x_n, z_n). Since for all $x \in C_2$

$$\begin{aligned} d(x, y; \text{Gr } F) &\leq d(y, F(x)) \\ &= d(x + y, C_1), \end{aligned}$$

Proposition 2.4.3 in Clarke [5] implies that (x_n, z_n) is a local minimum of the function

$$\begin{aligned} f_n(x, y) = \|y - y_n\| &+ 4[d(x + y; C_1) + d(x; C_2)] \\ &+ s_n(\|x - x_n\| + \|y - z_n\|). \end{aligned}$$

Thus by subdifferential calculus rules [10–11]

$$0 \in \hat{\partial}_G f_n(x_n, z_n)$$
$$\subset \{0\} \times S(z_n, y_n) + 4\left[\bigcup_{y^* \in \hat{\partial}_G d(x_n + z_n, C_1)} (y^*, y^*) + \hat{\partial}_G d(x_n, C_2) \times \{0\}\right]$$
$$+ s_n(B_X^* \times B_X^*),$$

where $S(z_n, y_n) = \{z^* \in X^*: \|z^*\| = 1 \text{ and } \langle z^*, z_n - y_n\rangle = \|z_n - y_n\|\}$. Thus there are $z_n^* \in S(z_n, y_n)$, $y_n^* \in \hat{\partial}_G d(x_n + z_n, C_1)$, and $x_n^* \in \hat{\partial}_G d(x_n, C_2)$ such that

$$\|z_n^* + 4y_n^*\| \leq s_n \quad \text{and} \quad \|y_n^* + x_n^*\| \leq s_n/4.$$

So $y_n^* \in \hat{\partial}_G d(x_n + z_n, C_1)$, $x_n^* \in \hat{\partial}_G d(x_n, C_2)$, and

$$\|x_n^*\| + \|y_n^*\| \geq (1 - s_n)/4 \quad \text{and} \quad \|x_n^* + y_n^*\| \leq s_n/4.$$

These inequalities contradict (i), since by (4.2), $x_n \in C_2$ and $x_n + z_n \in C_1$. ∎

Ioffe [11] has proved the last part of Proposition 4.1 for the case in which C_1 is epi-Lipschitzian at x_0. He used the following result.

LEMMA 4.2. *Suppose that $S \subset X$ is closed and epi-Lipschitzian at x_0. Then there are an $h \in X$, $\|h\| = 1$, and $r > 0$ such that*

$$x^* \in \hat{\partial}_G d(x, S) \Rightarrow \langle x^*, h\rangle \leq -r,$$

provided $x \notin S$ and x sufficiently close to x_0.

Note that we can use Proposition 2.2 instead of Lemma 4.2 and Ioffe's techniques, which are based on Ekeland variational principle [7], to prove Proposition 4.1.

COROLLARY 4.3. *Let all the hypotheses of Proposition 4.1 be satisfied, with $\hat{\partial}_G$-independence replaced by ∂_C-independence. Then the results of Proposition 4.1 are valid.*

Proof. Observe that $\hat{\partial}_G d(x, D) \subset \partial_C d(x, D)$ for all x and apply Proposition 4.1. ∎

We now use Proposition 4.1 to establish the intersection theorem.

THEOREM 4.4. *Under assumptions of Proposition 4.1 there exists a real number $k > 0$ such that*

$$\hat{\partial}_G d(x_0, C_1 \cap C_2) \subset k[\hat{\partial}_G d(x_0, C_1) + \hat{\partial}_G d(x_0, C_2)].$$

Proof. Let $f(x) = d(x, C_1 \cap C_2)$ and $g(x) = d(x, C_2) + d(x, C_1)$. Let $x^* \in \hat{\partial}_G f(x_0)$. Then by (2.1), we have for all $L \in \mathcal{F}(X)$ the existence of nets $x_i \xrightarrow{C_1 \cap C_2} x_0$, $\varepsilon_i \to 0^+$, with $\varepsilon_i < 1$, and $x_i^* \to x^*$ such that

$$\|x_i^*\| \leq 2 \qquad \text{and} \qquad x_i^* \in \partial_{\varepsilon_i}^- f_{x_i+L}(x_i).$$

Let $\varepsilon > 0$ be given. By Lemma 1 in Ioffe [9], x_i is a local minimum of the function

$$x \to f_{x_i+L}(x) - \langle x_i^*, x - x_i \rangle + (\varepsilon + \varepsilon_i)\|x - x_i\|. \tag{4.3}$$

By Proposition 4.1 there exist $k > 0$ and $r > 0$ such that

$$d(x, C_2 \cap C_1) \leq k(d(x, C_1) + d(x, C_2))$$

for all $x \in x_0 + rB_X$. Combining this and (4.3) we find that x_i is a local minimum of the function

$$x \to kg_{x_i+L}(x) - \langle x_i^*, x - x_i \rangle + (\varepsilon + \varepsilon_i)\|x - x_i\|.$$

Thus

$$x_i^* \in \partial_{(\varepsilon+\varepsilon_i)}^- kg_{x_i+L}(x_i)$$

and, hence,

$$\begin{aligned} x^* &\in k\hat{\partial}_G g(x_0) \\ &\subset k\,[\hat{\partial}_G d(x_0, C_1) + \hat{\partial}_G d(x_0, C_2)]. \quad \blacksquare \end{aligned}$$

Remark. Ioffe [11] has obtained this result under the hypothesis that C_1 is epi-Lipschitzian. But every epi-Lipschitz set is compactly epi-Lipschitzian and the converse does not hold.

COROLLARY 4.5. *If assumptions of Theorem* 4.4 *are satisfied, then*

$$\hat{N}_G(C_1 \cap C_2, x_0) \subset \hat{N}_G(C_1, x_0) + \hat{N}_G(C_2, x_0).$$

As a consequence of Theorem 4.4 we obtain the following result which extends the main theorem of Clarke and Raïssi [6].

COROLLARY 4.6. *Let all the hypotheses of Theorem 4.4 be satisfied. Then there exists a real number $k > 0$ such that*

$$\partial_C d(x_0, C_1 \cap C_2) \subset k\,[\partial_C d(x_0, C_1) + \partial_C d(x_0, C_2)].$$

Proof. By Theorem 4.4 there exists $k > 0$ such that

$$\begin{aligned}\hat{\partial}_G d(x_0, C_1 \cap C_2) &\subset k\,[\hat{\partial}_G d(x_0, C_1) + \hat{\partial}_G d(x_0, C_2)] \\ &\subset k\,[\partial_C d(x_0, C_1) + \partial_C d(x_0, C_2)].\end{aligned}$$

Since $\partial_C d(x_0, C_1) + \partial_C d(x_0, C_2)$ is convex and weak-star compact it follows that

$$\text{cl}^*\text{conv}\, \hat{\partial}_G d(x_0, C_1 \cap C_2) \subset k\,[\partial_C d(x_0, C_1) + \partial_C d(x_0, C_2)]$$

and, hence,

$$\partial_C d(x_0, C_1 \cap C_2) \subset k\,[\partial_C d(x_0, C_1) + \partial_C d(x_0, C_2)]. \quad \blacksquare$$

Remark. Note that this result has been established by Clarke and Raïssi [6], but with X a Hilbert space and C_1 a compact set.

5. MARGINAL FUNCTION

Intersection formulae can readily be applied to produce estimates for subdifferentials of the marginal function

$$m(x) := \inf\{f(x, y) \colon y \in F(x)\}, \tag{5.1}$$

where f is an extended-real-valued function on $X \times Y$ and F is a multivalued function from X into Y. For all $x \in \text{dom}\, m$ we consider the set of minimizers

$$M(x) = \{y \in F(x) \colon m(x) = f(x, y)\}.$$

To estimate the subdifferential of m we need a stability assumption of the type

There exist a norm-compact set $K \subset Y$ and a neighbourhood V of x_0
such that for all $x \in V$, $M(x) \neq \varnothing$ and $M(x) \subset K + p(x)B_Y$, (5.2)
where p is a real-valued function on V satisfying $\lim_{x \to x_0} p(x) = 0$.

Remarks. (1) It is easy to see that, under the hypothesis (5.2) and the fact that f is lower semicontinuous and Gr F is closed, the marginal function m is lower semicontinuous at x_0.

(2) Similar conditions to (5.2) have been used in the study of the directional derivative and the Clarke's subdifferential of the marginal function m (see, for example, [8, 25–27]).

THEOREM 5.1. *Let the multivalued function F have closed graph, let f be lower semicontinuous around any point (x_0, y_0) with $y_0 \in F(x_0)$, and let* (5.2) *hold. Suppose that, for all $y_0 \in M(x_0)$, one of the following conditions holds:*

(i) epi f *and* Gr $F \times \mathbb{R}$ *are strongly $\hat{\partial}_G$-independent at $(x_0, y_0, f(x_0, y_0))$,*

(ii) epi f *is compactly epi-Lipschitzian at $(x_0, y_0, f(x_0, y_0))$ and* epi f *and* Gr $F \times \mathbb{R}$ *are $\hat{\partial}_G$-independent at $(x_0, y_0, f(x_0, y_0))$,*

(iii) Gr F *is compactly epi-Lipschitzian at (x_0, y_0) and* epi f *and* Gr $F \times \mathbb{R}$ *are $\hat{\partial}_G$-independent at $(x_0, y_0, f(x_0, y_0))$.*

Then

$$\hat{\partial}_G m(x_0) \subset \bigcup_{y_0 \in M(x_0)} \bigcup_{(x^*, y^*) \in \hat{\partial}_G f(x_0, y_0)} \{x^* + \hat{D}_G^* F(x_0, y_0)(y^*)\}$$

and

$$\hat{\partial}_G^\infty m(x_0) \subset \bigcup_{y_0 \in M(x_0)} \bigcup_{(x^*, y^*) \in \hat{\partial}_G^\infty f(x_0, y_0)} \{x^* + \hat{D}_G^* F(x_0, y_0)(y^*)\}.$$

Remark. The results of Theorem 5.1 remain true if we replace the stability assumption (5.2) by the following one:

There exist a continuous selection s of the multivalued function M on some closed ball $x_0 + rB_X$.

In this case, for $y_0 = s(x_0)$,

$$\hat{\partial}_G m(x_0) \subset \bigcup_{(x^*, y^*) \in \hat{\partial}_G f(x_0, y_0)} \{x^* + \hat{D}_G^* F(x_0, y_0)(y^*)\}$$

and

$$\hat{\partial}_G^\infty m(x_0) \subset \bigcup_{(x^*, y^*) \in \hat{\partial}_G^\infty f(x_0, y_0)} \{x^* + \hat{D}_G^* F(x_0, y_0)(y^*)\}.$$

We will prove our theorem in two steps. The first step reduces the proof to that of a simpler estimate.

LEMMA 5.2. *Set* $C_1 = \operatorname{epi} f$ *and* $C_2 = \operatorname{Gr} F \times \mathbb{R}$. *Let the multivalued function F have closed graph, let f be lower semicontinuous around any point* (x_0, y_0) *with* $y_0 \in F(x_0)$, *and let* (5.2) *hold. Then for all* $(x^*, \beta) \in \hat{\partial}_G d(x_0, m(x_0), \operatorname{epi} m)$ *there exist* $y_0 \in M(x_0)$ *and* $k > 0$ *such that*

$$(x^*, 0, \beta) \in k\hat{\partial}_G d(x_0, y_0, f(x_0, y_0), C_1 \cap C_2).$$

Proof. Set $g(x, r) = d(x, r, \operatorname{epi} m)$ and $h(x, y, r) = d(x, y, r, C_1 \cap C_2)$. Then, by (2.1), we have for all $L \in \mathcal{F}(X)$ the existence of nets $(x_i, r_i) \xrightarrow{\operatorname{eip} m} (x_0, m(x_0))$, $\varepsilon_i \to 0^+$, with $\varepsilon_i < 1$, and $(x_i^*, \beta_i) \to (x^*, \beta)$ such that

$$(x_i^*, \beta_i) \in \partial_{\varepsilon_i}^- g_{(x_i, r_i) + L \times \mathbb{R}}(x_i, r_i).$$

Let $\varepsilon > 0$ be given. By Lemma 1 in Ioffe [9] and Proposition 2.4.3 in Clarke [5], (x_i, r_i) is a local minimum of the function

$$\begin{aligned}(x, r) \to g(x, r) &- \langle x_i^*, x - x_i \rangle - \langle \beta_i, r - r_i \rangle \\ &+ (\varepsilon + \varepsilon_i)(\|x - x_i\| + |r - r_i|) + 4d(x, x_i + L)\end{aligned}$$

and, hence, there exists $s_i > 0$ such that

$$-\langle x_i^*, x - x_i \rangle - \langle \beta_i, r - r_i \rangle + (\varepsilon + \varepsilon_i)(\|x - x_i\| + |r - r_i|) + 4d(x, x_i + L) \geq 0$$

for all $(x, r) \in \operatorname{epi} m \cap ((x_i, r_i) + s_i B_{X \times \mathbb{R}})$. By (5.2), there exists $y_i \in F(x_i)$ satisfying

$$m(x_i) = f(x_i, y_i)$$

and, extracting a subnet if necessary, we may suppose that $y_i \to y_0$. Because $\operatorname{Gr} F$ is closed, we have $y_0 \in F(x_0)$. Further, since $m(x_i) = f(x_i, y_i)$, $m(x_i) \leq r_i$, and f is lower semicontinuous around (x_0, y_0), it follows that

$$y_0 \in F(x_0) \qquad \text{and} \qquad m(x_0) \geq f(x_0, y_0).$$

Thus $y_0 \in M(x_0)$. On the one hand, $(x_i, y_i, r_i) \in C_1 \cap C_2$, and, on the other hand, if $(x, y, r) \in C_1 \cap C_2$ then $(x, r) \in \operatorname{epi} m$. So invoking (5.3) and Proposition 2.4.3 in Clarke [5] we obtain that (x_i, y_i, r_i) is a local minimum of the function

$$(x, y, r) \to 6d(x, y, r, C_1 \cap C_2) - \langle x_i^*, x - x_i \rangle - \langle \beta_i, r - r_i \rangle$$
$$+ (\varepsilon + \varepsilon_i)(\|x - x_i\| + |r - r_i|) + 4d(x, x_i + L).$$

It follows that for all $E \in \mathcal{F}(Y)$,

$$(x_i^*, 0, \beta_i) \in \partial^-_{(\varepsilon+\varepsilon_i)} 6h_{(x_i, y_i, r_i)+L\times E\times \mathbb{R}}(x_i, y_i, r_i)$$

and, hence,

$$(x^*, 0, \beta) \in 6\hat{\partial}_G h(x_0, y_0, f(x_0, y_0)). \quad \blacksquare$$

The second step shows that the sets C_1 and C_2 are $\hat{\partial}_G$-independent (resp. strongly $\hat{\partial}_G$-independent) at x_0 and satisfy the intersection formulae.

LEMMA 5.3. *Let the multivalued function F have closed graph and let f be lower semicontinuous around* (x_0, y_0) with $y_0 \in F(x_0)$. *Suppose that condition* (i) (*resp.* (ii)) (*resp.* (iii)) *holds. Then C_1 and C_2 are strongly $\hat{\partial}_G$-independent at* $(x_0, y_0, f(x_0, y_0))$ (*resp. C_1 is compactly epi-Lipschitzian at* $(x_0, y_0, f(x_0, y_0))$ *and C_1 and C_2 are $\hat{\partial}_G$-independent at this point*) (*resp. C_2 is compactly epi-Lipschitzian at* $(x_0, y_0, f(x_0, y_0))$ *and C_1 and C_2 are $\hat{\partial}_G$-independent at this point*). *In addition there exists $a > 0$ such that*

$$\hat{\partial}_G d(x_0, y_0, f(x_0, y_0), C_1 \cap C_2) \subset a\,[\hat{\partial}_G d(x_0, y_0, f(x_0, y_0), \operatorname{epi} f)$$
$$+ \hat{\partial}_G d(x_0, y_0, \operatorname{Gr} F) \times \{0\}].$$

Proof. It suffices to see that

$$\hat{\partial}_G d(x_0, y_0, f(x_0, y_0), C_1) = \hat{\partial}_G d(x_0, y_0, f(x_0, y_0), \operatorname{epi} f)$$

and

$$\hat{\partial}_G d(x_0, y_0, f(x_0, y_0), C_2) = \hat{\partial}_G d(x_0, y_0, \operatorname{Gr} F) \times \{0\}$$

and apply Theorem 4.4. $\blacksquare$

Proof of Theorem 5.1. Let $u^* \in \hat{\partial}_G m(x_0)$. Then there exist $\lambda > 0$ and $(x^*, \beta) \in \hat{\partial}_G d(x_0, m(x_0), \operatorname{epi} m)$ such that

$$u^* = \lambda x^* \quad \text{and} \quad \lambda\beta = -1.$$

By Lemma 5.2 there exist a real number $k > 0$ and a point $y_0 \in M(x_0)$ such that

$$(x^*, 0, \beta) \in k\hat{\partial}_G d(x_0, y_0, f(x_0, y_0), C_1 \cap C_2)$$

and by Lemma 5.3 there exist $a > 0$, $(z^*, y^*, \gamma) \in \hat{\partial}_G d(x_0, y_0, f(x_0, y_0), \operatorname{epi} f)$ and $(v^*, p^*) \in \hat{\partial}_G d(x_0, y_0, \operatorname{Gr} F)$ satisfying

$$x^* = ak(z^* + v^*),$$
$$0 = ak(y^* + p^*),$$

and

$$\beta = ak\gamma.$$

Thus

$$u^* = \lambda x^* = \lambda ak(z^* + v^*),$$
$$-1 = \lambda\beta = \lambda ak\gamma,$$

and

$$\lambda aky^* = -\lambda akp^*.$$

Hence

$$(\lambda akz^*, \lambda aky^*) \in \hat{\partial}_G f(x_0, y_0)$$

and

$$u^* \in (\lambda akz^*) + \hat{D}_G F(x_0, y_0)(\lambda aky^*).$$

The proof of the second part is similar to that of the first one. ∎

It is easy to obtain a number of corollaries of Theorem 5.1 by considering certain special forms for f and F. In particular, the theorem implies the following result which was first obtained in Mordukhovich [20–22] in finite dimensions.

COROLLARY 5.4. *Let the hypotheses of Theorem* 5.1 *be satisfied with* $f(x, y) = g(y)$. *Then*

$$\hat{\partial}_G m(x_0) \subset \bigcup_{y_0 \in M(x_0)} \bigcup_{y^* \in \hat{\partial}_G g(y_0)} \hat{D}^*_G F(x_0, y_0)(y^*)$$

and

$$\hat{\partial}_G^{\infty} m(x_0) \subset \bigcup_{y_0 \in M(x_0)} \bigcup_{y^* \in \hat{\partial}_G^{\infty} g(y_0)} \hat{D}_G^* F(x_0, y_0)(y^*).$$

In the case where F is a single-valued function we obtain the following generalized chain rules.

THEOREM 5.5. *Let F be a single-valued function from X into Y which is continuous around x_0 and let g be an extended-real-valued function on Y which is lower semicontinuous around $y_0 = F(x_0) \in \operatorname{dom} g$. Suppose that one of the following conditions holds:*

(i′) $X \times \operatorname{epi} g$ *and* $\operatorname{Gr} F \times \mathbb{R}$ *are strongly* $\hat{\partial}_G$*-independent at* $(x_0, y_0, g(y_0))$,

(ii′) $\operatorname{epi} g$ *is compactly epi-Lipschitzian at* $(y_0, g(y_0))$ *and* $X \times \operatorname{epi} g$ *and* $\operatorname{Gr} F \times \mathbb{R}$ *are* $\hat{\partial}_G$*-independent at* $(x_0, y_0, g(y_0))$,

(iii′) $\operatorname{Gr} F$ *is compactly epi-Lipschitzian at* (x_0, y_0) *and* $X \times \operatorname{epi} g$ *and* $\operatorname{Gr} F \times \mathbb{R}$ *are* $\hat{\partial}_G$*-independent at* $(x_0, y_0, g(y_0))$.

Then

$$\hat{\partial}_G(g \circ F)(x_0) \subset \bigcup_{y^* \in \hat{\partial}_G g(y_0)} \hat{D}_G^* F(x_0, y_0)(y^*)$$

and

$$\hat{\partial}_G^{\infty}(g \circ F)(x_0) \subset \bigcup_{y^* \in \hat{\partial}_G^{\infty} g(y_0)} \hat{D}_G^* F(x_0, y_0)(y^*).$$

Remark. Theorem 5.5 implies the estimates

$$\hat{\partial}_G(g \circ F)(x_0) \subset \bigcup_{y^* \in \hat{\partial}_G g(y_0)} \hat{\partial}_G(y^* \circ F)(x_0)$$

and

$$\hat{\partial}_G^{\infty}(g \circ F)(x_0) \subset \bigcup_{y^* \in \hat{\partial}_G^{\infty} g(y_0)} \hat{\partial}_G(y^* \circ F)(x_0)$$

whenever F belongs to the large class of strongly compactly Lipschitzian functions [15], since following Jourani and Thibault [16],

$$(x^*, -y^*) \in \hat{N}_G(\operatorname{Gr} F, x_0, F(x_0)) \Leftrightarrow x^* \in \hat{\partial}_G(y^* \circ F)(x_0).$$

In particular, when F is strictly differentiable at x_0 (that is, when

$$\lim_{\substack{x \to x_0 \\ x' \to x_0}} \frac{\|F(x) - F(x') - \nabla F(x_0)(x - x')\|}{\|x - x'\|} = 0,$$

where $\nabla F(x_0)$ is the Fréchet derivative of F at x_0), we have both

$$\hat{\partial}_G(g \circ F)(x_0) \subset \hat{\partial}_G g(y_0) \circ \nabla F(x_0)$$

and

$$\hat{\partial}_G^\infty(g \circ F)(x_0) \subset \hat{\partial}_G^\infty g(y_0) \circ \nabla F(x_0).$$

COROLLARY 5.6. *Let F and g be as in Theorem 5.5, with* epi g *compactly epi-Lipschitzian at $(y_0, g(y_0))$. Suppose that*

$$y^* \in \hat{\partial}_G^\infty g(y_0) \quad \text{and} \quad 0 \in \hat{D}_G F(x_0, y_0)(y^*) \Rightarrow y^* = 0.$$

Then the results of Theorem 5.5 are valid.

Proof. It suffices to show that the sets $C_1 := X \times \operatorname{epi} g$ and $C_2 := \operatorname{Gr} F \times \mathbb{R}$ are $\hat{\partial}_G$-independent. So let $(x^*, y^*, r) \in \hat{\partial}_G d(x_0, y_0, g(y_0), C_1) \cap (-\hat{\partial}_G d(x_0, y_0, g(y_0), C_2))$. As

$$\hat{\partial}_G d(x_0, y_0, g(y_0), C_1) = \{0\} \times \hat{\partial}_G d(y_0, g(y_0), \operatorname{epi} g)$$

and

$$\hat{\partial}_G d(x_0, y_0, g(y_0), C_2) = \hat{\partial}_G d(x_0, y_0, \operatorname{Gr} F) \times \{0\},$$

we have $x^* = 0$, $r = 0$. Thus,

$$y^* \in \hat{\partial}_G^\infty g(x_0) \quad \text{and} \quad 0 \in \hat{D}_G F(x_0, y_0)(y^*)$$

which implies by assumption that $y^* = 0$, and the proof is complete. ∎

Remark. Corollary 5.6 was first obtained by Ioffe [11, Theorem 7.5], under the hypothesis that g is directionally Lipschitz at y_0 or, equivalently, that epi g is epi-Lipschitz [24] at $(y_0, g(y_0))$. But (following Borwein [3]) every epi-Lipschtiz set is compactly epi-Lipschitzian and in finite dimensions each subset is compactly epi-Lipschitzian at all its points, and this is not the case for the epi-Lipschitz sets.

As a consequence of Theorem 5.5 we obtain the following chain rule for Clarke's subdifferential.

COROLLARY 5.7. *Under the assumptions of Theorem* 5.5 *we have*

$$\partial_C(g \circ F)(x_0) \subset \mathrm{cl}^* \bigcup_{y^* \in \partial_C g(y_0)} D_C^* F(x_0, y_0)(y^*).$$

Proof. Let $x^* \in \hat{\partial}_G(g \circ F)(x_0)$ and $u^* \in \hat{\partial}_G^\infty(g \circ F)(x_0)$. Then, by Theorem 5.5, there exist $y^* \in \hat{\partial}_G g(y_0)$ and $v^* \in \hat{\partial}_G^\infty g(y_0)$ such that

$$(x^*, -y^*) \in \hat{N}_G(\mathrm{Gr}\, F, x_0, y_0), \qquad (u^*, -v^*) \in \hat{N}_G(\mathrm{Gr}\, F, x_0, y_0).$$

As $\hat{N}_G(\mathrm{Gr}\, F, x_0, y_0) \subset N_C(\mathrm{Gr}\, F, x_0, y_0)$ and $N_C(\mathrm{Gr}\, F, x_0, y_0)$ is a closed convex cone it follows that

$$(x^* + u^*, -(y^* + v^*)) \in N_C(\mathrm{Gr}\, F, x_0, y_0)$$

and, hence,

$$x^* + u^* \in D_C^* F(x_0, y_0)(y^* + v^*), \qquad y^* + v^* \in \hat{\partial}_G g(y_0) + \hat{\partial}_G^\infty g(y_0).$$

Invoking (2.2), we conclude that

$$\partial_C(g \circ F)(x_0) \subset \mathrm{cl}^* \bigcup_{y^* \in \partial_C g(y_0)} D_C^* F(x_0, y_0)(y^*). \blacksquare$$

To complete this section we consider F of the special form

$$F(x) = \{y \in C\colon g(x, y) \in D\}, \tag{5.4}$$

where $g\colon X \times Y \to Z$ is a Lipschitz function at all points (x_0, y_0) with $y_0 \in F(x_0)$, Z is a Banach space, and C and D are two closed subsets of Y and Z, respectively. Suppose that there exist two neighbourhoods V and W of x_0 and y_0, respectively, and a real number $a > 0$ such that

$$d(y, F(x)) \leq a d(g(x, y), D) \tag{5.5}$$

for all $x \in V$ and $y \in W \cap C$. Then there exists $k > 0$ such that

$$d(x, y; \operatorname{Gr} F) \leq k\,[d(g(x, y), D) + d(y, C)]$$

for all $x \in V$ and $y \in W$. Thus, using (2.1), we get

$$\begin{aligned}\hat{\partial}_G d(x_0, y_0; \operatorname{Gr} F) &\subset k\hat{\partial}(d(g(\cdot, \cdot), D) + d(\cdot, C))(x_0, y_0)\\ &\subset \bigcup_{z^* \in \hat{N}_G(D, g(x_0, y_0))} \hat{D}_G g(x_0, y_0)(z^*) + \{0\} \times \hat{N}_G(C, y_0).\end{aligned}$$

These arguments establish the following corollary of Theorem 5.1.

COROLLARY 5.8. *Let f be as in Theorem* 5.1, *let F be given by* (5.4), *and let* (5.2) *hold. Suppose that* (5.5) *holds for all* $y_0 \in M(x_0)$. *Then*

$$\begin{aligned}\hat{\partial}_G m(x_0) \subset \bigcup_{y_0 \in M(x_0)} \bigcup_{(x^*, y^*) \in \hat{\partial}_G f(x_0, y_0)} &\{x^* + u^* \colon (u^*, -y^*)\\ &\in \bigcup_{z^* \in \hat{N}_G(D, g(x_0, y_0))} \hat{D}_G^* g(x_0, y_0)(z^*) + \{0\} \times \hat{N}_G(C, y_0)\}.\end{aligned}$$

6. Comparison of Independence Conditions

In finite dimensions, Aubin and Ekeland [1] establish (1.1) under the assumption

$$T_C(C_1, x_0) - T_C(C_2, x_0) = X. \tag{6.1}$$

Their result can easily be extended to infinite dimensions and without any additional effort we can prove the following corollary of Theorem 4.4.

COROLLARY 6.1. *Let* C_1 *and* C_2 *be two closed subsets of* X, *with* C_1 *compactly epi-Lipschitzian at* $x_0 \in C_1 \cap C_2$. *Then* (6.1) *implies* (1.1). *Therefore,*

$$N_C(C_1 \cap C_2, x_0) \subset N_C(C_1, x_0) + N_C(C_2, x_0). \tag{6.2}$$

Proof. It is not difficult to show that (6.1) implies the $\hat{\partial}_G$-independence of C_1 and C_2. So using Corollary 4.6 and the fact that $N_C(C_i, x_0) = \mathrm{cl}^*(\mathbb{R}_+ \partial_C d(x_0, C_i))$ (see [5]) we get

$$N_C(C_1 \cap C_2, x_0) \subset \mathrm{cl}^*[N_C(C_1, x_0) + N_C(C_2, x_0)].$$

Theorem 6.3 in Borwein [3] implies that

$$\mathrm{cl}^*[N_C(C_1, x_0) + N_C(C_2, x_0)] = N_C(C_1, x_0) + N_C(C_2, x_0)$$

and, hence,

$$N_C(C_1 \cap C_2, x_0) \subset N_C(C_1, x_0) + N_C(C_2, x_0). \quad \blacksquare$$

Remark. Corollary 6.1 is established in [28] under the hypothesis that C_1 and C_2 are epi-Lipschitz-like at x_0. But every epi-Lipschitz-like set is compactly epi-Lipschitz and the converse cannot hold (see Example 4.1 in [3]).

In infinite dimensions, Rockafellar [24] shows that the assumption

$$\begin{array}{l} \operatorname{int} T_C(C_1, x_0) \cap T_C(C_2, x_0) \neq \emptyset \\ C_1 \text{ is epi-Lipschitzian at } x_0 \end{array} \tag{6.3}$$

implies (6.2).

The following lemma shows that our $\hat{\partial}_G$-independence condition is weaker than (6.1), (6.3), and the ∂_C-independence condition.

LEMMA 6.2. *The following implications hold:*

$$\begin{aligned} (6.3) \Rightarrow (6.1) &\Rightarrow \partial_C d(x_0, C_1) \cap (-\partial_C d(x_0, C_2)) = \{0\} \\ &\Rightarrow \hat{N}_G(C_1, x_0) \cap (-\hat{N}_G(C_2, x_0)) = \{0\} \\ &\Leftrightarrow \hat{\partial}_G d(x_0, C_1) \cap (-\hat{\partial}_G d(x_0, C_2)) = \{0\}. \end{aligned}$$

Proof. The implication (6.3) $\Rightarrow$ (6.1) is immediate.

For (6.1) $\Rightarrow \partial_C d(x_0, C_1) \cap (-\partial_C d(x_0, C_2)) = \{0\}$, it suffices to see that (6.1) $\Rightarrow N_C(C_1, x_0) \cap (-N_C(C_2, x_0)) = \{0\}$ and $\partial_C d(x_0, C_i) \subset N_C(C_i, x_0)$.

Finally, the ∂_C-independence condition implies the $\hat{\partial}_G$-independence condition, since $\hat{\partial}_G d(x_0, C_i) \subset \partial_C d(x_0, C_i)$. $\blacksquare$

The following example shows that the implication (6.3) $\Rightarrow$ (6.1) is strict.

EXAMPLE 6.3. Take $C_1 = \{0\} \times \mathbb{R}$ and $C_2 = \mathbb{R} \times \{0\}$ and $x_0 = (0, 0)$. Then (6.1) holds while $\operatorname{int} T_C(C_1, x_0) = \operatorname{int} T_C(C_2, x_0) = \emptyset$.

The implication (6.1) $\Rightarrow \partial_C d(x_0, C_1) \cap (-\partial_C d(x_0, C_2)) = \{0\}$ is also strict.

EXAMPLE 6.4. [6, 23]. Let $C_1 = [-1, 1] \times \{0\} \times [-1, 1]$ and $C_2 = \operatorname{epi} f$, where

$$f(x, y) = \begin{cases} y^2/2|x|, & \text{if } x \neq 0 \\ 0, & \text{if } x = y = 0 \\ \infty, & \text{if } x = 0, y \neq 0, \end{cases}$$

and let $x_0 = (0, 0, 0)$. Then [23]

$$\partial_C d(x_0, C_1) = \{0\} \times [-1, 1] \times \{0\},$$

$$\partial_C d(x_0, C_2) = \operatorname{conv}\left[\left\{(x, y, z) \in B_{\mathbb{R}^3}: z \leq \frac{-y}{2|x|}, x \neq 0\right\}\right.$$
$$\left.\cup \{(x, y, z) \in B_{\mathbb{R}^3}: x = y = 0, z \leq 0\}\right],$$
$$N_C(C_1, x_0) = \{0\} \times \mathbb{R} \times \{0\},$$

and

$$N_C(C_2, x_0) = \{(x, y, z) \in \mathbb{R}^3: z \leq 0\}.$$

So $\partial_C d(x_0, C_1) \cap (-\partial_C d(x_0, C_2)) = \{0\}$ while $N_C(C_1, x_0) \cap (-N_C(C_2, x_0)) = \{0\} \times \mathbb{R} \times \{0\}$.

Finally we give an example in which the converse of the implication $\partial_C d(x_0, C_1) \cap (-\partial_C d(x_0, C_2)) = \{0\} \Rightarrow \hat{\partial}_G d(x_0, C_1) \cap (-\hat{\partial}_G d(x_0, C_2)) = \{0\}$ does not hold.

EXAMPLE 6.5. Let $C_1 = \{0\} \times \mathbb{R}$, $C_2 = \{(x, y) \in R^2: y = |x|\}$, and $x_0 = (0, 0)$. Then

$$\hat{\partial}_G d(x_0, C_1) = [-1, 1] \times \{0\},$$
$$\hat{\partial}_G d(x_0, C_2) = \{(x, y) \in B_{\mathbb{R}^2}: y = |x|\} \cup \{(x, y) \in B_{\mathbb{R}^2}: y \leq -|x|\},$$
$$\partial_C d(x_0, C_1) = \operatorname{conv} \hat{\partial}_G d(x_0, C_1) = [-1, 1] \times \{0\},$$

and

$$\partial_C d(x_0, C_2) = B_{\mathbb{R}^2}.$$

So $\hat{\partial}_G d(x_0, C_1) \cap (-\hat{\partial}_G d(x_0, C_2)) = \{(0, 0)\}$ while $\partial_C d(x_0, C_1) \cap (-\partial_C d(x_0, C_2)) = [-1, 1] \times \{0\}$.

ACKNOWLEDGMENTS

The author expresses his appreciation to the anonymous referees for careful reading and many helpful comments that improved the original manuscript. He also thanks Philip D. Loewen for having helped him to improve the quality of the English in the paper.

REFERENCES

1. J. P. AUBIN AND I. EKELAND, "Applied Nonlinear Analysis," Wiley-Interscience, New York, 1984.
2. J. M. BORWEIN, Stability and regular point of inequality systems, *J. Optim. Theory Appl.* **48,** 1986, 9–52.
3. J. M. BORWEIN, Epi-lipschitz-like sets in Banach space: Theorems and examples, *Nonlinear Anal.* **11** (1987), 1207–1217.
4. J. M. BORWEIN AND H. M. STROJWAS, Tangential approximations, *Nonlinear Anal.* **9** (1985), 1347–1366.
5. F. H. CLARKE, "Optimization and Nonsmooth Analysis," Wiley, New York, 1983.
6. F. H. CLARKE AND N. RAÏSSI, Formules d'intersection en analyse non lisse, *Ann. Sci. Math. Québec* **14** (1990), 121–129.
7. I. EKELAND, Nonconvex minimization problems, *Bull. Amer. Math. Soc.* **1** (1979), 443–474.
8. J.-B. HIRIART-URRUTY, Gradients généralisés de fonctions marginales, *SIAM J. Control Optim.* **16** (1978), 301–316.
9. A. D. IOFFE, Approximate subdifferentials and applications 1: The finite dimensional theory, *Trans. Amer. Math. Soc.* **281** (1984), 289–316.
10. A. D. IOFFE, Approximate subdifferentials and applications 2: Functions on locally convex spaces, *Mathematika* **33** (1986), 111–128.
11. A. D. IOFFE, Approximate subdifferentials and applications 3: Metric theory, *Mathematika* **36** (1989), 1–38.
12. A. JOURANI, Regularity and strong sufficient optimality conditions in differentiable optimization problems, *Numer. Funct. Anal. Optim.* **14** (1993), 69–87.
13. A. JOURANI, Metric regularity and second order necessary optimality conditions for minimization problems under inclusion constraints, *J. Optim. Theory Appl.* **81** (1994), 97–120.
14. A. JOURNAI AND L. THIBAULT, The use of metric graphical regularity in approximate subdifferential calculus rule in finite dimensions, *Optimization* **21** (1990), 509–519.
15. A. JOURANI AND L. THIBAULT, Approximations and metric regularity in mathematical programming in Banach space, *Math. Oper. Res.* **18** (1993), 390–401.
16. A. JOURNAI AND L. THIBAULT, Approximate subdifferentials of composite functions, *Bull. Austral. Math. Soc.* **47** (1993), 443–455.
17. A. JOURNAI AND L. THIBAULT, Metric regularity for strongly compactly Lipschitzian mappings, *Nonlinear Anal.* **24** (1995), 229–240.
18. A. JOURNAI AND L. THIBAULT, Extensions of subdifferential calculus rules in Banach spaces and applications, *Can. J. Math.* (1995), to appear.
19. P. D. LOEWEN, Limits of Fréchet normals in nonsmooth analysis, *in* "Optimization and Nonlinear Analysis" (A Ioffe, M. Marcus, and S. Reich, Eds.), pp. 178–188, Pitman Res. Notes Math. Ser., Longman Sci. Tech., Harlow, 1990.
20. B. S. MORDUKHOVICH, Nonsmooth analysis with nonconvex generalized differentials and adjoint mappings, *Dokl. Akad. Nauk BSSR* (1984), 976–979.

21. B. S. MORDUKHOVICH, "Approximation Methods in Problems of Optimization and Control," Nauka, Moscow, 1988.
22. B. S. MORDUKHOVICH, Complete characterization of openness, metric regularity, and Lipschitzian properties of multifunctions, *Tran. Amer. Math. Soc.*, to appear.
23. N. RAÏSSI, "Analyse proximale en optimization," Ph.D. thesis, Université de Montréal, 1987.
24. R. T. ROCKAFELLAR, Directionally Lipschitzian functions and subdifferential calculus, *Proc. London Math. Soc.* **39** (1979), 331–355.
25. R. T. ROCKAFELLAR, Directional differentiability of the optimal value function in a nonlinear programming problem, *Math. Programming, Study* **21** (1984), 213–226.
26. R. T. ROCKAFELLAR, Extension of subgradient calculus rules with applications to optimization, *Nonlinear Anal.* **9** (1985), 665–698.
27. L. THIBAULT, On subdifferentials of optimal value functions, *SIAM J. Control Optim.* **29** (1991), 1019–1036.
28. D. E. WARD, Convex subcones of the contingent cone in nonsmooth calculus and optimization, *Trans. Amer. Math. Soc.* **302** (1987), 661–682.
29. D. E. WARD AND J. M. BORWEIN, Nonsmooth calculus in finite dimensions, *SIAM J. Control Optim.* **25** (1987), 1312–1340.