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Coderivatives of multivalued mappings, locally compact cones and metric regularity

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1. Introduction

In recent years, there have been several papers showing the utility of metric regularity/or Lipschitz behaviour/or openness at a linear rate of multivalued mappings in the study of optimization problems (see [41, 1, 13, 37, 25, 43, 44, 31–35, 7, 16–21, 24], and references therein). Conditions ensuring metric regularity have been expressed in terms of subdifferentials or tangent cones. In infinite dimensions, most of these conditions for nonsmooth cases require verification around the point but not at the point, and moreover they are generally stated in a complicated way. Thus, to overcome this difficulty, Ioffe [13] introduced the so-called *finite codimension property* in terms of topologically closed approximate subdifferentials and normal cones. Using the notion introduced by Borwein and Strojwas [4, 5] as a point of departure, we introduced in [24] a new class of multivalued mappings between Banach spaces called *partially compactly epi-Lipschitzian* to extend the easily verifiable (finite dimensional) sufficient criteria of Ioffe [9] and Mordukhovich [30–33] to infinite dimensions. This class of multivalued mappings obeys to the following important condition: if F is a multivalued mapping between two Banach spaces X and Y belonging to this class and if (x_0, y_0) is in the graph GrF of F, then [24] there are norm-compact sets $H \subset X$ and

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$K \subset Y$ and neighbourhoods V and W of x_0 and y_0, respectively, such that

$$\begin{cases} \forall (x,y) \in (V \times W) \cap GrF, \quad \forall (x^*,y^*) \in \mathbf{R}_+ \partial_A d(x,y,GrF), \\ \|y^*\| \le \max_{h \in H} \langle x^*, h \rangle + \max_{k \in K} \langle y^*, k \rangle, \end{cases} \tag{1}$$

where $d(x,S)$ denotes the distance function from x to S and $\partial_A f(x)$ denotes the A-subdifferential [10, 11] of f at x. Note that our paper [24] was the first work using partial compactness notion in the study of metric regularity or openness.

Using condition (1) and the finite codimension property by Ioffe [13], Mordukhovich and Shao [34] introduced later another condition in terms of Fréchet normal cones. Note that in the proof of our main result in [24] we only used the following convergence condition:

$$\begin{cases} \text{If } ((x_i,y_i)) \subset \operatorname{graph}(F) \text{ and } (x_i^*, y_i^*) \in \mathbf{R}_+ \partial_A d(x_i,y_i,GrF) \\ \text{are such that } (x_i,y_i) \to (x_0,y_0), \ \|x_i^*\| \to 0 \text{ and } y_i^* \xrightarrow{w^*} 0 \\ \text{then } \|y_i^*\| \to 0 \end{cases} \tag{2}$$

to give sufficient criteria at a point and not around it. Such conditions are stated in the remark following Theorem 2.12 of our paper [23] and are also used in Ioffe [15], Mordukhovich and Shao [34,35] and Penot [38,39]. Note also that it is easy seen by (1) that any partially compactly epi-Lipschitzian multivalued mapping satisfies condition (2). But the class of partially compactly epi-Lipschitzian multivalued mappings does not include the convex case. We mean that if zero is an interior point of the range of a graph convex multivalued mapping F then F is not necessarily partially compactly epi-Lipschitzian, but it satisfies (2).

The class of multivalued mappings satisfying (2) also includes that of normally locally compact ones. Following Loewen [29], we say that a multivalued mapping F between two Banach spaces X and Y is graphically partially normally locally compact at (x_0,y_0) in its graph if there exist a locally compact cone K^* and neighbourhoods V and W of x_0 and y_0 such that

$$\forall (x,y) \in V \times W \cap GrF, \quad N(GrF,x,y) \subset X^* \times K^*, \tag{3}$$

where $N(S,x)$ denotes some normal cone to a set S at x. The cone K^* is said to be locally weak-star compact (for short locally compact) if for every point of K^* there exists a weak-star closed neighbourhood of this point such that the intersection of K^* and this neighbourhood is weak-star compact. Note that if a cone has nonempty interior then its negative polar is locally compact (see Section 2 for other examples). Loewen [29] showed that in locally compact cones one has the following equivalence

$$x_i^* \xrightarrow{w^*} 0 \quad \text{iff} \quad \|x_i^*\| \to 0. \tag{4}$$

This class of cones plays an important role in the study of necessary optimality conditions (see [7, 8, 18–20, 23, 24], and references therein). In his paper [29], Loewen gave the following important example of locally compact cones:

$$K(H) := \left\{ x^*: \ \|x^*\| \le \max_{h \in H} \langle x^*, h\rangle \right\}, \tag{5}$$

where H is a norm-compact set. In [19], it is shown that when C is a closed convex cone then its negative polar is locally compact iff it is contained in $K(H)$ for some norm-compact set H.

In this paper, among our main interests we intend to employ an abstract subdifferential to obtain a verifiable criterion for metric regularity of infinite dimensional multivalued mappings satisfying (2) and to study some important properties of multivalued mappings that are separately partially compactly epi-Lipshitzian. We use some subclasses of partially compactly epi-Lipschitzian multivalued mappings to give necessary and sufficient conditions for metric regularity of infinite-dimensional multivalued mappings in the form first introduced by Mordukhovich [30–33] in finite dimensions. We then obtain that the metric regularity at (x_0, y_0) of a multivalued mapping F with F^{-1} uniformly compactly epi-Lipschitzian implies that the kernel of the approximate coderivative of F at (x_0, y_0) is reduced to zero. The same result was established when X is finite-dimensional (hence F^{-1} is automatically uniformly compactly epi-Lipschitzian) by different method by Mordukhovich and Shao [36] with the limiting Fréchet coderivative. Using recent results established in [3], we show that for every epigraphically normally locally compact function f, i.e. satisfying Eq. (3) with GrF replaced by the epigraph of this function, the A-subdifferential [10, 11] is sequentially generated by epsilon Dini subdifferentials in the sense

$$\partial_A f(x_0) = seq - \limsup_{\substack{x \xrightarrow{f} x_0 \\ \varepsilon \to 0^+}} \partial_\varepsilon^- f(x)$$

provided that the space is weakly compactly generated (WCG) (see [6]). Examples of locally compact cones and conditions ensuring Eq. (2) are given. The results obtained can be applied to extend calculus rules by Ioffe [10, 11] and Jourani and Thibault [23].

Our notation is rather standard. X and Y are *Banach spaces* with topological duals X^* and Y^*. The closed unit balls of X and X^* will be denoted by B_X and B_X^*. The distance function $d(x, S)$ from x to S is

$$d(x, S) = \inf_{u \in S} \|x - u\|.$$

The graph of a multivalued mapping $F : X \rightrightarrows Y$ and the epigraph of a function f are, respectively, denoted by GrF and $\operatorname{epi} f$

$$GrF = \{(x, y) \in X \times Y: \ y \in F(x)\}$$
$$\operatorname{epi} f = \{(x, r) \in X \times R: \ f(x) \le r\}.$$

The inverse multivalued mapping F^{-1} of F is defined by

$$F^{-1}(y) = \{x \in X: \ y \in F(x)\}.$$

We write $x \xrightarrow{f} x_0$ and $x \xrightarrow{S} x_0$ to express $x \to x_0$ with $f(x) \to f(x_0)$ and $x \to x_0$ with $x \in S$.

For an extended real-valued function f on X and a subset S of X we write $f_{x+S}(x) = f(x)$ if $x \in S$ and $f_{x+S}(x) = +\infty$ if $x \notin S$. Finally, the indicator function Ψ_S of S is defined by $\Psi_S(x) = 0$ if $x \in S$ and $\Psi_S(x) = +\infty$ if $x \notin S$. For a multivalued mapping Φ from a metric space M into X^*, we write

$$\limsup_{x \to x_0} \Phi(x) = \{x^* \in X^*\colon\ \exists x_i \to x_0,\ \exists x_i^* \xrightarrow{w^*} x^* / x_i^* \in \Phi(x_i), \forall i\}$$

the set of limits of such nets.

2. Examples of normal cones which are locally compact

This section and the following ones are devoted to the study of some properties of several normal cones. We start with sets whose normal cones are locally compact. We have already pointed out the importance of this property to establish optimality conditions for optimization problems with constraints. As it will be seen in the sequel this property is also useful for the proofs of conditions ensuring metric regularity.

Example 1 (*Approximate normal cones*)**.** The *A-subdifferential* of a lower semicontinuous extended real-valued function on X at x_0 in the domain of f ($\operatorname{dom} f := \{x \in X\colon |f(x)| < \infty\}$) is the set [10]

$$\partial_A f(x_0) = \bigcap_{L \in \mathscr{T}(X)} \limsup_{\substack{x \xrightarrow{f} x_0 \\ \varepsilon \to 0^+}} \partial_\varepsilon^- f_{x+L}(x),$$

where $\mathscr{T}(X)$ denotes the collection of all weak trustworthy subspaces of X and $\partial_\varepsilon^- f(x)$ denotes the lower Dini ε-subdifferential of f at x

$$\partial_\varepsilon^- f(x) = \{x^* \in X^*\colon\ \langle x^*, h\rangle \le d^- f(x,h) + \varepsilon\|h\|,\ \forall h\}.$$

Here $d^- f(x,h)$ denotes the lower directional derivative of f at x in the direction h given by

$$d^- f(x,h) = \liminf_{\substack{u \to h \\ t \to 0^+}} t^{-1}(f(x+tu) - f(x)).$$

Following Ioffe [12], X is called a weak trustworthy space (for short WT-space) if for all lower semicontinuous functions f_1 and f_2 on X, each $x \in \operatorname{dom}(f_1 + f_2)$ and each $\varepsilon > 0$

$$\partial_\varepsilon^-(f_1 + f_2)(x) \subset \limsup_{\substack{x_j \xrightarrow{f_j} x \\ j=1,2}} (\partial_\varepsilon^-(f_1(x_1) + \partial_\varepsilon^-(f_2(x_2)).$$

In this class of spaces, the definition of approximate subdifferential can be written [10]

$$\partial_A f(x_0) = \limsup_{\substack{x \xrightarrow{f} x_0 \\ \varepsilon \to 0^+}} \partial_\varepsilon^- f(x).$$

The *approximate normal cone* to a closed set $S \subset X$ containing x_0 is the set

$$N_A(S, x_0) = \partial_A \Psi_S(x_0)$$

and $x^* \in \partial_A f(x_0)$ iff $(x^*, -1) \in N_A(epi f, x_0, f(x_0))$.

Following Borwein and Strojwas [4, 5] a set $S \subset X$ is said to be *compactly epi-Lipschitzian* at $x_0 \in S$ if there exist $r > 0$ and a norm-compact set $H \subset X$ such that

$$S \cap (x_0 + rB_X) + tr\, B_X \subset S - tH, \quad \forall t \in]0, r[. \tag{6}$$

Using ideas in Jourani and Thilbault [25] and Loewen [29] the following proposition has been proved by Jourani [19].

Proposition 2.1. *Let X be a WT-space and let $S \subset X$ be a closed and compactly epi-Lipschitzian set at $x_0 \in S$. Then there are $s > 0$ and $h_1, \ldots, h_m \in X$ such that for all $x \in (x_0 + sB_X) \cap S$.*

(a) *$x^* \in N_A(S, x)$ iff there are nets $\varepsilon_i \to 0^+$, $x_i^* \xrightarrow{w^*} x^*$, $x_i \xrightarrow{S} x_0$, $c > 0$ such that for all i*

$$\|x_i^*\| \le c \quad \textit{and} \quad x_i^* \in \partial_{\varepsilon_i}^- \Psi_S(x_i)$$

and

(b) *for all $x^* \in N_A(S, x)$*

$$\|x^*\| \le \max_{j=1,\ldots,m} \langle x^*, h_j \rangle$$

and hence $N_A(S, x)$ is locally compact.

In a joint work in [16] (see Ch. 3, p. 21), we considered (see also [38]) the notion of *b-approximate normal cone* $N_A^b(S, x_0)$ to a closed set S at $x_0 \in S$ in the following manner: $x^* \in N_A^b(S, x_0)$ iff for each $L \in T(X)$ there exist nets $\varepsilon_i \to 0^+$, $x_i^* \xrightarrow{w^*} x^*$, $x_i \xrightarrow{S} x_0$, $c > 0$ satisfying for all i

$$\|x_i^*\| \le c \quad \text{and} \quad x_i^* \in \partial_{\varepsilon_i}^- \Psi_{S \cap (x_i + L)}(x_i).$$

It is a very slight variant of Ioffe's original definition and it is always contained in $N_A(S, x_0)$.

In fact, we need only the following variant $N_A^\beta(\,)$ defined by $x^* \in N_A^\beta(S, x_0)$ iff for each $L \in T(X)$ there exist nets $\varepsilon_i \to 0^+$, $x_i^* \xrightarrow{w^*} x^*$, $l_i^* \in L^\perp \xrightarrow{w^*} 0$, $x_i \xrightarrow{S} x_0$, $c > 0$ satisfying for all i

$$\|x_i^*\| \le c \quad \text{and} \quad x_i^* + l_i^* \in \partial_{\varepsilon_i}^- \Psi_{S \cap (x_i + L)}(x_i).$$

(Here $L^\perp := \{l^* \in X^*\colon \langle l^*, h \rangle = 0, \ \forall h \in L\}$).

When X is a WT-space and S a closed and compactly epi-Lipschitzian set at $x_0 \in S$, it follows from Proposition 2.1 that

$$N_A^b(S,x_0) = N_A^\beta(S,x_0) = N_A(S,x_0).$$

Now we use the notion of β or b-approximate normal cone to formulate the following result for any Banach space.

Proposition 2.2. *Let S be a closed and compactly epi-Lipschitzian set at $x_0 \in S$. Then there exist $s > 0$ and a norm-compact set $K \subset X$ such that for all $x \in (x_0 + sB_X) \cap S$ and all $x^* \in N_A^\beta(S,x)$,*

$$\|x^*\| \le \max_{h \in K} \langle x^*, h \rangle.$$

Consequently, for each $x \in (x_0 + sB_X) \cap S$, $N_A^b(S,x)$ and $N_A^\beta(S,x)$ are weak-star locally bounded.

Proof. Let r and H be as in Eq. (6) and set $s = r/2$. Let $x \in (x_0 + sB_X) \cap S$ and $x^* \in N_A^\beta(S,x)$. Let $b \in rB_X$ and let L be a separable subspace of X containing b and H. Then following Ioffe [12], L is a weak trustworthy subspace of X and hence there are $c > 0$ and nets $x_i \xrightarrow{S} x_0$ (with $x_i \in x + sB_X$), $x_i^* \xrightarrow{w^*} x^*$, $l_i^* \in L^\perp \xrightarrow{w^*} 0$, $\varepsilon_i \to 0^+$ such that for all i

$$\|x_i^*\| \le c \quad \text{and} \quad x_i^* + l_i^* \in \partial_{\varepsilon_i}^- \Psi_{S \cap (x_i + L)}(x_i)$$

which easily ensures

$$\langle x_i^* + l_i^*, h \rangle \le \varepsilon_i \|h\|, \quad \forall h \in K(S \cap (x_i + L), x_i), \tag{7}$$

where $K(C,x)$ denotes the contingent cone to C at $x \in C$, that is the set of all $h \in X$ for which there exist $t_n \to 0^+$ and $h_n \to h$ with $x + t_n h_n \in C$ for all integers n. By Eq. (6), there exists $h_i^n \in H$ such that $x_i + t_n(b + h_i^n) \in S$ for all n, where $t_n := \min(n^{-1}, s)$. Extracting subsequences (if necessary) we may assume that $h_i^n \to h_i$, with $h_i \in H$ and hence it follows, from the definition of L, that

$$b + h_i \in K(S \cap (x_i + L), x_i).$$

Using Eq. (7), we get

$$\langle x_i^*, b + h_i \rangle \le \varepsilon_i \|b + h_i\|.$$

But we may also suppose (h_i) converges to some $h_0 \in H$. Then the last inequality and the boundedness of (x_i^*) ensures

$$\langle x^*, b \rangle \le \langle x^*, -h_0 \rangle$$

and hence

$$r\|x^*\| \le \max_{h \in H} \langle x^*, -h \rangle.$$

The weak-star local boundedness of $N_A^{\beta}(S,x)$ follows from the first part and the result by Loewen [29] recalled in Eq. (5). □

Remark. *The result of this proposition also holds for the* (*Mordukhovich–Kruger*) *limiting Fréchet normal cone since it is contained in the b-approximate normal cone.*

Example 2 (*Clarke normal cone*). The Clarke tangent cone $T_c(S,x)$ to a set $S\subset X$ at $x\in S$ is the set of all $h\in X$ for which we have for all sequences $x_n \xrightarrow{S} x$ and $t_n \to 0^+$ the existence of $h_n \to h$ such that $x_n + t_n h_n \in S$, for all integers n. The Clarke normal cone $N_c(S,x)$ is the set

$$N_c(S,x) = \{x^* \in X^*:\ \langle x^*, h\rangle \leq 0,\ \forall h \in T_c(S,x)\}.$$

Following Borwein [2], a set S is said to be *epi-Lipschitz-like* at $x\in S$ if there exist $r>0$ and a convex set Ω such

- $\Omega^0 := \{x^* \in X^*:\ \langle x^*, h\rangle \leq 1,\ \forall h\in\Omega\}$ is locally compact and
- $S\cap(x+rB_X)+t\Omega\subset S$, for all $t\in]0,r[$.

Note that every epi-Lipschitz-like set is compactly epi-Lipschitzian and the converse is in general false (see [2]).

Proposition 2.3. *Let S be epi-Lipschitz-like at $x_0\in S$. Then there exist $s>0$ and a norm-compact set $H\subset X$ such that for all $x\in(x_0+sB_X)\cap S$ and all $x^*\in N_c(S,x)$*

$$\|x^*\| \leq \max_{h\in H}\langle x^*, -h\rangle.$$

Consequently for all $x\in(x_0+sB_X)\cap S$, $N_c(S,x)$ is locally compact.

Proof. The inequality has been established in [23] (see also [18]). □

3. Weak-star sequential upper-semicontinuity of normal cones and subdifferentials

We begin by recalling that a Banach space Z is weakly compactly generated (WCG) provided that there is a weakly compact set K such that $Z = \mathrm{cl}(\mathrm{span}\, K)$ (for more details see [6]). For this class of spaces, Ioffe [10] showed that the A-normal cone of any closed set S containing x_0 can be expressed as follows:

$$N_A(S,x_0) = \limsup_{\substack{x \xrightarrow{S} x_0 \\ \varepsilon\to 0^+}} \partial_\varepsilon^- \Psi_S(x),$$

since every WCG-space is a weak trustworthy space (see [12]).

We will need the following result whose proof follows from an obvious adaptation of that of Proposition 3.3 in Mordukhovich and Shao [34].

Proposition 3.1. *Let (M,d) be a metric space with $u_0 \in M$, X be a WCG-space and let F be a multivalued mapping between M and X^*. Equip $M \times X^*$ with the $d \times$ weak-star topology and assume that there exists a weak-star locally bounded, weak-star closed set $K^* \subset X^*$ and a real-valued function $\rho : M \to [0, +\infty[$ with $\lim_{u \to x_0} \rho(u) = 0$ such that*

$$F(u) \subset K^* + \rho(u) B_{X^*}, \quad \forall u \in M.$$

Then

$$\limsup_{u \to u_0} F(u) = seq - \limsup_{u \to u_0} F(u).$$

We can now prove the following theorem which is in the line of Theorem 3.4 in [29] (see also Proposition 3.4 in [34] and Proposition 12 in [39]).

Theorem 3.2. *Let X be a WCG-space and S be a closed nonempty subset of X containing x_0. Suppose that there exists a weak-star locally bounded, weak-star closed set $K^* \subset X^*$, a real number $s > 0$ and a function $\rho :]0, \infty[\to [0, \infty[$ such that* $\lim_{\varepsilon \to 0^+} \rho(\varepsilon) = 0$ *and*

$$\partial_\varepsilon^- \Psi_S(x) \subset K^* + \rho(\varepsilon) B_{X^*}, \quad \forall x \in (x_0 + sB_X) \cap S, \quad \forall \varepsilon \in [0, s].$$

Then

$$N_A(S, x_0) = seq - \limsup_{\substack{x \xrightarrow{S} x_0 \\ \varepsilon \to 0^+}} \partial_\varepsilon^- \Psi_S(x).$$

Proof. Setting $M = (x_0 + sB_X) \cap S \times [0, s]$ and $F(x, \varepsilon) = \partial_\varepsilon^- \Psi_S(x)$, it suffices to apply Proposition 3.1. $\square$

Using this theorem and Propositions 2.1 and 2.3 we obtain the following results.

Corollary 3.3. *Let X be a WCG-space and S be a closed subset of X containing x_0. Suppose that S is compactly epi-Lipschitzian at x_0. Then the conclusion of Theorem 3.2 holds.*

Proof. By Loewen's result recalled in Eq. (5) and Proposition 2.1 there exist $s > 0$ and a weak-star closed and weak-star locally compact cone K^* such that $N_A(S, x) \subset K^*$ for any $x \in x_0 + sB_X$. Fix any such x and any $\varepsilon > 0$. Then for $f := \psi_S$ and $x^* \in \partial_\varepsilon^- f(x)$ one has for any $h \in X$

$$\langle x^*, h \rangle \le d^- f(x, h) + \varepsilon \|h\| = d^-(f + \|. - x\|)(x, h)$$

and hence

$$x^* \in \partial^-(f + \|. - x\|)(x) \subset \partial_A(f + \|. - x\|)(x) \subset N_A(S, x) + \varepsilon B_{X^*}.$$

So $\partial_\varepsilon^- \psi_S(x) \subset K^* + \varepsilon B_{X^*}$ and hence the assumptions of Theorem 3.2 are satisfied. $\square$

The following lemma will be used in the next corollary. See Ioffe [14] and Kruger [27] for a similar result about Fréchet subdifferentials.

Lemma 3.4. *Let f be a function from a normed space X into $R \cup \{+\infty\}$. If $(x^*, -1) \in \partial_\varepsilon^- \Psi_{epi\, f}(x, f(x))$ with $\varepsilon \in [0, 1[$, then $x^* \in \partial_{\varepsilon'}^- f(x)$ where $\varepsilon' := \varepsilon(1 + \|x^*\|)/(1 - \varepsilon)$.*

Proof. It is well known (and not difficult to see) that $\Psi_{epi\, f}^-((x, f(x)), .)$ is the indicator function of $epi\, d^- f(x, .)$. Fix any $v \in X$ with $d^- f(x, v) < \infty$. As $(x^*, -1) \in \partial_\varepsilon^- \Psi_{epi\, f}(x, f(x))$, one has for any real number $r \geq d^- f(x, v)$

$$\langle x^*, v\rangle - r - \varepsilon|r| \leq \varepsilon\|v\|,$$

which implies that $d^- f(x, v)$ is finite and

$$\langle x^*, v\rangle \leq d^- f(x, v) + \varepsilon(\|v\| + |d^- f(x, v)|).$$

So considering both cases $d^- f(x, v) \geq 0$ and $d^- f(x, v) < 0$, one gets for $s := \text{sign}\,(d^- f(x, v))$

$$(1 + s\varepsilon)\langle x^*, v\rangle \leq (1 + s\varepsilon) d^- f(x, v) + \varepsilon(1 + \|x^*\|)\|v\|,$$

and hence for $\varepsilon_s' := \varepsilon(1 + \|x^*\|)/(1 + s\varepsilon)$ and $\varepsilon' := \varepsilon(1 + \|x^*\|)/(1 - \varepsilon)$ one obtains

$$\langle x^*, v\rangle \leq d^- f(x, v) + \varepsilon_s'\|v\| \leq d^- f(x, v) + \varepsilon'\|v\|.$$

This ensures $x^* \in \partial_{\varepsilon'}^- f(x)$ and completes the proof of the lemma.

Remark. *Note that one obviously has $(x^*, -1) \in \partial_\varepsilon^- \Psi_{epi\, f}(x, f(x))$ whenever $x^* \in \partial_\varepsilon^- f(x)$.*

Corollary 3.5. *Let X be a WCG-space and $f : X \to R \cup \{\infty\}$ be a lower semicontinuous function whose epigraph epi f is compactly epi-Lipschitzian at $(x_0, f(x_0))$. Then*

$$\partial_A f(x_0) = seq - \limsup_{\substack{x \xrightarrow{f} x_0 \\ \varepsilon \to 0^+}} \partial_\varepsilon^- f(x).$$

Proof. Fix $x^* \in \partial_A f(x_0)$. Then $(x^*, -1) \in N_A(\text{epi}\, f, x_0, f(x_0))$ and hence by Corollary 3.3 there exist sequences $(\varepsilon_n) \to 0^+$ with $\varepsilon_n < 1$ and $(x_n^*, -r_n) \xrightarrow{w^*} (x^*, -1)$ with $r_n > 0$, $(x_n^*, -r_n) \in \partial_{\varepsilon_n}^- \Psi_{epi\, f}(x_n, s_n)$ and $(x_n, s_n) \to (x_0, f(x_0))$. Then $s_n \geq f(x_n)$ and $f(x_n) \to f(x_0)$ since f is lower semicontinuous and $s_n \to f(x_0)$. So we may suppose that $f(x_n)$ is finite for all integers $n \geq 1$. Moreover, for any $(v, \lambda) \in X \times R$ and any $t > 0$ one has for $S := epi\, f$

$$t^{-1}[\Psi_S(x_n + tv, s_n + t\lambda) - \Psi_S(x_n, s_n)] \leq t^{-1}[\Psi_S(x_n + tv, f(x_n) + t\lambda) - \Psi_S(x_n, f(x_n))]$$

and hence

$$\partial_{\varepsilon_n}^- \Psi_S(x_n, s_n) \subset \partial_{\varepsilon_n}^- \Psi_S(x_n, f(x_n)).$$

It follows that, for $\gamma_n := r_n^{-1}\varepsilon_n$, one obtains

$$(r_n^{-1}x_n^*, -1) \in \partial_{\gamma_n}\Psi_S(x_n, f(x_n))$$

and hence Lemma 3.4 completes the proof. □

One may also show (see [35]) that, under assumptions of Corollary 3.5,

$$\partial_{Fré} f(x_0) \overset{\text{def}}{=} seq - \limsup_{\substack{x \xrightarrow{f} x_0 \\ \varepsilon \to 0^+}} \hat{\partial}^{\varepsilon}_{Fré} f(x) = \limsup_{\substack{x \xrightarrow{f} x_0 \\ \varepsilon \to 0^+}} \hat{\partial}^{\varepsilon}_{Fré} f(x),$$

where

$$\hat{\partial}^{\varepsilon}_{Fré} f(x) = \left\{ x^* \in X^*:\ \liminf_{h \to 0} \|h\|^{-1}(f(x+h) - f(x) - \langle x^*, h\rangle) \geq -\varepsilon \right\}.$$

$\hat{\partial}^{\varepsilon}_{Fré} f(x)$ and $\partial_{Fré} f(x)$ are, respectively, called the set of Fréchet ε-subgradients and the limiting Fréchet subdifferential (in the sense of [31]) of f at x. When $\varepsilon = 0$ the set $\hat{\partial}^{\varepsilon}_{Fré} f(x)$ is called the set of Fréchet subgradients of f at x and is denoted by $\hat{\partial}_{Fré} f(x)$.

In the following corollary we give conditions ensuring that the A-subdifferential and the limiting Fréchet subdifferential coincide (see also [3, 14, 35]). Note that in Remark 9.4(b) in [35] it is stated that the result also holds for a more general class of non-Lipschitzian functions.

Corollary 3.6. *Let X be an Asplund space which is WCG (e.g., a separable Asplund space or a reflexive space) and $f : X \to R$ be a locally Lipschitzian function at x_0. Then*

$$\partial_A f(x_0) = \partial_{Fré} f(x_0).$$

Proof. The proof is similar to that given by Ioffe [14, Lemma 4]. First note that since f is locally Lipschitz at x_0 its epigraph is compactly epi-Lipschitzian at $(x_0, f(x_0))$ (see [2]).

Obviously $\partial_{Fré} f(x_0) \subset \partial_A f(x_0)$. Conversely, let V^* be any weak-star neighbourhood of 0 in X^*. Then there exists a finite-dimensional subspace L of X such that $L^{\perp} \subset V^*$, where $L^{\perp} = \{x^* \in X^*:\ \langle x^*, h\rangle = 0,\ \forall h \in L\}$. So let $x^* \in \partial_A f(x_0)$. Then there are, by Corollary 3.5, sequences $x_n \to x_0$, $\varepsilon_n \to 0^+$ and $x_n^* \to x^*$ such that $\|x_n^*\| \leq R$, and $x_n^* \in \partial^-_{\varepsilon_n} f(x_n)$ for all n. By Lemma 1 in [9] and since f is locally Lipschitzian with constant $k_f > 0$ the function

$$x \to f(x) - \langle x_n^*, x - x_n\rangle + 2\varepsilon_n\|x - x_n\| + (R + k_f + 2\varepsilon_n)d(x, x_n + L)$$

attains a local minimum at x_n. As in [14, Lemma 4] and using calculus rules in [35], one gets that $x^* \in \partial_{Fré} f(x_0) + L^{\perp} \subset \partial_{Fré} f(x_0) + V^*$. Hence, using the comments following Corollary 3.5 one obtains $x^* \in \partial_{Fré} f(x_0)$.

Theorem 3.7. *Let X be a WCG-space and S be a closed subset of X containing x_0. Suppose that S is epi-Lipschitz-like at x_0. Then*

$$\limsup_{x \xrightarrow{S} x_0} N_c(S,x) = seq - \limsup_{x \xrightarrow{S} x_0} N_c(S,x).$$

4. Partially ∂_A-coderivatively compact multivalued mappings

In this section we consider the notion of partially coderivatively compact multivalued mappings. This notion will be used in Section 6 to give sufficient conditions ensuring the metric regularity of multivalued mappings.

Let F be a multivalued mapping between X and Y and let $(x_0, y_0) \in GrF$. The ∂_A-coderivative of F at (x_0, y_0) is the multivalued mapping $D_A^*F(x_0, y_0)$ defined from Y^* into X^* by

$$D_A^*F(x_0, y_0)(y^*) = \{x^* \in X^*\colon\ (x^*, -y^*) \in R_+\partial_A d(x_0, y_0, GrF)\}.$$

Example 3 (*Case F is a strongly compactly Lipschitzian mapping***).** Following Jourani and Thibault [21] a mapping F is said to be strongly compactly Lipschitzian at x_0 if there exist a function $r : X \times X \to R_+$ and a multivalued mapping S from X into $\mathrm{Comp}(Y)$, where $\mathrm{Comp}(Y)$ denotes the collection of all norm-compact subsets of Y, satisfying the following properties

(i) $\lim_{x \to x_0\, h \to 0} r(x,h) = 0$;

(ii) there exists $\mu > 0$ such that for all $h \in \mu B_X$, $x \in x_0 + \mu B_X$ and $t \in]0, \mu[$

$$t^{-1}(F(x+th) - F(x)) \in S(h) + \|h\| r(x,th) B_Y;$$

(iii) $S(0) = \{0\}$ and S is upper semicontinuous.

Thibault [42] showed that every strongly compactly Lipschitzian mapping is locally Lipschitzian and the converse holds whenever Y is finite dimensional.

It is shown in Jourani and Thibault [22] (after Ioffe [11] for mappings with compact prederivatives) that the A-coderivative of a strongly compactly Lipschitzian mapping F can be characterized as follows:

$$D_A^*F(x_0, F(x_0))(y^*) = \partial_A(y^* \circ F)(x_0).$$

In the case F is strictly differentiable at x_0 one gets

$$D_A^*F(x_0, F(x_0))(y^*) = y^* \circ \nabla F(x_0).$$

We stated in the introduction that condition (2) is what is needed in the proof of our main result in [24]. Here we must precise that our purpose in [24] was first to assume more easily verifiable properties of the multivalued mapping F which do not consider the normal cones. Conditions in the line of condition (2) are also stated in the remark following Theorem 2.12 of our paper [23] and are also used in [15, 34, 35, 39]. They also appear in a clear way in Proposition 3.10 in [34]. Now we give the following definition (see also [15, 34, 39]).

Definition 4.1. A multivalued mapping $F: X \rightrightarrows Y$ is said to be partially ∂_A-coderivatively compact at $(x_0, y_0) \in GrF$ if for every net $((x_i, y_i)) \subset GrF$ converging to (x_0, y_0) and every net $((x_i^*, y_i^*))$ satisfying $x_i^* \in D_A^* F(x_i, y_i)(y_i^*)$, $\|x_i^*\| \to 0$ and $y_i^* \xrightarrow{w^*} 0$ we have $\|y_i^*\| \to 0$.

More generally for any presubdifferential ∂ (see Section 6) one may define the notion of partially ∂-coderivative compactness by replacing ∂_A and D_A^* by ∂ and D_∂^*, where D_∂^* is the coderivative associated to the presubdifferential ∂ and defined by

$$x^* \in D_\partial^* F(x, y)(y^*) \Leftrightarrow (x^*, -y^*) \in R_+ \partial d(x, y, GrF)$$

for any $(x, y) \in GrF$.

We will be mainly concerned in this section by the approximate subdifferential ∂_A and the limiting Fréchet subdifferential which are presubdifferential on any Banach space and any Asplund space respectively.

Definition 4.2. F is said to be metrically regular at $(x_0, y_0) \in GrF$ if there exist $r > 0$ and $a > 0$ such that

$$d(x, F^{-1}(y) \leq ad(y, F(x))$$

for all $(x, y) \in (x_0 + rB_X) \times (y_0 + rB_Y)$ with $d(y, F(x)) \leq r$.

We start with the following result which is due to Mordukhovich and shao [36]. For the convenience of the reader we give a proof.

Proposition 4.3. *Let $\hat{D}_{Fr\acute{e}}^*$ denote the Fréchet coderivative. Let $F: \rightrightarrows Y$ be a multivalued mapping with closed graph between normed spaces X and Y. If F is metrically regular at $(x_0, y_0) \in GrF$, then there exist $c > 0$ and $\rho > 0$ such that for all $(x, y) \in GrF \cap (x_0 + \rho B_X) \times (y_0 + \rho B_Y)$, $y^* \in Y^*$ and $x^* \in \hat{D}_{Fr\acute{e}}^* F(x, y)(y^*)$*

$$c\|y^*\| \leq \|x^*\|.$$

Proof. By metric regularity fix $a \geq 1$ and $\rho > 0$ such that

$$d(x, F^{-1}(y)) \leq ad(y, F(x)) \tag{8}$$

for all $(x, y) \in (x_0 + \rho B_X) \times (y_0 + \rho B_Y)$ with $d(y, F(x)) \leq 2\rho$. Let $(x, y) \in GrF \cap (x_0 + \rho B_X) \times (y_0 + \rho B_Y)$, $y^* \in Y^*$ and $x^* \in \hat{D}_{Fr\acute{e}}^* F(x, y)(y^*)$. Fix any $\varepsilon > 0$ and any $\gamma > 1$. Then there exists $r \in]0, \rho[$ such that for all $u \in x + 2rB_X$, $v \in y + 2rB_Y$ with $(u, v) \in GrF$

$$\langle x^*, u - x\rangle - \langle y^*, v - y\rangle \leq \varepsilon(\|u - x\| + \|v - y\|). \tag{9}$$

Now fix any $v \in y + a^{-1}rB_Y$. Then $d(y, F(x)) \leq a^{-1}r \leq \rho$ and hence by Eq. (8) there exists $u \in F^{-1}(v)$ such that

$$\|x - u\| \leq \gamma ad(v, F(x)) \leq \gamma a\|v - y\| \leq \gamma aa^{-1}r = \gamma r.$$

By Eq. (9) we have

$$\langle y^*, y - v\rangle \leq (\gamma\|x^*\| + \varepsilon(\gamma + a^{-1}))r$$

and, since this holds for any $v \in y + a^{-1}rB_Y$, one gets

$$a^{-1}r\|y^*\| \leq (\gamma\|x^*\| + \varepsilon(\gamma + a^{-1}))r$$

and hence

$$a^{-1}\|y^*\| \leq \gamma\|x^*\| + \varepsilon(\gamma + a^{-1})$$

which ensures that

$$a^{-1}\|y^*\| \leq \|x^*\|.$$

In the sequel, we will say that a multivalued mapping F between X and Y is ∂_A-coderivatively regular at $(x_0, y_0) \in GrF$ if $D_A^*F(x_0, y_0) = \hat{D}^*_{Fré}F(x_0, y_0)$. Note that this class includes that of all convex multivalued mappings. The following corollaries are then consequences of Proposition 4.3.

Corollary 4.4. *Let F be a multivalued mapping with closed graph between X and Y. If F is metrically regular and ∂_A-coderivatively regular at $(x_0, y_0) \in GrF$, then F is partially ∂_A-coderivatively compact at (x_0, y_0).*

Taking Robinson Theorem [40] into account we also have

Corollary 4.5. *If F is a multivalued mapping between X and Y with closed and convex graph such that y_0 is an internal point of its range, then F is partially ∂_A-coderivatively compact at all points (x_0, y_0) with $x_0 \in F^{-1}(y_0)$.*

We are going to study the partial coderivative compactness with respect to the approximate subdifferential.

Note first that it follows from Eq. (4) if a multivalued mapping F satisfies the following property:

$$(D_A^*F(x, y))^{-1}(x^*) \subset K^*, \quad \forall (x, y) \in V \cap GrF,\ x^* \in U$$

for some locally compact cone K^* and neighbourhoods V and U of $(x_0, y_0) \in GrF$ and $0 \in X^*$ respectively, then F is partially A-coderivatively compact at $(x_0, y_0) \in GrF$.

In order to give other sufficient conditions ensuring partially ∂_A-coderivative compactness of multivalued mappings in the nonconvex case, let us recall that [24] F is said to be *partially compactly epi-Lipschitzian* at $(x_0, y_0) \in GrF$ if there exist $r > 0$, $b > 0$ and two norm-compact sets $H \subset X$ and $K \subset Y$ such that

$$GrF \cap ((x_0, y_0) + rB_{X\times Y}) + t(\{0\} \times B_Y) \subset GrF - t(H \times K), \quad \forall t \in]0, r[.$$

F is said to be *uniformly compactly epi-Lipschitzian* at $(x_0, y_0) \in GrF$ if the last relation holds for $H = \{0\}$.

As in Proposition 2.1, we may establish the following result (see [24]).

Proposition 4.6. *If F is partially compactly epi-Lipschitzian at $(x_0,y_0)\in GrF$, then there exist $r>0$ and two norm-compact sets $H\subset X$ and $K\subset Y$ such that*

$$\|y^*\|\le \max_{h\in H}\langle x^*,h\rangle+\max_{k\in K}\langle y^*,k\rangle$$

for all $(x,y)\in(x_0+rB_X)\times(y_0+rB_Y)\cap GrF$ and $(x^*,y^*)\in\partial_A d(x,y,GrF)$. Moreover F is partially ∂_A-coderivatively compact.

In the following theorem we prove relationship between metric regularity, partially compact epi-Lipschitzness and partially ∂_A-coderivative compactness properties.

Theorem 4.7. *Let F be a multivalued mapping between X and Y and let $(x_0,y_0)\in GrF$. Consider the following assertions:*
(a) *F^{-1} is partially compactly epi-Lipschitzian at (y_0,x_0);*
(b) *F is metrically regular at (x_0,y_0);*
(c) *GrF is compactly epi-Lipschitzian at (x_0,y_0);*
(d) *F is partially compactly epi-Lipschitzian at (x_0,y_0);*
(e) *F is partially ∂_A-coderivatively compact at (x_0,y_0).*
Then (a) + (b) $\Rightarrow$ (c) $\Leftrightarrow$ (a) + (d) *and* (d) $\Rightarrow$ (e).

Proof. (a) + (b) $\Rightarrow$ (c): Since F is metrically regular at (x_0,y_0) there exist $r>0$ and $a>0$ such that

$$d(x,F^{-1}(y)\le ad(y,F(x)) \tag{10}$$

for all $x\in x_0+rB_X$ and $y\in y_0+rB_Y$ with $d(y,F(x))\le r$. As F^{-1} is partially compactly epi-Lipschitzian at (y_0,x_0) there exist $s>0$ and $c>0$ and two norm-compact sets $H\subset X$ and $K\subset Y$ such that

$$GrF^{-1}\cap(y_0+sB_Y)\times(x_0+sB_X)+t(\{0\}\times cB_X)\subset GrF^{-1}-t(K\times H) \tag{11}$$

for all $t\in]0,s[$. Set $\eta=\min(r,s/(1+2a))$. One may assume that $3a\le c$ (see the remark below). Fix $(x,y)\in GrF\cap(x_0+\eta B_X)\times(y_0+\eta B_Y)$, $t\in]0,\eta[$, $b_1\in B_X$ and $b_2\in B_Y$. Then $y+tb_2\in y_0+rB_Y$ and $d(y+tb_2,F(x))\le r$ and hence, by Eq. (10) (since $d(y+tb_2,F(x))\le t$), there exists $b'\in B_X$ such that $u:=x+2atb'\in F^{-1}(y+tb_2)$. Note that $(y+tb_2,u)\in GrF^{-1}\cap(y_0+sB_Y)\times(x_0+sB_X)$. So, by Eq. (11),

$$(y+tb_2,u)+t(0,-2ab'+ab_1)\in GrF^{-1}-t(K\times H)$$

or equivalently

$$(x,y)+t(ab_1,b_2)\in GrF-t(H\times K).$$

Thus, for all $t\in]0,\eta[$,

$$GrF\cap(x_0+\eta B_X)\times(y_0+\eta B_Y)+t\min(1,a)(B_X\times B_Y)\subset GrF-t(H\times K)$$

and hence GrF is compactly epi-Lipschitzian at (x_0,y_0).

(c) ⇒ (a) + (d) is obvious and (d) ⇒ (e) follows from Proposition 4.3. So let us prove the implication (a) + (d) ⇒ (c). Since F is partially compactly epi-Lipschitzian at (x_0, y_0) there are $s>0$, $c>0$ and norm-compact sets H_1 in X and K_1 in Y such that

$$GrF \cap (x_0 + sB_X) \times (y_0 + sB_Y) + t(\{0\} \times cB_Y) \subset GrF - t(H_1 \times K_1) \tag{12}$$

for all $t \in]0, s[$. Set $\eta = s/(2(|H_1| + 1 + |K_1|))$. Let $(x, y) \in GrF \cap (x_0 + \eta B_X) \times (y_0 + \eta B_Y)$, $t \in]0, \eta[$, $b_1 \in cB_X$ and $b_2 \in cB_Y$. By Eq. (10) there exist $h_1 \in H_1$ and $k_1 \in K_1$ such that $(x, y) + t(h_1, b_2 + k_1) \in GrF$. Then, by the choice of η, $(x, y) + t(h_1, b_2 + k_1) \in GrF \cap (x_0 + sB_X) \times (y_0 + sB_Y)$. So, by Eq. (11), there exist $h_2 \in H$ and $k_2 \in K$ such that

$$(x, y) + t(b_1 + h_1 + h_2, b_2 + k_1 + k_2) \in GrF$$

and hence

$$(x, y) + t(b_1, b_2) \in GrF - t((H + H_1) \times (K + K_1))$$

which ensures that GrF is compactly epi-Lipschitzian at (x_0, y_0).

Remark. *In the proof of Theorem* 4.7 *we assumed that* $3a \le c$. *Indeed, if* $3a > c$, *one has from Eq.* (11) *and from the inequality* $sc/3a < s$

$$GrF^{-1} \cap \left(y_0 + \frac{sc}{3a} B_Y\right) \times \left(x_0 + \frac{sc}{3a} B_X\right) + \frac{tc}{3a}(\{0\} \times 3aB_X)$$
$$\subset GrF^{-1} - \frac{tc}{3a}\frac{3a}{c}(K \times H)$$

for all $t \in]0, s[$ *and hence for* $s' = sc/3a$, $c' = 3a$, $K' = (3a/c)K$ *and* $H' = (3a/c)H$ *one obtains*

$$GrF^{-1} \cap (y_0 + s'B_Y) \times (x_0 + s'B_X) + t(\{0\} \times c'B_X) \subset GrF^{-1} - t(K' \times H')$$

for all $t \in]0, s'[$. *So we may assume that Eqs.* (10) *and* (11) *hold with* $3a \le c$.

5. Kernels of coderivatives of multivalued mappings

The sufficient conditions in Section 6 for the metric regularity will require the nullity of the kernel of the coderivative. We are going to prove first that the nullity of the kernel is also necessary for large classes of multivalued mappings.

Theorem 5.1. *Let F be a multivalued mapping between X and Y whose inverse multivalued mapping F^{-1} is uniformly compactly epi-Lipschitzian at (y_0, x_0). Suppose that F is metrically regular at (x_0, y_0). Then* $\operatorname{Ker} D^*_A F(x_0, y_0) = \{0\}$.

Proof. Let $y^* \in \operatorname{Ker} D^*_A F(x_0, y_0)$. Then there exists $\lambda > 0$ and $(0, v^*) \in \partial_A d(x_0, y_0, GrF)$ such that $y^* = \lambda v^*$. From the metric regularity of F we get the existence of $r > 0$ and $a > 0$ such that

$$d(x, F^{-1}(y)) \le ad(yF(x)) \tag{13}$$

for all $(x, y) \in (x_0, y_0) + rB_{X\times Y}$ with $d(y, F(x)) \le r$, and since F^{-1} is uniformly compactly epi-Lipschitzian at (y_0, x_0) there exist $s>0$, $c>0$ and a norm-compact set $H \subset X$ such that

$$F^{-1}(y) \cap (x_0 + sB_X) + tcB_X \subset F^{-1}(y) - tH \tag{14}$$

for all $y \in y_0 + sB_Y$ and $t \in]0, s[$. As in Theorem 4.6 one may suppose $c \ge 2a$.

Set $\eta = \min(r, s)$. Let $b \in B_Y$ and L and M two separable spaces containing, respectively, H and b. Since $(0, v^*) \in \partial_A d(x_0, y_0, GrF)$, there are nets $(x_i, y_i) \xrightarrow{GrF} (x_0, y_0)$ and $\varepsilon_i \to 0^+$ and a bounded net $(x_i^*, y_i^*) \xrightarrow{w^*} (0, v^*)$ such that

$$\langle x_i^*, h\rangle + \langle y_i^*, k\rangle \le \varepsilon_i \|(h,k)\| \tag{15}$$

for all $(h,k) \in K(GrF \cap ((x_i + L) \times (y_i + M)), x_i, y_i)$.

Let $\gamma \in]1, 2[$ and $t_n \to 0^+$. Then for i and n sufficiently large, there exists, by (13), $b' \in B_X$ such that $x_i + a\gamma t_n b' \in F^{-1}(y_i + t_n b)$. Now, by Eq. (14), there is $h_{n,i} \in H$ with

$$x_i + t_n(a\gamma' - a\gamma' + h_{n,i}) \in F^{-1}(y_i + t_n b),$$

or equivalently

$$(x_i, y_i) + t_n(h_{n,i}, b) \in GrF \cap (x_i + L) \times (y_i + M).$$

Extracting subsequence if necessary we may assume that $h_{n,i} \to h_i$, with $h_i \in H$, and hence $(h_i, b) \in K(GrF \cap ((x_i + L) \times (y_i + M)), x_i, y_i)$. Thus, by Eq. (15),

$$\langle y_i^*, b\rangle \le \varepsilon_i(|H| + 1) + \max_{h\in H} \langle x_i^*, h\rangle$$

and hence

$$\|y_i^*\| \le \varepsilon_i(|H| + 1) + \max_{h\in H} \langle x_i^*, h\rangle.$$

Using the boundedness of (x_i^*) and the lower semicontinuity of $\| \ \|$ and passing to the limit on i we obtain $\|v^*\| = 0$ and the proof is complete.

Remark. *Theorem* 5.1 *also holds for the limiting Fréchet coderivative. Note that for the limiting Fréchet coderivative and with X finite-dimensional the result was proved in* [33].

Consider now the kernel of the Fréchet coderivative. The following result is a direct consequence of Proposition 4.3.

Proposition 5.2. *Let X and Y be two normed spaces and let F be a multivalued mapping between X and Y. If F is metrically regular at* $(x_0, y_0) \in GrF$, *then* $\operatorname{Ker} \hat{D}^*_{Fré} F(x_0, y_0) = \{0\}$.

As in Section 4 some corollaries can be stated for coderivatively regular multivalued mappings.

6. Abstract ∂-coderivative of multivalued mappings

The results of Theorems 4.7 and 5.1 state that if F is metrically regular at (x_0, y_0) and F^{-1} is uniformly compactly epi-Lipschitzian at (y_0, x_0) (note that it is the case whenever X is finite-dimensional), then the following assertions hold:

(i) F is partially ∂_A-coderivatively compact at (x_0, y_0);

(ii) $Ker\, D_A^* F(x_0, y_0) = \{0\}$.

To illustrate the converse, let us recall [45, 46] that an abstract presubdifferential on X is any operator ∂ which satisfies the following properties: for any function $f : X \to R \cup \{\infty\}$, any convex continuous function $g : X \to R$ and any $x \in X$

(P_1) $\partial f(x) \subset X^*$ and $\partial f(x) = \emptyset$ if $f(x) = \infty$;

(P_2) $\partial g(x)$ coincides with the subdifferential in the sense of convex analysis;

(P_3) $0 \in \partial f(x)$ whenever x is a local minimum for f;

(P_4) $\partial f(x) = \partial h(x)$ whenever f and h coincide around x, and

(P_5) if f is lower semicontinuous near x in its domain, for any $x^* \in \partial(f + g)(x)$ and any $\varepsilon > 0$ there exist $\delta > 0$ and some u and v with $\|u - x\| \le \delta$,

$$|f(u) - f(x)| \le \delta, \|v - x\| \le \delta, |f(v) - f(x)| \le \delta,$$

$$x^* \in \partial f(u) + \partial g(v) + \varepsilon B_{X^*}.$$

Recall that the ∂-coderivative of a multivalued mapping F at (x_0, y_0) associated to a presubdifferential ∂ is denoted by $D_\partial^* F(x_0, y_0)$ and defined as follows:

$$D_\partial^* F(x_0, y_0)(y^*) = \{x^* \in X^* : (x^*, -y^*) \in R_+ \partial d(x_0, y_0, GrF)\}.$$

Recall also that F is partially ∂-coderivatively compact at (x_0, y_0) if Definition 4.1 holds for this presubdifferential.

Before giving our results in this section let us note that if f is locally Lipschitzian at x_0, then

$$\partial_A f(x_0) \subset \limsup_{x \xrightarrow{f} x_0} \partial f(x).$$

Indeed, let $x^* \in \partial_A f(x_0)$. Then [9, Lemma 1] for all finite-dimensional subspace L of X there are nets $x_i \xrightarrow{f} x_0$ and $x_i^* \xrightarrow{w*} x^*$ such that for all $\varepsilon > 0$ and i sufficiently large $\|x_i^*\| \le (k_f + \varepsilon)$ and the function

$$x \to f(x) - \langle x_i^*, x - x_i \rangle + 2(k_f + \varepsilon) d(x, x_i + L)$$

attains a local minimum at x_i. So, by the definition of ∂

$$x_i^* \in \limsup_{x \xrightarrow{f} x_i} \partial f(x_i) + 2(k_f + \varepsilon) B_{X^*} \cap L^\perp$$

where $L^\perp = \{x^* \in X^* : \langle x^*, x \rangle = 0, \forall x \in L\}$. So for any finite-dimensional subspace L of X one has $x^* \in \limsup_{x \xrightarrow{f} x_0} \partial f(x) + L^\perp$ and hence $x^* \in \limsup_{x \xrightarrow{f} x_0} \partial f(x)$.

The same proof leads to following formula for limiting Fréchet subdifferential (see also Theorem 9.7 in [34])

$$\partial_{Fré} f(x_0) \subset seq - \limsup_{x \xrightarrow{f} x_0} \partial f(x)$$

provided that f is locally Lipschitzian at x_0.

It is easy to verify that the result by Jourani [18] proved for A-subdifferential holds for any presubdifferential satisfying (P_1)–(P_5).

Theorem 6.1. *Let F be a multivalued mapping between X and Y with closed graph and ∂ be a presubdifferential over $X \times Y$. Let $(x_0, y_0) \in GrF$ and $k > 1$. Suppose that there are $c > 0$ and $r > 0$ such that for all $(x, y) \in GrF \cap (x_0 + rB_X) \times (y_0 + rB_Y)$, for all $y^* \in Y^*$, with $\|y^*\| = 1$, and all $(x^*, z^*) \in k\partial d(x, y, GrF)$*

$$\max(\|x^*\|, \|y^* + z^*\|) \geq c. \tag{16}$$

Then F is metrically regular at (x_0, y_0).

As a consequence of this theorem we have the following result.

Theorem 6.2. *Let F be a multivalued mapping between X and Y with closed graph containing (x_0, y_0). Suppose that F is partially ∂-coderivatively compact at (x_0, y_0) for some presubdifferential ∂ included in ∂_A and suppose $Ker\, D_A^* F(x_0, y_0) = \{0\}$. Then F is metrically regular at (x_0, y_0).*

Proof. It suffices to show that Eq. (16) is satisfied and to apply Theorem 6.1. So suppose the contrary. Then there are sequences $(x_n, y_n) \xrightarrow{GrF} (x_0, y_0), y_n^* \in Y^*$, with $\|y_n^*\| = 1$, and $(x_n^*, z_n^*) \in k\partial d(x_n, y_n, GrF)$ such that

$$\max(\|x_n^*\|, \|y_n^* + z_n^*\|) \leq 1/n. \tag{17}$$

As (z_n^*) is bounded, there is a subnet $(z_{n_i}^*)$ of (z_n^*) such that $z_{n_i}^* \xrightarrow{w^*} z^*$. We get, by (17), that $\|x_n^*\| \to 0$ and $z^* \in Ker\, D_A^* F(x_0, y_0)$ and by the nullity of the kernel of $D_A^* F(x_0, y_0)$ we have $z^* = 0$. Thus, we obtain $(x_{n_i}, y_{n_i}) \xrightarrow{GrF} (x_0, y_0)$ and $(x_{n_i}^*, z_{n_i}^*) \in R_+ \partial d(x_{n_i}, y_{n_i}, GrF)$ with $\|x_{n_i}^*\| \to 0$ and $z_{n_i}^* \xrightarrow{w^*} 0$ and since F is partially ∂-coderivatively compact at (x_0, y_0) we get $\|z_{n_i}^*\| \to 0$. But Eq. (17) implies that

$$1 - 1/n_i \leq \|z_{n_i}^*\| \leq 1 + 1/n_i$$

and consequently $\|z_{n_i}^*\| \to 1$. This contradiction completes the proof. $\square$

Taking the comments at the beginning of this section and Theorem 6.2 into account we get the following corollary (see also [34] for the limiting Fréchet coderivative and X finite dimensional).

Corollary 6.3. *Let F be a multivalued mapping between X and Y with closed graph containing (x_0, y_0). Assume that X is finite-dimensional (or more generally F^{-1} is uniformly compactly epi-Lipschitzian at (y_0, x_0)). Then F is metrically regular at (x_0, y_0) if and only if F is partially ∂_A-coderivatively compact at (x_0, y_0) and* $\ker D_A^* F(x_0, y_0) = \{0\}$.

Using in the proof above the weak-star sequential compactness of the unit ball of the dual of any Asplund space, we get the following theorem by Mordukhovich and Shao [34] stated for partially normally compact multivalued mappings. Note that their proof still holds for partially coderivatively compact multivalued mappings with respect to the Fréchet subgradients.

Theorem 6.4 (Mordukhovich and Shao [34]). *Let X and Y be Asplund spaces and F be a multivalued mapping between X and Y with closed graph. Suppose that F is sequentially partially coderivatively compact at $(x_0, y_0) \in GrF$ with respect to the Fréchet subgradients (i.e. Definition* 4.1 *holds for the Fréchet subgradient set with sequences instead of nets) and $Ker\, D_{Fré}^* F(x_0, y_0) = \{0\}$. Then F is metrically regular at (x_0, y_0).*

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References

[1] J.M. Borwein, Stability and regular point of inequality systems, J. Optim. Theory Appl. 48 (1986) 9–52.
[2] J.M. Borwein, Epi-Lipschitz-like sets in Banach space: theorems and examples, Nonlinear Anal. Theory Meth. Appl. 11 (1987) 1207–1217.
[3] J.M. Borwein, S.P. Fitzpatrick, Weak-star sequential compactness and bornological limit derivatives, CECM Research Report 93–102, 1993.
[4] J.M. Borwein, H.M. Strojwas, Tangential approximations, Nonlinear Anal. Theory Meth. Appl. 9 (1985) 1347–1366.
[5] J.M. Borwein, H.M. Strojwas, Proximal analysis and boundaries of closed sets in Banach space, Part I, Can. J. Math. 38 (1986) 431–452.
[6] J. Diestel, Geometry of Banach Spaces, Lecture Notes in Mathematics, vol. 485, Springer, New York, 1975.
[7] B. El Abdouni, Thèse d'état, Université Mohamed V, Rabat, 1994.
[8] B. El Abdouni, L. Thibault, Lagrange multipliers for Pareto nonsmooth programming problems in Banach spaces, Optimization 26 (1992) 277–285.
[9] A.D. Ioffe, Approximate subdifferentials and applications 1: the finite dimensional theory, Trans. Amer. Math. Soc. 281 (1984) 289–316.
[10] A.D. Ioffe, Approximate subdifferentials and applications 2: Functions on locally convex spaces, Mathematika 33 (1986) 111–128.
[11] A.D. Ioffe, Approximate subdifferentials and applications 3: Metric theory, Mathematika 36 (1989) 1–38.
[12] A.D. Ioffe, On subdifferentiability spaces, Ann NY Acad. Sci. 4 (1983) 107–120.

[13] A.D. Ioffe, On the local surjection property, Nonlinear Anal. Theory Meth. Appl. 11 (1987) 565–592.
[14] A.D. Ioffe, Proximal analysis and approximate subdifferentials, J. London Math. Soc. 41 (1990) 175–192.
[15] A.D. Ioffe, Codirectional compactness, metric regularity and subdifferential calculus, to appear.
[16] A. Jourani, Régularité métrique et ses applications en programmation mathématique, Thesis, Université de Pau, France, 1989.
[17] A. Jourani, Metric regularity and second order necessary optimality conditions for minimization problems under inclusion constraints, J. Optim. Theory Appl. 81 (1994) 97–120.
[18] A. Jourani, Qualification conditions for multivalued functions in Banach spaces with applications to nonsmooth vector optimization problems, Math. Prog. 66 (1994) 1–23.
[19] A. Jourani, Compactly epi-Lipschitzian sets and A-subdifferential in WT-spaces, Optimization 34 (1995) 1–17.
[20] A. Jourani, Intersection formulae and the marginal function in Banach spaces, J. Math. Anal. Appl. 192 (1995) 867–891.
[21] A. Jourani, L. Thibault, Approximations and metric regularity in mathematical programming in Banach space, Math. Oper. Res. 18 (1993) 390–401.
[22] A. Jourani, L. Thibault, Approximate subdifferentials of composite functions, Bull. Austral. Math. Soc. 47 (1993) 443–455.
[23] A. Jourani, L. Thibault, Extensions of subdifferential calculus rules in Banach spaces and applications (Preprint 1993), Can. J. Math. 48 (1996) 834–848.
[24] A. Jourani, L. Thibault, Verifiable conditions for openness and metric regularity of multivalued mappings in Banach spaces (Preprint 1993) Trans. Amer. Math. Soc. 347 (1995) 1255–1268.
[25] A. Jourani, L. Thibault, Metric regularity for strongly compactly Lipschitzian mappings (Preprint 1993), Nonlinear Anal. Theory Meth. Appl. 24 (1995) 229–240.
[26] A. Ya. Kruger, Properties of generalized differentials, Sib. Math. J. 26 (1985) 822–832.
[27] A. Ya. Kruger, A covering theorem for set-valued mappings, Optimization 19 (1988) 763–780.
[28] A. Ya. Kruger, B.S. Mordukhovich, Extremum points and the Euler equation in nonsmooth optimization, Dokl. Akad. Nauk. BSSR 24 (1980) 684–687.
[29] P.D. Loewen, Limits of Fréchet normals in nonsmooth analysis, in: A.D. Ioffe et al. (Eds.), Optimization and Nonlinear Analysis, Pitman Research Notes Math. Ser., Vol. 244, Pitman, London, 1992, pp. 178–188.
[30] B.S. Mordukhovich, Nonsmooth analysis with nonconvex generalized differentials and adjoint mappings, Dokl. Akad. Nauk BSSR 28 (1984) 976–979.
[31] B.S. Mordukhovich, Lipschitz stability of constraint systems and generalized equations, Nonlinear Anal. Theory Meth. Appl. 22 (1994) 173–206.
[32] B.S. Mordukhovich, Stability theory for parametric generalized equations and variational inequalities via nonsmooth analysis, Trans. Amer. Math. Soc. 343 (1994) 609–657.
[33] B.S. Mordukhovich, Complete characterization of openness, metric regularity, and Lipschitzian properties of multifunctions, Trans. Amer. Math. Soc. 340 (1993) 1–35.
[34] B.S. Mordukhovich, Y. Shao, Stability of set-valued mappings in infinite dimensions: point criteria and applications, preprint (1994), SIAM J. Control Optim. 35 (1997) 285–314.
[35] B.S. Mordukhovich, Y. Shao, Nonsmooth sequential analysis in Asplund spaces (preprint, 1994), Trans. Amer. Math. Soc. 348 (1996) 1235–1280.
[36] B.S. Mordukhovich, Y. Shao, Differential characterizations of covering, metric regularity, and Lipschitzian properties of multifunctions between Banach spaces, Nonlinear Anal. Theory Meth. Appl. 25 (1995) 1401–1424.
[37] J.P. Penot, Metric regularity, openness and Lipschitzian behavior of multifunctions, Nonlinear Anal. Theory Meth. Appl. 13 (1989) 629–643.
[38] J.-P. Penot, Miscellaneous incidences of convergence theories in optimization and nonlinear analysis I: behavior of solutions, Set-Val. Anal. 2 (1994) 259–274.
[39] J.-P. Penot, Compactness property, openness criteria and coderivatives, to appear.
[40] S.M. Robinson, Regularity and stability for convex multivalued functions, Math. Oper. Res. 1 (1976) 130–143.

[41] R.T. Rockafellar, Lipschitz property of multifunctions, Nonlinear Anal. Theory Meth. Appl. 9 (1985) 867–885.
[42] L. Thibault, Subdifferentials of compactly Lipschitzian vector-valued functions, Travaux du Sém. d'Analyse Convexe 8 (1978) Fasc. I, Montpellier.
[43] L. Thibault, Lagrange–Kuhn–Tucker multipliers for Pareto optimization problems, in: A. Ioffe, M. Marcus, S. Reich (Eds.), Optimization and Nonlinear Analysis, Pitman Research Notes in Mathematics Series, Pitman, London, 1992.
[44] L. Thibault, On subdifferentials of optimal value functions, SIAM J. Control Optim. 29 (1991) 1019–1036.
[45] L. Thibault, A note on the Zagrodny mean value theorem (Preprint, 1994), Optimization 35 (1995) 127–130.
[46] L. Thibault, D. Zagrodny, Integration of subdifferentials of lower semicontinuous functions in Banach spaces (Preprint 1992), J. Math. Anal. Appl. 189 (1995) 33–58.