Intersection Formulae and the Marginal Function in Banach Spaces

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Intersection formulae are central to the development of subdifferential calculus and the differentiation of marginal functions. In this paper, we reexamine the connection between independence conditions and intersection formulae. Then we apply the formulae to a general parametric mathematical programming problem in which the constraints are defined by multivalued functions. These results allow us to obtain generalized chain rules for composite functions. Corollaries of this work include several well-known intersection formulae and calculus rules. © 1995 Academic Press, Inc.

1. INTRODUCTION

Given two subsets \( C_1 \) and \( C_2 \) of a Banach space \( X \) and a point \( x_0 \in C_1 \cap C_2 \), our objective in this paper is to study conditions ensuring the inclusion

\[
\partial d(x_0, C_1 \cap C_2) \subseteq k[\partial d(x_0, C_1) + \partial d(x_0, C_2)],
\]

where \( k \) is a positive real number and \( \partial d(x_0, C) \) is the subdifferential of the distance function \( d(x, C) \) to \( C \) at \( x_0 \) (see Sections 2 and 3). This formula plays an important role in optimization theory, namely in subdifferential calculus rules (see Rockafellar [24], Mordukhovich [20–22], Ward and Borwein [29], Ioffe [10–11], Jourani and Thibault [14, 18]), in the study of necessary optimality conditions (see Clarke [5], Ioffe [9], Raïssi [23], Journai [12]), and in the differentiability of the marginal function (see Rockafellar [25–26], Mordukhovich [20–22], Borwein [2], Thibault [27]).
Several conditions have been proposed to ensure formula (1.1). Some are stated in terms of the subdifferential of the distance function, and others in terms of the tangent cone or the normal cone.

In finite dimensions, Aubin and Ekeland [1, pp. 441] (see also Ward and Borwein [29]) establish formula (1.1) under the condition that the difference of the tangent cones to $C_1$ and $C_2$ at $x_0$ is equal to the space $X$. In Banach spaces, Rockafellar [24] shows that formula (1.1) holds whenever the tangent cone to $C_2$ at $x_0$ meets the interior of the tangent cone to $C_1$ at $x_0$ and $C_1$ is epi-Lipschitzian at $x_0$. Ioffe [11] obtains formula (1.1) under the two assumptions that

$$\partial d(x_0, C_1) \cap (-\partial d(x_0, C_2)) = \{0\}$$

and that $C_1$ is epi-Lipschitzian at $x_0$. In Hilbert spaces, Clarke and Raïssi [6] establish inclusion (1.1) by assuming that $C_1$ is compact and requiring that

for all $\forall \varepsilon > 0 \exists \delta > 0$, $\exists$ neighbourhood $V$ of $x_0$ such that

$$\forall x_i \in C_i \cap V, \forall x_i^* \in \partial d(x_i, C_i) \quad i = 1, 2,$$

satisfying $\|x_i^*\| + \|x_2^*\| \geq \varepsilon$ then $\|x_1^* + x_2^*\| \geq \delta$.

In [28], Ward has extended the results in [1, 29] to the case in which $X$ is a Banach space and $C_1$ and $C_2$ are epi-Lipschitz-like at $x_0$ (see [3–4]). This class of sets includes epi-Lipschitzian sets and finite dimensional sets.

In infinite dimensional spaces most of the criteria ensuring inclusion (1.1) require that either $C_1$ or $C_2$ is epi-Lipschitzian, or $C_1$ and $C_2$ are epi-Lipschitz-like, or they require certain conditions to hold at all points in a neighbourhood of $x_0$. Moreover they are stated in a complicated way. Recently, Jourani and Thibault [18] have shown that condition (1.2) implies formula (1.1) when $C_1$ is compactly epi-Lipschitzian at $x_0$ in the sense of Borwein and Strojwas [4]. Let us note (following Borwein [3]) that the class of compactly epi-Lipschitzian sets includes epi-Lipschitzian sets, epi-Lipschitz-like sets, and finite dimensional sets.

The purposes of this paper are threefold. First, we show that, when $C_1$ is compactly epi-Lipschitzian, condition (1.2) is equivalent to condition (1.3) and both are equivalent to a third condition introduced in Section 3 below. Second, we extend the result of Clarke and Raïssi [6] to general Banach spaces without any compactness assumption on $C_1$. The results obtained generalize those of Ioffe [11] and Rockafellar [24]. Finally we apply our results to estimate the subdifferential of the marginal function $m$ defined by

$$m(x) := \inf \{f(x, y) : y \in F(x)\},$$
where $f$ is an extended-real-valued function on $X \times Y$, $X$ and $Y$ are Banach spaces, and $F$ is a multivalued function from $X$ into $Y$. These results allow us to derive chain rules for nonsmooth functions. Such rules includes sum formulae as well as the calculus rules in [11, 20–22].

Throughout the paper, we will let $X$ and $Y$ be Banach spaces equipped with the norm $\|\|$ and $X^*$ and $Y^*$ their topological dual spaces endowed with the weak-star topology $w^*$. We denote by $\langle \cdot, \cdot \rangle$ the pairing between the space and its topological dual and by $B_X, B_X^*, \ldots$, the closed unit balls of $X, X^*, \ldots$. For an extended-real-valued function $f$ on $X$, we write

$$
\text{epi } f := \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}
$$

and

$$
\text{dom } f := \{x \in X : |f(x)| < +\infty\}.
$$

For a multivalued function $F$ from $X$ into $Y$, we denote the graph of $F$ by $\text{Gr } F$, that is,

$$
\text{Gr } F = \{(x, y) \in X \times Y : y \in F(x)\}.
$$

The abbreviations for interior, weak-star closure, and convex hull are "int," "cl\*," and "conv." The distance function $d(x, S)$ to a given set $S$ is

$$
d(x, S) = \inf_{u \in S} \|x - u\|.
$$

We write $x \overset{f}{\to} x_0$ and $x \overset{S}{\to} x_0$ to express $x \to x_0$ with $f(x) \to f(x_0)$ and $x \to x_0$ with $x \in S$, respectively. Finally, if not specified, the norm in a product of two Banach spaces is defined by $\|(u, v)\| = \|u\| + \|v\|$.

2. Preliminaries

In this paper we will use the notations and definitions in Ioffe [10–11] and Clarke [5]. Let $f$ be a function from $X$ into $\mathbb{R} := \mathbb{R} \cup \{-\infty, +\infty\}$ and let $\varepsilon > 0$. We write for any subset $S$ of $X$

$$
f_S(x) = \begin{cases} f(x), & \text{if } x \in S, \\ +\infty, & \text{otherwise}. \end{cases}
$$
Recall that for $x \in \text{dom } f$ the lower Dini directional derivative, the Dini subdifferential and the Dini $\varepsilon$-subdifferential of $f$ at $x$ are defined by

$$d^{-}f(x, h) = \liminf_{t \downarrow 0} \frac{f(x + th) - f(x)}{t}$$

$$\partial^{-}f(x) = \{x^* \in X^*: \langle x^*, h \rangle \leq d^{-}f(x, h), \forall h \in X\}.$$

and

$$\partial^{-}_\varepsilon f(x) = \{x^* \in X^*: \langle x^*, h \rangle \leq d^{-}f(x, h) + \varepsilon\|h\|, \forall h \in X\}.$$

If $x \not\in \text{dom } f$ we put $\partial^{-}f(x) = \partial^{-}_\varepsilon f(x) = \emptyset$.

Let $\mathcal{F}(X)$ be the family of all finite dimensional subspaces of $X$ and let $f$ be a lower semicontinuous function at $x_0 \in \text{dom } f$. The $A$-subdifferential of $f$ at $x_0$ (see Ioffe [10–11]) is defined by

$$\partial_A f(x_0) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{x \rightarrow x_0} \partial^{-} f_{x+L}(x),$$

where

$$\limsup_{x \rightarrow x_0} \partial^{-} f_{x+L}(x) = \{x^* \in X^*: x^* = w^* - \lim x_i^*, x_i^* \in \partial^{-} f_{x_i+L}(x_i), x_i \rightarrow x_0\},$$

that is, the set of weak-star limits of all such nets.

The normal cone generated by $\partial_A d(x_0, S)$ is denoted by $\hat{N}_G(S, x_0)$, that is,

$$\hat{N}_G(S, x_0) = \bigcup_{\lambda > 0} \lambda \partial_A d(x_0, S).$$

The $\hat{G}$-subdifferential $\hat{G}_f(x_0)$ of $f$ at $x_0 \in \text{dom } f$ and its singular counterpart $\hat{G}^\varepsilon_f(x_0)$ are given by (see [10–11])

$$\hat{G}_f(x_0) = \{x^* \in X^*: (x^*, -1) \in \hat{N}_G(\text{epi } f; x_0, f(x_0))\},$$

$$\hat{G}^\varepsilon_f(x_0) = \{x^* \in X^*: (x^*, 0) \in \hat{N}_G(\text{epi } f; x_0, f(x_0))\}.$$

Following Clarke [5], a vector $h \in X$ will be in the Clarke tangent cone $T_C(S, x_0) \cap S$ at $x_0$ if for any sequence $(x_n) \subset S$ converging to $x_0$ and any $t_n \rightarrow 0^+$ there exists $h_n \rightarrow h$ such that for all positive integers $n$,

$$x_n + t_n h_n \in S.$$
The Clarke normal cone $N_C(S, x_0)$ to $S$ at $x_0$ is the polar of $T_C(S, x_0)$; i.e.,

$$N_C(S, x_0) = \{ x^* \in X^* : (x^*, x) \leq 0, \forall x \in T_C(S, x_0) \}.$$  

Clarke's subdifferential of $f$ at $x_0$ is given by

$$\partial_C f(x_0) = \{ x^* \in X^* : (x^*, -1) \in N_C(\text{epi } f; x_0, f(x_0)) \}.$$  

Suppose finally that $F$ is a multivalued function from $X$ into $Y$. The $\hat{G}$-coderivative and the Clarke coderivative of $F$ at $(x_0, y_0) \in \text{Gr } F$, respectively, are the multivalued functions of $y^* \in Y^*$ defined by

$$\hat{D}_G F(x_0, y_0)(y^*) = \{ x^* \in X^* : (x^*, -y^*) \in \hat{N}_G(\text{Gr } F, x_0, y_0) \}$$  

and

$$D_C F(x_0, y_0)(y^*) = \{ x^* \in X^* : (x^*, -y^*) \in N_C(\text{Gr } F, x_0, y_0) \}.$$  

The following theorem lists some of the important properties of the $\hat{G}$-subdifferential.

**Theorem 2.1.** [10–11]. Let $f$ be an extended-real-valued function on $X$ which is lower semicontinuous around $x_0 \in \text{dom } f$ and let $S$ be a closed subset of $X$ containing $x_0$:

$$\partial_A d(x_0, S) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{x \to x_0 \in x \to x_0} \partial^- d_{x+L}(x, S), \quad (2.1)$$

$$\partial_C f(x_0) = \text{cl}^* \text{conv}(\hat{G}_f(x_0) + \hat{G}_f^e(x_0)). \quad (2.2)$$

If $f$ is Lipschitz near $x_0$ with Lipschitz constant $k_f$,

$$\hat{G}_f(x_0) = \partial_A f(x_0) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{x \to x_0 \in x \to x_0} [\partial^- f_{x+L}(x) \cap (k_f + \varepsilon) B_x^+], \quad (2.3)$$

$$\hat{G}_f(x_0) = \limsup_{x \to x_0} \hat{G}_f(x) \quad \text{(upper semicontinuity condition).} \quad (2.4)$$

A thorough discussion of these concepts can be found in [10–11, 5]. We mention here that if $f$ is convex, then the $\hat{G}$-subdifferential coincides with the subdifferential of convex analysis. The singular $\hat{G}$-subdifferential reduces to zero whenever $f$ is Lipschitz near $x_0$. 

Following Borwein and Strojwas [4], we call a set $S \subset X$ compactly epi-Lipschitzian at $x_0 \in S$ if there exist a norm-compact set $K$ and a scalar $r > 0$ such that

$$S \cap (x_0 + rB_X) + \text{tr } B_X \subset S - tK, \quad \forall t \in ]0, r[.$$

The following proposition gives us an important property of such sets.

**Proposition 2.2.** [17]. If $S$ is compactly epi-Lipschitzian at $x_0 \in S$, then there exist $k_1, \ldots, k_m \in X$ and $r > 0$ such that

$$\|x^*\| \leq \max_{i=0,\ldots,m} |\langle x^*, k_i \rangle|$$

for all $x \in x_0 + rB_X$ and $x^* \in \delta_G d(x, S)$.

**Proof.** Let $K$ be a compact subset of $X$ and $r \in ]0, 1[$ such that

$$S \cap (x_0 + 3rB_X) + \text{tr } B_X \subset S - tK, \text{ for all } t \in ]0, r[.$$

Choose an open neighbourhood $V$ of $x_0$ with $V \subset x_0 + rB_X$ and $d(x, S) \leq r$ for all $x \in V$. For all $\varepsilon \in ]0, \frac{1}{2}[$ there exist $h_1, \ldots, h_m \in K$ such that

$$K \subset \bigcup_{i=1}^m (h_i + \varepsilon rB_X).$$

For each $x \in V$ and each $t \in ]0, r]$ we may select some $p(x, t) \in S$ such that

$$\|x - p(x, t)\| \leq d(x, S) + t^2. \quad (2.5)$$

By the choice of $V$ we have for each $x \in V$ and each $t \in ]0, r]$

$$\|x_0 - p(x, t)\| \leq \|x - p(x, t)\| + \|x - x_0\|$$

$$< d(x, S) + t^2 + r < 3r$$

and, hence, $p(x, t) \in x_0 + 3rB_X$. Fix any $x \in V$ and any $x^* \in \partial_G d(x, S)$. Fix any $b \in B_X$ and any $L \in \mathcal{F}(X)$ with $\{b, h_1, \ldots, h_m\} \subset L$. We may write $x^* = w^* - \lim_{j \in J} x_j^*$ with $x_j^* \in \partial - d_{S_j + L}(x_j, S)$ and $x_j \rightarrow x$. Choose $j_0 \in J$ such that $x_j \in V$ for any $j \in J, j \geq j_0$. Let $(t_n) \subset ]0, r]$ converging to zero. For each $n \in \mathbb{N}$ and each $j \in J, j \geq j_0$ choose $h_{(n,j)} \in \{h_1, \ldots, h_m\}$ and $b_{n,j} \in B_X$ with

$$p(x_j, t_n) + t_n(rb + h_{(n,j)} + \varepsilon rb_{n,j}) \in S. \quad (2.6)$$
For each $j \in J$, $j \geq j_0$, we may suppose that $h_{i(n,j)} = h_{q(j)}$, for all $n \in \mathbb{N}$. Then for each $j \in J$, $j \geq j_0$ (recalling that $d_{x_j + L}(x, S) = d(x, S)$ if $x \in x_j + L$ and $d_{x_j + L}(x, S) = +\infty$ if $x \not\in x_j + L$), we have

\[ t_n^{-1}[d_{x_j + L}(x_j + t_n(rb + h_{q(j)}), S) - d_{x_j + L}(x_j, S)] \]
\[ = t_n^{-1}[d(x_j + t_n(rb + h_{q(j)}), S) - d(x_j, S)] \]
\[ \leq t_n^{-1}[d(p(x_j, t_n) + t_n(rb + h_{q(j)}), S) + \|x_j - p(x_j, t_n)\|] - d(x_j, S)] \]
\[ \leq t_n^{-1}[d(p(x_j, t_n) + t_n(rb + h_{q(j)}), S) + t_n \quad \text{(by (2.5))} \]
\[ \leq t_n + er\|b_{n,j}\| + t_n^{-1}d(p(x_j, t_n) + t_n(rb + h_{q(j)} + erb_{n,j}), S) \]
\[ \leq t_n + er \quad \text{(by (2.6))}. \]

So for each $j \geq j_0$ we have

\[ d^{-1}d_{x_j + L}(\cdot, S)(x_j; rb + h_{q(j)}) \leq er \]

which implies $\langle x_j^*, rb + h_{q(j)} \rangle \leq er$ and, hence,

\[ r(x_j^*, b) \leq er + \max_{i=1,\ldots,m} |\langle x_j^*, h_i \rangle|. \]

Therefore we get for each $\varepsilon \in ]0, \frac{1}{2}[$ and each $b \in B_X$

\[ \langle x^*, b \rangle \leq \varepsilon + r^{-1} \max_{i=1,\ldots,m} |\langle x^*, h_i \rangle| \]
\[ \leq \varepsilon + r^{-1} \sup_{h \in K} |\langle x^*, h \rangle| \]

and, hence,

\[ \|x^*\| \leq \frac{1}{r(1 - \varepsilon)} \max_{i=1,\ldots,m} |\langle x^*, h_i \rangle| \]

and the proposition is proved. \( \blacksquare \)

**Remarks.** (1) It follows from this proposition that in $\delta_G d(x, S)$, weak-star and strong convergences of nets to zero are equivalent, that is,

\[ x_j^* \overset{w^*}{\to} 0 \quad \text{iff} \|x_j^*\| \to 0. \]
(2) Loewen [19] has shown that in weakly locally compact cones (of reflexive Banach spaces), weak-star and strong convergences of sequences to zero are equivalent.

3. Independence Conditions

**Definition 3.1.** Let \( C_1 \) and \( C_2 \) be two nonempty subsets of \( X \) and let \( x_0 \in C_1 \cap C_2 \). Let \( \partial = \hat{\partial}_G \) or \( \partial = \partial C \).

(a) \( C_1 \) and \( C_2 \) and \( \partial \)-independent at \( x_0 \) [11] if (1.2) holds for \( \partial \).

(b) \( C_1 \) and \( C_2 \) are strongly \( \partial \)-independent at \( x_0 \) [6] if (1.3) holds for \( \partial \).

**Remark.** Since the approximate subdifferential of Lipschitz functions is always a subset of the Clarke subdifferential, \( \partial_G \)-independence implies \( \hat{\partial}_G \)-independence, and this implication can be strict (see Example 6.5).

**Proposition 3.2.** Let \( C_1 \) and \( C_2 \) be two nonempty subsets of \( X \), with \( C_1 \) compactly epi-Lipschitzian at \( x_0 \in C_1 \cap C_2 \). Then the following assertions are equivalent:

(i) For all \( \varepsilon > 0 \) there exists \( \delta > 0 \) and a neighbourhood \( V \) of \( x_0 \) such that for all \( x_i \in V \), \( x_i^* \in \hat{\partial}_G d(x_i, C_i) \), \( i = 1, 2 \), satisfying \( \| x_i^* \| + \| x_i^* + x_i^* \| \geq \varepsilon \) then \( \| x_i^* + x_i^* \| \geq \delta \).

(ii) \( C_1 \) and \( C_2 \) are \( \hat{\partial}_G \)-independent at \( x_0 \).

(iii) \( C_1 \) and \( C_2 \) are \( \partial_G \)-independent at \( x_0 \).

**Proof.** The implications (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) are obvious.

(iii) \( \Rightarrow \) (i) Suppose that (i) is false. Then there are \( \varepsilon > 0 \), \( u_n \to x_0 \), \( v_n \to x_0 \), \( u_n^* \in \hat{\partial}_G d(u_n, C_1) \), and \( v_n^* \in \hat{\partial}_G d(v_n, C_2) \) such that

\[
\| u_n^* \| + \| v_n^* \| \geq \varepsilon \quad \text{and} \quad \| u_n^* + v_n^* \| \leq 1/n. \tag{3.1}
\]

Since the function \( d(x, C) \) is Lipschitz with constant 1, the sequences \( (u_n^*) \) and \( (v_n^*) \) are bounded. So, extracting subnets if necessary, we may suppose that \( u_n^* \to u^* \) and \( v_n^* \to -u^* \). By (2.4),

\[
u^* \in \hat{\partial}_G d(C_1, x_0) \cap (-\hat{\partial}_G d(C_2, x_0)).
\]

We shall arrive at a contradiction with (iii) if we show that \( u^* \neq 0 \). So suppose that \( u^* = 0 \). Then Proposition 2.2 implies the existence of \( k_1, \ldots, k_m \) in \( X \) (not depending on \( n \)) satisfying

\[
\| u_n^* \| \leq \max_{i=1, \ldots, m} | \langle u_n^*, k_i \rangle |.
\]
Thus \( \|u_n^*\| \to 0 \). Now, by (3.1), \( \|u_n^*\| \to 0 \) (because \(-1/n + \|u_n^*\| \leq \|u_n^*\| \leq \|u_n^*\| + 1/n\)) and so \( \varepsilon \leq 0 \) which leads to the contradiction and terminates the proof.

**Remark.** Assuming that \( C_1 \) is epi-Lipschitz-like at \( x_0 \) in the sense of Borwein [3] and using the result of Jourani and Thibault [18], instead of Proposition 2.2, we may show that the result of Proposition 3.2 is valid if we replace \( \hat{\partial}_G \) by \( \partial_C \).

To close this section let us give an other characterization of the \( \hat{\partial}_G \)-independence condition.

**Proposition 3.3.** Let \( C_1 \) and \( C_2 \) be two closed subsets of \( X \), with \( x_0 \in C_1 \cap C_2 \). Then \( C_1 \) and \( C_2 \) are \( \hat{\partial}_G \)-independent at \( x_0 \) iff

\[
\hat{N}_G(C_1, x_0) \cap (-\hat{N}_G(C_2, x_0)) = \{0\}.
\]

**Proof.** First note that if \( t \in [0, 1] \) and \( u^* \in \hat{\partial}_G d(x_0, C) \) then \( tu^* \in \hat{\partial}_G d(x_0, C) \). Let \( x^* \in [\hat{N}_G(C_1, x_0) \cap (-\hat{N}_G(C_2, x_0))] \). Then there are \( r > 0 \) and \( s > 0 \) such that \( x^*/r \in \hat{\partial}_G d(x_0, C_1) \) and \( -x^*/s \in \hat{\partial}_G d(x_0, C_2) \). Suppose, for example, that \( r \geq s \). Then \(-sx^*/rs \in \hat{\partial}_G d(x_0, C_2) \) and, hence,

\[
x^*/r \in \hat{\partial}_G d(x_0, C_1) \cap (-\hat{\partial}_G d(x_0, C_2))
\]

which implies that \( x^* = 0 \).

The converse is immediate, since \( \hat{\partial}_G d(x_0, C) \subset \hat{N}_G(C, x_0) \).

4. Intersection Formulae

We start this section by establishing the following important result which represents the key to the proof of our intersection theorem.

**Proposition 4.1** Let \( C_1 \) and \( C_2 \) be two closed subsets of \( X \), with \( x_0 \in C_1 \cap C_2 \). Suppose that one of the following assumptions is satisfied:

(i) \( C_1 \) and \( C_2 \) are strongly \( \hat{\partial}_G \)-independent at \( x_0 \), or

(ii) \( C_1 \) is compactly epi-Lipschitzian at \( x_0 \) and \( C_1 \) and \( C_2 \) are \( \hat{\partial}_G \)-independent at \( x_0 \).

Then there exist \( r > 0 \) and \( a > 0 \) such that

\[
d(x, C_2 \cap (C_1 - y)) \leq a \, d(x + y, C_1) \tag{4.1}
\]

for all \( x \in C_2 \cap (x_0 + rB_X) \) and \( y \in rB_X \). In particular,
$$d(x, C_2 \cap C_1) \leq (a + 1)(d(x, C_1) + d(x, C_2))$$

for all $x \in x_0 + rB_x$.

Proof. Suppose, for example, that (i) holds. Consider the multi-valued function

$$F(x) = \begin{cases} -x + C_1, & \text{if } x \in C_2 \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then (4.1) is equivalent to

$$d(x, F^{-1}(y)) \leq a d(y, F(x))$$

for all $x \in x_0 + rB_x$ and $y \in rB_x$, since $F^{-1}(y) = C_2 \cap (C_1 - y)$. Suppose that (4.1) is false. Then, by Lemma 1.2 of Jourani [12], there are $s_n \downarrow 0$, with $s_n \leq 1$, $x_n \to x_0$, $z_n \to 0$, $y_n \to 0$, and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$(x_n, z_n) \in \text{Gr } F, \quad y_n \not\in F(x_n), \quad (4.2)$$

and for all $(x, y) \in \text{Gr } F$,

$$\|z_n - y_n\| \leq \|y - y_n\| + s_n(\|x - x_n\| + \|y - z_n\|)$$

and, hence, the function

$$(x, y) \to \|y - y_n\| + 2d(x, y; \text{Gr } F) + s_n(\|x - x_n\| + \|y - z_n\|)$$

attains a local minimum at $(x_n, z_n)$. Since for all $x \in C_2$

$$d(x, y; \text{Gr } F) \leq d(y, F(x)) = d(x + y, C_1),$$

Proposition 2.4.3 in Clarke [5] implies that $(x_n, z_n)$ is a local minimum of the function

$$f_n(x, y) = \|y - y_n\| + 4[d(x + y; C_1) + d(x; C_2)]$$

$$+ s_n(\|x - x_n\| + \|y - z_n\|).$$

Thus by subdifferential calculus rules [10–11]
\[ 0 \in \hat{\partial}_G f_n(x_n, z_n) \]
\[ \subset \{0\} \times S(z_n, y_n) + 4 \left[ \bigcup_{y^* \in \hat{\partial}_G d(x_n + z_n, C_2) \times \{0\}} (y^*, y^*) + \hat{\partial}_G d(x_n, C_2) \times \{0\} \right] + s_n(B_X^* \times B_X^*), \]

where \( S(z_n, y_n) = \{z^* \in X^*: \|z^*\| = 1 \text{ and } \langle z^*, z_n - y_n \rangle = \|z_n - y_n\|\}. \) Thus there are \( z_n^* \in S(z_n, y_n), y_n^* \in \hat{\partial}_G d(x_n + z_n, C_1), \) and \( x_n^* \in \hat{\partial}_G d(x_n, C_2) \) such that

\[ \|z_n^* + 4y_n^*\| \leq s_n \quad \text{and} \quad \|y_n^* + x_n^*\| \leq s_n/4. \]

So \( y_n^* \in \hat{\partial}_G d(x_n + z_n, C_1), x_n^* \in \hat{\partial}_G d(x_n, C_2), \) and

\[ \|x_n^*\| + \|y_n^*\| \geq (1 - s_n)/4 \quad \text{and} \quad \|x_n^* + y_n^*\| \leq s_n/4. \]

These inequalities contradict (i), since by (4.2), \( x_n \in C_2 \) and \( x_n + z_n \in C_1. \)

Ioffe [11] has proved the last part of Proposition 4.1 for the case in which \( C_1 \) is epi-Lipschitzian at \( x_0. \) He used the following result.

**Lemma 4.2.** Suppose that \( S \subset X \) is closed and epi-Lipschitzian at \( x_0. \) Then there are an \( h \in X, \|h\| = 1, \) and \( r > 0 \) such that

\[ x^* \in \hat{\partial}_G d(x, S) \Rightarrow \langle x^*, h \rangle \leq -r, \]

provided \( x \not\in S \) and \( x \) sufficiently close to \( x_0. \)

Note that we can use Proposition 2.2 instead of Lemma 4.2 and Ioffe’s techniques, which are based on Ekeland variational principle [7], to prove Proposition 4.1.

**Corollary 4.3.** Let all the hypotheses of Proposition 4.1 be satisfied, with \( \hat{\partial}_G \)-independence replaced by \( \partial_C \)-independence. Then the results of Proposition 4.1 are valid.

**Proof.** Observe that \( \hat{\partial}_G d(x, D) \subset \partial_C d(x, D) \) for all \( x \) and apply Proposition 4.1.

We now use Proposition 4.1 to establish the intersection theorem.

**Theorem 4.4.** Under assumptions of Proposition 4.1 there exists a real number \( k > 0 \) such that

\[ \hat{\partial}_G d(x_0, C_1 \cap C_2) \subset k [\hat{\partial}_G d(x_0, C_1) + \hat{\partial}_G d(x_0, C_2)]. \]
Proof. Let \( f(x) = d(x, C_1 \cap C_2) \) and \( g(x) = d(x, C_2) + d(x, C_1) \). Let \( x^* \in \partial G f(x_0) \). Then by (2.1), we have for all \( L \in F(X) \) the existence of nets \( x_i \stackrel{C_1 \cap C_2}{\longrightarrow} x_0, \varepsilon_i \to 0^+ \), with \( \varepsilon_i < 1 \), and \( x^*_i \to x^* \) such that
\[
\|x^*_i\| \leq 2 \quad \text{and} \quad x^*_i \in \partial_{x_i}^- f_{x_i + L}(x_i).
\]
Let \( \varepsilon > 0 \) be given. By Lemma 1 in Ioffe [9], \( x_i \) is a local minimum of the function
\[
x \mapsto f_{x_i + L}(x) - \langle x^*_i, x - x_i \rangle + (\varepsilon + \varepsilon_i)\|x - x_i\|.
\]
(4.3)

By Proposition 4.1 there exist \( k > 0 \) and \( r > 0 \) such that
\[
d(x, C_2 \cap C_1) \leq k(d(x, C_1) + d(x, C_2))
\]
for all \( x \in x_0 + rB_X \). Combining this and (4.3) we find that \( x_i \) is a local minimum of the function
\[
x \mapsto kg_{x_i + L}(x) - \langle x^*_i, x - x_i \rangle + (\varepsilon + \varepsilon_i)\|x - x_i\|.
\]
Thus
\[
x^*_i \in \partial_{(\varepsilon + \varepsilon_i)k} g_{x_i + L}(x_i)
\]
and, hence,
\[
x^* \in k\partial G g(x_0)
\subset k [\partial G d(x_0, C_1) + \partial G d(x_0, C_2)].
\]

Remark. Ioffe [11] has obtained this result under the hypothesis that \( C_1 \) is epi-Lipschitzian. But every epi-Lipschitz set is compactly epi-Lipschitzian and the converse does not hold.

Corollary 4.5. If assumptions of Theorem 4.4 are satisfied, then
\[
\hat{N}_G(C_1 \cap C_2, x_0) \subset \hat{N}_G(C_1, x_0) + \hat{N}_G(C_2, x_0).
\]

As a consequence of Theorem 4.4 we obtain the following result which extends the main theorem of Clarke and Raïssi [6].
Corollary 4.6. Let all the hypotheses of Theorem 4.4 be satisfied. Then there exists a real number \( k > 0 \) such that

\[
\partial_c d(x_0, C_1 \cap C_2) \subseteq k \left[ \partial_c d(x_0, C_1) + \partial_c d(x_0, C_2) \right].
\]

Proof. By Theorem 4.4 there exists \( k > 0 \) such that

\[
\hat{\partial}_G d(x_0, C_1 \cap C_2) \subseteq k \left[ \hat{\partial}_G d(x_0, C_1) + \hat{\partial}_G d(x_0, C_2) \right]
\]

\[
\subseteq k \left[ \partial_c d(x_0, C_1) + \partial_c d(x_0, C_2) \right].
\]

Since \( \partial_c d(x_0, C_1) + \partial_c d(x_0, C_2) \) is convex and weak-star compact it follows that

\[
\text{cl}^{\text{conv}} \hat{\partial}_G d(x_0, C_1 \cap C_2) \subseteq k \left[ \partial_c d(x_0, C_1) + \partial_c d(x_0, C_2) \right]
\]

and, hence,

\[
\partial_c d(x_0, C_1 \cap C_2) \subseteq k \left[ \partial_c d(x_0, C_1) + \partial_c d(x_0, C_2) \right]. \quad \blacksquare
\]

Remark. Note that this result has been established by Clarke and Raïssi [6], but with \( X \) a Hilbert space and \( C_1 \) a compact set.

5. Marginal Function

Intersection formulae can readily be applied to produce estimates for subdifferentials of the marginal function

\[
m(x) := \inf \{ f(x, y) : y \in F(x) \}, \tag{5.1}
\]

where \( f \) is an extended-real-valued function on \( X \times Y \) and \( F \) is a multivalued function from \( X \) into \( Y \). For all \( x \in \text{dom} \ m \) we consider the set of minimizers

\[
M(x) = \{ y \in F(x) : m(x) = f(x, y) \}.
\]

To estimate the subdifferential of \( m \) we need a stability assumption of the type

There exist a norm-compact set \( K \subseteq Y \) and a neighbourhood \( V \) of \( x_0 \) such that for all \( x \in V \), \( M(x) \neq \emptyset \) and \( M(x) \subseteq K + p(x)B_Y \). \tag{5.2}

where \( p \) is a real-valued function on \( V \) satisfying \( \lim_{x \to x_0} p(x) = 0 \).
Remarks. (1) It is easy to see that, under the hypothesis (5.2) and the fact that $f$ is lower semicontinuous and $\text{Gr} \ F$ is closed, the marginal function $m$ is lower semicontinuous at $x_0$.

(2) Similar conditions to (5.2) have been used in the study of the directional derivative and the Clarke’s subdifferential of the marginal function $m$ (see, for example, [8, 25–27]).

Theorem 5.1. Let the multivalued function $F$ have closed graph, let $f$ be lower semicontinuous around any point $(x_0, y_0)$ with $y_0 \in F(x_0)$, and let (5.2) hold. Suppose that, for all $y_0 \in M(x_0)$, one of the following conditions holds:

(i) $\text{epi} \ f$ and $\text{Gr} \ F \times \mathbb{R}$ are strongly $\delta_G$-independent at $(x_0, y_0, f(x_0, y_0))$.

(ii) $\text{epi} \ f$ is compactly $\text{epi}$-Lipschitzian at $(x_0, y_0, f(x_0, y_0))$ and $\text{epi} \ f$ and $\text{Gr} \ F \times \mathbb{R}$ are $\delta_G$-independent at $(x_0, y_0, f(x_0, y_0))$.

(iii) $\text{Gr} \ F$ is compactly $\text{epi}$-Lipschitzian at $(x_0, y_0)$ and $\text{epi} \ f$ and $\text{Gr} \ F \times \mathbb{R}$ are $\delta_G$-independent at $(x_0, y_0, f(x_0, y_0))$.

Then

$$\hat{\delta}_G m(x_0) \subset \bigcup_{y_0 \in M(x_0)} \bigcup_{(x^*, y^*) \in \hat{\delta}_G f(x_0, y_0)} \{x^* + \hat{D}^*_G F(x_0, y_0)(y^*)\}$$

and

$$\hat{\delta}_G^* m(x_0) \subset \bigcup_{y_0 \in M(x_0)} \bigcup_{(x^*, y^*) \in \hat{\delta}_G^* f(x_0, y_0)} \{x^* + \hat{D}^*_G F(x_0, y_0)(y^*)\}.$$  

Remark. The results of Theorem 5.1 remain true if we replace the stability assumption (5.2) by the following one:

There exist a continuous selection $s$ of the multivalued function $M$ on some closed ball $x_0 + rB_x$.

In this case, for $y_0 = s(x_0)$,

$$\hat{\delta}_G m(x_0) \subset \bigcup_{(x^*, y^*) \in \hat{\delta}_G f(x_0, y_0)} \{x^* + \hat{D}^*_G F(x_0, y_0)(y^*)\}$$

and

$$\hat{\delta}_G^* m(x_0) \subset \bigcup_{(x^*, y^*) \in \hat{\delta}_G^* f(x_0, y_0)} \{x^* + \hat{D}^*_G F(x_0, y_0)(y^*)\}.$$
We will prove our theorem in two steps. The first step reduces the proof to that of a simpler estimate.

**Lemma 5.2.** Set $C_1 = \text{epi} \ f$ and $C_2 = \text{Gr} \ F \times \mathbb{R}$. Let the multivalued function $F$ have closed graph, let $f$ be lower semicontinuous around any point $(x_0, y_0)$ with $y_0 \in F(x_0)$, and let (5.2) hold. Then for all $(x^*, \beta) \in \hat{\delta}_G d(x_0, m(x_0), \text{epi} \ m)$ there exist $y_0 \in M(x_0)$ and $k > 0$ such that

$$(x^*, 0, \beta) \in k\hat{\delta}_G d(x_0, y_0, f(x_0, y_0), C_1 \cap C_2).$$

**Proof.** Set $g(x, r) = d(x, r, \text{epi} \ m)$ and $h(x, y, r) = d(x, y, r, C_1 \cap C_2)$. Then, by (2.1), we have for all $L \in \mathcal{F}(X)$ the existence of nets $(x_i, r_i) \xrightarrow{\text{epi} \ m} (x_0, m(x_0)), \varepsilon_i \to 0^+$, with $\varepsilon_i < 1$, and $(x_i^*, \beta_i) \to (x^*, \beta)$ such that

$$(x_i^*, \beta_i) \in \partial^{-} g_{(x_i, r_i) + L \times \mathbb{R}}(x_i, r_i).$$

Let $\varepsilon > 0$ be given. By Lemma 1 in Ioffe [9] and Proposition 2.4.3 in Clarke [5], $(x_i, r_i)$ is a local minimum of the function

$$(x, r) \mapsto g(x, r) - \langle x_i^*, x - x_i \rangle - \langle \beta_i, r - r_i \rangle$$

$$+ (\varepsilon + \varepsilon_i)(\|x - x_i\| + \|r - r_i\|) + 4d(x, x_i + L)$$

and, hence, there exists $s_i > 0$ such that

$$-\langle x_i^*, x - x_i \rangle - \langle \beta_i, r - r_i \rangle + (\varepsilon + \varepsilon_i)(\|x - x_i\| + \|r - r_i\|) + 4d(x, x_i + L) \geq 0$$

for all $(x, r) \in \text{epi} \ m \cap ((x_i, r_i) + s_i B_{X \times \mathbb{R}})$. By (5.2), there exists $y_i \in F(x_i)$ satisfying

$$m(x_i) = f(x_i, y_i)$$

and, extracting a subnet if necessary, we may suppose that $y_i \to y_0$. Because $\text{Gr} \ F$ is closed, we have $y_0 \in F(x_0)$. Further, since $m(x_i) = f(x_i, y_i)$, $m(x_i) \leq r_i$, and $f$ is lower semicontinuous around $(x_0, y_0)$, it follows that

$$y_0 \in F(x_0) \quad \text{and} \quad m(x_0) \geq f(x_0, y_0).$$

Thus $y_0 \in M(x_0)$. On the one hand, $(x_i, y_i, r_i) \in C_1 \cap C_2$, and, on the other hand, if $(x, y, r) \in C_1 \cap C_2$ then $(x, r) \in \text{epi} \ m$. So invoking (5.3) and Proposition 2.4.3 in Clarke [5] we obtain that $(x_i, y_i, r_i)$ is a local
minimum of the function

\[(x, y, r) \rightarrow 6d(x, y, r, C_1 \cap C_2) - \langle x_i^*, x - x_i \rangle - \langle \beta_i, r - r_i \rangle + (\epsilon + \varepsilon)\|x - x_i\| + |r - r_i| + 4d(x, x_i + L).\]

It follows that for all \(E \in \mathfrak{F}(Y),\)

\[(x_i^*, 0, \beta_i) \in \partial_{(\epsilon + \varepsilon)}h_{(x_i, y_i, r_i), L \times E \times \mathbb{R}}(x_i, y_i, r_i)\]

and, hence,

\[(x^*, 0, \beta) \in 6\partial_G h(x_0, y_0, f(x_0, y_0)).\]

The second step shows that the sets \(C_1\) and \(C_2\) are \(\hat{\partial}_G\)-independent (resp. strongly \(\hat{\partial}_G\)-independent) at \(x_0\) and satisfy the intersection formulae.

**Lemma 5.3.** Let the multivalued function \(F\) have closed graph and let \(f\) be lower semicontinuous around \((x_0, y_0)\) with \(y_0 \in F(x_0)\). Suppose that condition (i) (resp. (ii)) (resp. (iii)) holds. Then \(C_1\) and \(C_2\) are strongly \(\hat{\partial}_G\)-independent at \((x_0, y_0, f(x_0, y_0))\) (resp. \(C_1\) is compactly epi-Lipschitzian at \((x_0, y_0, f(x_0, y_0))\) and \(C_1\) and \(C_2\) are \(\hat{\partial}_G\)-independent at this point) (resp. \(C_2\) is compactly epi-Lipschitzian at \((x_0, y_0, f(x_0, y_0))\) and \(C_1\) and \(C_2\) are \(\hat{\partial}_G\)-independent at this point). In addition there exists \(a > 0\) such that

\[\hat{\partial}_G d(x_0, y_0, f(x_0, y_0), C_1 \cap C_2) \subset a [\hat{\partial}_G d(x_0, y_0, f(x_0, y_0), \text{epi } f) + \hat{\partial}_G d(x_0, y_0, \text{Gr } F) \times \{0\}].\]

**Proof.** It suffices to see that

\[\hat{\partial}_G d(x_0, y_0, f(x_0, y_0), C_1) = \hat{\partial}_G d(x_0, y_0, f(x_0, y_0), \text{epi } f)\]

and

\[\hat{\partial}_G d(x_0, y_0, f(x_0, y_0), C_2) = \hat{\partial}_G d(x_0, y_0, \text{Gr } F) \times \{0\}\]

and apply Theorem 4.4. \(\blacksquare\)

**Proof of Theorem 5.1.** Let \(u^* \in \hat{\partial}_G m(x_0)\). Then there exist \(\lambda > 0\) and \((x^*, \beta) \in \hat{\partial}_G d(x_0, m(x_0), \text{epi } m)\) such that

\[u^* = \lambda x^* \quad \text{and} \quad \lambda \beta = -1.\]
By Lemma 5.2 there exist a real number $k > 0$ and a point $y_0 \in M(x_0)$ such that

$$(x^*, 0, \beta) \in k\hat{\delta}_G d(x_0, y_0, f(x_0, y_0), C_1 \cap C_2)$$

and by Lemma 5.3 there exist $a > 0$, $(z^*, y^*, \gamma) \in \hat{\delta}_G d(x_0, y_0, f(x_0, y_0), \text{epi } f)$ and $(v^*, p^*) \in \hat{\delta}_G d(x_0, y_0, \text{Gr } F)$ satisfying

$$x^* = ak(z^* + v^*),$$
$$0 = ak(y^* + p^*),$$

and

$$\beta = ak\gamma.$$

Thus

$$u^* = \lambda x^* = \lambda ak(z^* + v^*),$$
$$-1 = \lambda \beta = \lambda ak\gamma,$$

and

$$\lambdaaky^* = -\lambda akp^*.$$

Hence

$$(\lambda az^*, \lambda k\gamma^*) \in \hat{\delta}_G f(x_0, y_0)$$

and

$$u^* \in (\lambda az^*) + \hat{D}_G F(x_0, y_0)(\lambda k\gamma^*).$$

The proof of the second part is similar to that of the first one. \( \square \)

It is easy to obtain a number of corollaries of Theorem 5.1 by considering certain special forms for $f$ and $F$. In particular, the theorem implies the following result which was first obtained in Mordukhovich [20–22] in finite dimensions.

**Corollary 5.4.** Let the hypotheses of Theorem 5.1 be satisfied with $f(x, y) = g(y)$. Then

$$\hat{\delta}_G m(x_0) \subset \bigcup_{y_0 \in M(x_0)} \bigcup_{y^* \in \delta_G g(y_0)} \hat{D}_G F(x_0, y_0)(y^*)$$
and
\[ \hat{\partial}_G^\varepsilon m(x_0) \subseteq \bigcup_{y_0 \in M(x_0)} \bigcup_{y^* \in \hat{\partial}_G^\varepsilon g(y_0)} \hat{D}_G^\varepsilon F(x_0, y_0)(y^*) . \]

In the case where \( F \) is a single-valued function we obtain the following generalized chain rules.

**Theorem 5.5.** Let \( F \) be a single-valued function from \( X \) into \( Y \) which is continuous around \( x_0 \) and let \( g \) be an extended-real-valued function on \( Y \) which is lower semicontinuous around \( y_0 = F(x_0) \in \text{dom } g \). Suppose that one of the following conditions holds:

(i') \( X \times \text{epi } g \) and \( \text{Gr } F \times \mathbb{R} \) are strongly \( \hat{\partial}_G \)-independent at \((x_0, y_0, g(y_0))\),

(ii') \( \text{epi } g \) is compactly \( \text{epi-Lipschitzian at } (y_0, g(y_0)) \) and \( X \times \text{epi } g \) and \( \text{Gr } F \times \mathbb{R} \) are \( \hat{\partial}_G \)-independent at \((x_0, y_0, g(y_0))\),

(iii') \( \text{Gr } F \) is compactly \( \text{epi-Lipschitzian at } (x_0, y_0) \) and \( X \times \text{epi } g \) and \( \text{Gr } F \times \mathbb{R} \) are \( \hat{\partial}_G \)-independent at \((x_0, y_0, g(y_0))\).

Then
\[ \hat{\partial}_G(g \circ F)(x_0) \subseteq \bigcup_{y^* \in \hat{\partial}_G g(y_0)} \hat{D}_G^\varepsilon F(x_0, y_0)(y^*) \]

and
\[ \hat{\partial}_G^\varepsilon(g \circ F)(x_0) \subseteq \bigcup_{y^* \in \hat{\partial}_G^\varepsilon g(y_0)} \hat{D}_G^\varepsilon F(x_0, y_0)(y^*) . \]

**Remark.** Theorem 5.5 implies the estimates
\[ \hat{\partial}_G(g \circ F)(x_0) \subseteq \bigcup_{y^* \in \hat{\partial}_G g(y_0)} \hat{\partial}_G(y^* \circ F)(x_0) \]

and
\[ \hat{\partial}_G^\varepsilon(g \circ F)(x_0) \subseteq \bigcup_{y^* \in \hat{\partial}_G^\varepsilon g(y_0)} \hat{\partial}_G(y^* \circ F)(x_0) . \]
whenever \( F \) belongs to the large class of strongly compactly Lipschitzian functions [15], since following Jourani and Thibault [16],

\[(x^*, -y^*) \in \hat{N}_G(\text{Gr } F, x_0, F(x_0)) \iff x^* \in \hat{\delta}_G(y^* \circ F)(x_0).\]

In particular, when \( F \) is strictly differentiable at \( x_0 \) (that is, when

\[
\lim_{\substack{x \to x_0 \\ x' \to x_0}} \frac{\|F(x) - F(x') - \nabla F(x_0)(x - x')\|}{\|x - x'\|} = 0,
\]

where \( \nabla F(x_0) \) is the Fréchet derivative of \( F \) at \( x_0 \), we have both

\[
\hat{\delta}_G(g \circ F)(x_0) \subset \hat{\delta}_G g(y_0) \circ \nabla F(x_0)
\]

and

\[
\hat{\delta}_G^C(g \circ F)(x_0) \subset \hat{\delta}_G^C g(y_0) \circ \nabla F(x_0).
\]

**Corollary 5.6.** Let \( F \) and \( g \) be as in Theorem 5.5, with epi \( g \) compactly epi-Lipschitzian at \((y_0, g(y_0))\). Suppose that

\[
y^* \in \hat{\delta}_G^x g(y_0) \quad \text{and} \quad 0 \in \hat{D}_G F(x_0, y_0)(y^*) \Rightarrow y^* = 0.
\]

Then the results of Theorem 5.5 are valid.

**Proof.** It suffices to show that the sets \( C_1 := X \times \text{epi } g \) and \( C_2 := \text{Gr } F \times \mathbb{R} \) are \( \hat{\delta}_G \)-independent. So let \((x^*, y^*, r) \in \hat{\delta}_G d(x_0, y_0, g(y_0), C_1) \cap (-\hat{\delta}_G d(x_0, y_0, g(y_0), C_2))\). As

\[
\hat{\delta}_G d(x_0, y_0, g(y_0), C_1) = \{0\} \times \hat{\delta}_G d(y_0, g(y_0), \text{epi } g)
\]

and

\[
\hat{\delta}_G d(x_0, y_0, g(y_0), C_2) = \hat{\delta}_G d(x_0, y_0, \text{Gr } F) \times \{0\},
\]

we have \( x^* = 0, r = 0 \). Thus,

\[
y^* \in \hat{\delta}_G^x g(x_0) \quad \text{and} \quad 0 \in \hat{D}_G F(x_0, y_0)(y^*)
\]

which implies by assumption that \( y^* = 0 \), and the proof is complete.

**Remark.** Corollary 5.6 was first obtained by Ioffe [11, Theorem 7.5], under the hypothesis that \( g \) is directionally Lipschitz at \( y_0 \) or, equivalently,
that epi \( g \) is epi-Lipschitz [24] at \((y_0, g(y_0))\). But (following Borwein [3]) every epi-Lipschitz set is compactly epi-Lipschitzian and in finite dimensions each subset is compactly epi-Lipschitzian at all its points, and this is not the case for the epi-Lipschitz sets.

As a consequence of Theorem 5.5 we obtain the following chain rule for Clarke's subdifferential.

**Corollary 5.7.** Under the assumptions of Theorem 5.5 we have

\[
\partial^c(g \circ F)(x_0) \subset \text{cl}^* \bigcup_{y^* \in \partial^c g(y_0)} D^c_F(x_0, y_0)(y^*).
\]

**Proof.** Let \( x^* \in \partial^c G(g \circ F)(x_0) \) and \( u^* \in \partial^c G(g \circ F)(x_0) \). Then, by Theorem 5.5, there exist \( y^* \in \partial^c G g(y_0) \) and \( v^* \in \partial^c G g(y_0) \) such that

\[
(x^*, -y^*) \in \hat{N}_G(\text{Gr } F, x_0, y_0), \quad (u^*, -v^*) \in \hat{N}_G(\text{Gr } F, x_0, y_0).
\]

As \( \hat{N}_G(\text{Gr } F, x_0, y_0) \subset N_c(\text{Gr } F, x_0, y_0) \) and \( N_c(\text{Gr } F, x_0, y_0) \) is a closed convex cone it follows that

\[
(x^* + u^*, -(y^* + v^*)) \in N_c(\text{Gr } F, x_0, y_0)
\]

and, hence,

\[
x^* + u^* \in D^c_F(x_0, y_0)(y^* + v^*), \quad y^* + v^* \in \partial^c G g(y_0) + \partial^c G g(y_0).
\]

Invoking (2.2), we conclude that

\[
\partial^c (g \circ F)(x_0) \subset \text{cl}^* \bigcup_{y^* \in \partial^c G(y_0)} D^c_F(x_0, y_0)(y^*).
\]

To complete this section we consider \( F \) of the special form

\[
F(x) = \{ y \in C : g(x, y) \in D \}, \quad (5.4)
\]

where \( g : X \times Y \to Z \) is a Lipschitz function at all points \((x_0, y_0)\) with \( y_0 \in F(x_0) \), \( Z \) is a Banach space, and \( C \) and \( D \) are two closed subsets of \( Y \) and \( Z \), respectively. Suppose that there exist two neighbourhoods \( V \) and \( W \) of \( x_0 \) and \( y_0 \), respectively, and a real number \( a > 0 \) such that

\[
d(y, F(x)) \leq ad(g(x, y), D) \quad (5.5)
\]
for all \( x \in V \) and \( y \in W \cap C \). Then there exists \( k > 0 \) such that

\[
d(x, y; \text{Gr} \, F) \leq k \left[ d(g(x, y), D) + d(y, C) \right]
\]

for all \( x \in V \) and \( y \in W \). Thus, using (2.1), we get

\[
\hat{d}_G d(x_0, y_0; \text{Gr} \, F) \subset k \hat{d} \left( d(g(\cdot, \cdot), D) + d(\cdot, C) \right)(x_0, y_0)
\subset \bigcup_{z^* \in \hat{N}_G(x_0, y_0)} \hat{D}_G g(x_0, y_0)(z^*) + \{0\} \times \hat{N}_G(C, y_0).
\]

These arguments establish the following corollary of Theorem 5.1.

**Corollary 5.8.** Let \( f \) be as in Theorem 5.1, let \( F \) be given by (5.4), and let (5.2) hold. Suppose that (5.5) holds for all \( y_0 \in M(x_0) \). Then

\[
\hat{d}_G m(x_0) \subset \bigcup_{y_0 \in M(x_0)} \bigcup_{(x^*, y^*) \in \hat{d}_G(x_0, y_0)} \{x^* + u^*: (u^*, -y^*)
\subset \bigcup_{z^* \in \hat{N}_G(x_0, y_0)} \hat{D}_G g(x_0, y_0)(z^*) + \{0\} \times \hat{N}_G(C, y_0)\}.
\]

### 6. Comparison of Independence Conditions

In finite dimensions, Aubin and Ekeland [1] establish (1.1) under the assumption

\[
T_C(C_1, x_0) - T_C(C_2, x_0) = X.
\]  

(6.1)

Their result can easily be extended to infinite dimensions and without any additional effort we can prove the following corollary of Theorem 4.4.

**Corollary 6.1.** Let \( C_1 \) and \( C_2 \) be two closed subsets of \( X \), with \( C_1 \) compactly epi-Lipschitzian at \( x_0 \in C_1 \cap C_2 \). Then (6.1) implies (1.1). Therefore,

\[
N_C(C_1 \cap C_2, x_0) \subset N_C(C_1, x_0) + N_C(C_2, x_0).
\]  

(6.2)

**Proof.** It is not difficult to show that (6.1) implies the \( \hat{d}_G \)-independence of \( C_1 \) and \( C_2 \). So using Corollary 4.6 and the fact that \( N_C(C_i, x_0) = \text{cl}^*[\mathbb{R}_+ \partial_C d(x_0, C_i)] \) (see [5]) we get

\[
N_C(C_1 \cap C_2, x_0) \subset \text{cl}^*[N_C(C_1, x_0) + N_C(C_2, x_0)].
\]
Theorem 6.3 in Borwein [3] implies that
\[ \text{cl}^*[N_C(C_1, x_0) + N_C(C_2, x_0)] = N_C(C_1, x_0) + N_C(C_2, x_0) \]
and, hence,
\[ N_C(C_1 \cap C_2, x_0) \subseteq N_C(C_1, x_0) + N_C(C_2, x_0). \]

**Remark.** Corollary 6.1 is established in [28] under the hypothesis that \(C_1\) and \(C_2\) are epi-Lipschitz-like at \(x_0\). But every epi-Lipschitz-like set is compactly epi-Lipschitz and the converse cannot hold (see Example 4.1 in [3]).

In infinite dimensions, Rockafellar [24] shows that the assumption
\[ \text{int } T_C(C_1, x_0) \cap T_C(C_2, x_0) \neq \emptyset \]
\[ C_i \text{ is epi-Lipschitzian at } x_0 \] (6.3)
implies (6.2).

The following lemma shows that our \( \hat{\partial}_G \)-independence condition is weaker than (6.1), (6.3), and the \( \partial_C \)-independence condition.

**Lemma 6.2.** The following implications hold:

(6.3) \( \Rightarrow \) (6.1) \( \Rightarrow \) \( \hat{\partial}_G d(x_0, C_1) \cap (-\hat{\partial}_G d(x_0, C_2)) = \{0\} \)
\[ \Rightarrow \hat{N}_G(C_1, x_0) \cap (-\hat{N}_G(C_2, x_0)) = \{0\} \]
\[ \Leftrightarrow \hat{\partial}_G d(x_0, C_1) \cap (-\hat{\partial}_G d(x_0, C_2)) = \{0\}. \]

**Proof.** The implication (6.3) \( \Rightarrow \) (6.1) is immediate.

For (6.1) \( \Rightarrow \) \( \hat{\partial}_G d(x_0, C_1) \cap (-\hat{\partial}_C d(x_0, C_2)) = \{0\} \), it suffices to see that
\( (6.1) \Rightarrow N_C(C_1, x_0) \cap (-N_C(C_2, x_0)) = \{0\} \) and \( \hat{\partial}_C d(x_0, C_i) \subset N_C(C_i, x_0) \).

Finally, the \( \hat{\partial}_C \)-independence condition implies the \( \hat{\partial}_G \)-independence condition, since \( \hat{\partial}_G d(x_0, C_i) \subset \partial_C d(x_0, C_i) \).

The following example shows that the implication (6.3) \( \Rightarrow \) (6.1) is strict.

**Example 6.3.** Take \( C_1 = \{0\} \times \mathbb{R} \) and \( C_2 = \mathbb{R} \times \{0\} \) and \( x_0 = (0, 0) \). Then (6.1) holds while \( \text{int } T_C(C_1, x_0) = \text{int } T_C(C_2, x_0) = \emptyset \).

The implication (6.1) \( \Rightarrow \) \( \hat{\partial}_G d(x_0, C_1) \cap (-\hat{\partial}_G d(x_0, C_2)) = \{0\} \) is also strict.

**Example 6.4.** [6, 23]. Let \( C_1 = [-1, 1] \times \{0\} \times [-1, 1] \) and \( C_2 = \text{epi } f \), where
\[
    f(x, y) = \begin{cases} 
        y^2/2|x|, & \text{if } x \neq 0 \\
        0, & \text{if } x = y = 0 \\
        \infty, & \text{if } x = 0, y \neq 0,
    \end{cases}
\]

and let \( x_0 = (0, 0, 0) \). Then [23]

\[
    \partial_C d(x_0, C_1) = \{0\} \times [-1, 1] \times \{0\},
\]

\[
    \partial_C d(x_0, C_2) = \text{conv}\left[ \left\{ (x, y, z) \in B_{\mathbb{R}^3} : z \leq \frac{-y}{2|x|}, x \neq 0 \right\} \right.
    \]

\[
    \left. \cup \{(x, y, z) \in B_{\mathbb{R}^3} : x = y = 0, z \leq 0\} \right],
\]

\[
    N_C(C_1, x_0) = \{0\} \times \mathbb{R} \times \{0\},
\]

and

\[
    N_C(C_2, x_0) = \{(x, y, z) \in \mathbb{R}^3 : z \leq 0\}.
\]

So \( \partial_C d(x_0, C_1) \cap (-\partial_C d(x_0, C_2)) = \{0\} \) while \( N_C(C_1, x_0) \cap (-N_C(C_2, x_0)) = \{0\} \times \mathbb{R} \times \{0\} \).

Finally we give an example in which the converse of the implication \( \partial_C d(x_0, C_1) \cap (-\partial_C d(x_0, C_2)) = \{0\} \Rightarrow \hat{\partial}_G d(x_0, C_1) \cap (-\hat{\partial}_G d(x_0, C_2)) = \{0\} \) does not hold.

**Example 6.5.** Let \( C_1 = \{0\} \times \mathbb{R}, C_2 = \{(x, y) \in \mathbb{R}^2 : y = |x|\} \), and \( x_0 = (0, 0) \). Then

\[
    \hat{\partial}_G d(x_0, C_1) = [-1, 1] \times \{0\},
\]

\[
    \hat{\partial}_G d(x_0, C_2) = \{(x, y) \in B_{\mathbb{R}^2} : y = |x|\} \cup \{(x, y) \in B_{\mathbb{R}^2} : y \leq -|x|\},
\]

\[
    \partial_C d(x_0, C_1) = \text{conv} \hat{\partial}_G d(x_0, C_1) = [-1, 1] \times \{0\},
\]

and

\[
    \partial_C d(x_0, C_2) = B_{\mathbb{R}^2}.
\]

So \( \hat{\partial}_G d(x_0, C_1) \cap (-\hat{\partial}_G d(x_0, C_2)) = \{(0, 0)\} \) while \( \partial_C d(x_0, C_1) \cap (-\partial_C d(x_0, C_2)) = [-1, 1] \times \{0\} \).
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