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Limit Superior of Subdifferentials of Uniformly Convergent Functions

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Abstract. In this paper we show that the G – subdifferential of a lower semicontinuous function is contained in the limit superior of the G – subdifferential of lower semicontinuous uniformly convergent family to this function. It happens that this result is equivalent to the corresponding normal cones formulas for family of sets which converges in the sense of the bounded Hausdorff distance. These results extend to the infinite dimensional case those of Ioffe for C^2 – functions and of Benoist for Clarke's normal cone. As an application we characterize the subdifferential of any function which is bounded from below by a negative quadratic form in terms of its Moreau–Yosida proximal approximation.

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1. Introduction

In 1984, Ioffe [8] showed that in finite dimensional space the G – subdifferential $\partial_G f(x_0)$ of a function f at x_0 is smaler than any Warga's derivative containers [18] of f at x_0. He established the following result which is essentially due to Kruger–Mordukhovich [12–13] who proved it under some what stronger assumptions.

THEOREM 1.1. *Let f, f_n, $n = 1, 2, \dots$, be real-valued functions defined on some finite dimensional space X. Suppose $f_n, n = 1, 2, \dots$, are C^2 and the sequence (f_n) converges uniformly to f around x_0. Then*

$$\partial_G f(x_0) \subset \mathop{lim\,sup}_{\substack{n\to\infty \\ u_n \to x_0}} \{\nabla f_n(u_n)\}. \qquad \square$$

Here, as usual, $\nabla f_n(u_n)$ denotes the gradient of f_n at u_n.

In 1993, Benoist [4] extended this result to the uniformly convergent sequence (f_n) which are strictly differentiable. In fact he established the following more general result for normal cones to sets in finite dimension. His proof is essentially geometric.

THEOREM 1.2. *Let (C_n) be a sequence of nonempty closed subsets of some finite dimensional space X and C a nonempty closed subset of X such that for any bounded set $Q \subset X$*

$$\lim_{n\to\infty} haus_Q(C, C_n) = 0.$$

Then for each boundary point x_0 of C

$$N_G(C, x_0) \subset \limsup_{\substack{n\to\infty \\ u_n \in C_n \to x_0}} N_C(C_n, u_n). \qquad \square$$

Here $haus_Q(C, C_n)$ denotes the bounded Hausdorff distance between the sets C and C_n, $N_G(C_n, u_n)$ is the G-normal cone of Ioffe [10] and $N_C(C_n, u_n)$ is the Clarke's normal cone to C_n at u_n. Note that the Clarke normal cone contains the G-normal cone and this containment may be strict.

Our aim in this paper is twofold. First we extend both result to Banach spaces. Second we show that Theorem 1.1 remains true for any lower semicontinuous uniformly convergent sequence (f_n), by replacing $\nabla f_n(u_n)$ by $\partial_G f_n(u_n)$ and Theorem 1.2 is valid if we replace the Clarke normal cone by the G-normal cone. We show that both extension are equivalent. Note that our proof is completely different from the previous ones and it is essentially analytic and based on Ekeland variational principle [7].

The obtained results are used to characterize the subdifferential of lower semicontinuous functions which are bounded from below by a negative quadratic form in terms of their Moreau–Yosida proximal approximations.

2. Notations and Preliminaries

Throughout the paper X will be a Banach space, X^* its topological dual equipped with the weak-star topology w^*. We will denote by B_X the closed unit ball of X and by $d(\cdot, S)$ the distance function to a subset S of X

$$d(x, S) = \inf_{u\in S} \|x - u\|.$$

We will write $x \xrightarrow{f} x_0$ and $x \xrightarrow{S} x_0$ to express $x \to x_0$ with $f(x) \to f(x_0)$ and $x \to x_0$ with $x \in S$, respectively and we will denote by $epif$ the epigraph of a real valued extended function, i.e.,

$$epif = \{(x, r) : \quad f(x) \le r\}.$$

If C and D are nonempty subsets of X and if Q is a bounded subset of X, the bounded Hausdorff distance between C and D is denoted by

$$haus_Q(C, D) = \inf\{\varepsilon > 0 : Q \cap C \subset D + \varepsilon B_X \text{ and } Q \cap D \subset C + \varepsilon B_X\}.$$

If not specified the norm in a product of two Banach spaces is defined by $\|(a, b)\| = \|a\| + \|b\|$.

We will use the notations in Ioffe [9, 10].

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function in a neighborhood of $x_0 \in X$ with $f(x_0) < \infty$. The approximate subdifferential (see Ioffe [9, 10]), which is an extension to the context of Banach spaces of the concept introduced by Mordukhovich [14, 15] for finite dimensional spaces is defined by

$$\partial_A f(x_0) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{x \xrightarrow{f} x_0} \partial^- f_{x+L}(x)$$

where

$$\partial^- f(x) = \{x^* \in X^* : \langle x^*, h \rangle \leq d^- f(x; h), \forall h \in X\},$$

$$d^- f(x; h) = \liminf_{\substack{u \to h \\ t \downarrow 0}} t^{-1}(f(x + tu) - f(x)).$$

Here, for $S \subset X$, f_S denotes the function defined by

$$f_S(x) = \begin{cases} f(x) & \text{if } x \in S \\ +\infty & \text{otherwise,} \end{cases}$$

$\mathcal{F}(X)$ is the family of all finite dimensional subspaces of X and

$$\limsup_{x \xrightarrow{f} x_0} \partial^- f_{x+L}(x) = \{x^* \in X^* : x^* = w^* - \lim x_i^*, x_i^* \in \partial f_{x_i+L}(x_i), x_i \xrightarrow{f} x_0\},$$

that is the set of w^*-limits of all such nets.

The G-normal cone to S at $x_0 \in S$ is denoted by $N_G(S, x_0)$, that is,

$$N_G(S, x_0) = cl^* \bigcup_{\lambda > 0} \lambda \partial_A d(x_0, S).$$

The G-subdifferential $\partial_G f(x_0)$ of f at x_0 with $f(x_0) < \infty$ is given by (see [9, 10])

$$\partial_G f(x_0) = \{x^* \in X^* : (x^*, -1) \in N_G(\text{epi} f; x_0, f(x_0))\}.$$

Following Clarke [6], a vector $h \in X$ will be in the Clarke tangent cone $T_C(S, x_0)$ to S at x_0 if for any sequence $(x_n) \subset S$ converging to x_0 and any $t_n \to 0^+$ there exists $h_n \to h$ such that for all positive integers n

$$x_n + t_n h_n \in S.$$

The Clarke normal cone $N_C(S, x_0)$ to S at x_0 is the negative polar of $T_C(S, x_0)$, i.e.,

$$N_C(S, x_0) = \{x^* \in X^* : \langle x^*, x\rangle \le 0, \quad \forall x \in T_C(S, x_0)\}.$$

The following theorem lists some of the important properties of the G-subdifferential.

THEOREM 2.1 *[8–10]. Let f be an extended real-valued function on X which is lower semicontinuous around x_0, with $f(x_0) < \infty$ and let S be a closed subset of X containing x_0.*

(i) $N_C(C, x_0)$ *contains* $N_G(C, x_0)$.

(ii) If f is Lipschitz near x_0 with Lipschitz constant k_f

$$\partial_G f(x_0) = \partial_A f(x_0)x = \bigcap_{L \in \mathcal{F}(\chi)} \limsup_{\substack{x \xrightarrow{f} x_0 \\ \varepsilon \to 0^+}} \partial_\varepsilon^- f_{x+L}(x) \cap (k_f + \varepsilon)B_{X^*},$$

where

$$\partial_\varepsilon^- f(x) = \{x^* \in X^* : \langle x^*, h\rangle \le d^- f(x; h) + \varepsilon\|h\|, \forall h \in X\}.$$

(iii)

$$\partial_A f(x_0) = \limsup_{x \xrightarrow{f} x_0} \partial_A f(x) \quad (\textit{upper semicontinuity condition}).$$

(iv) If g is a locally Lipschitz function around x_0 then

$$\partial_G(f+g)(x_0) \subset \partial_G f(x_0) + \partial_G g(x_0).$$

(v) $x^* \in \partial_A d(C, x_0)$ iff there exist a family (L_i) of finite dimensional subspaces of X cofinal with $\mathcal{F}(X)$, $x_i^* \to x^*$, $x_i \xrightarrow{C} x_0$, $\varepsilon_i \to 0^+$ such that $x_i^* \in \partial_{\varepsilon_i}^- d_{x_i+L_i}(C, x_i) \cap (1+\varepsilon_i)B_{X^*}$ for all i.

A thorough discussion of these concepts can be found in [8–10] and [6]. We mention here that if f is convex, then the G-subdifferential coincides with the subdifferential of convex analysis.

3. Equivalence of the Extended Theorems

We begin by an extension of Ioffe's theorem [8] to the infinite dimensional spaces and for lower semicontinuous functions.

THEOREM 3.1. *Let f, f_j, $j \in J$ be lower semicontinuous real valued functions on X such that the net (f_j) converges uniformly to f around x_0. Then*

$$\partial_G f(x_0) \subset \limsup_{\substack{j \in J \\ u \to x_0 \\ f_j(u) \to f(x_0)}} \partial_G f_j(u). \qquad \square \tag{3.1}$$

The following theorem is a smooth version of the previous one.

THEOREM 3.2. *Let f, $f_j : X \to \mathbb{R}$, $j \in J$ be Lipschitz functions on X such that the net (f_j) converges uniformly to f around x_0. Then*

$$\partial_G f(x_0) \subset \limsup_{\substack{j \in J \\ u \to x_0}} \partial_G f_j(u). \qquad \square$$

Since the Clarke normal cone contains the G-normal cone, the following theorem is an extension of those of Benoist [3–5] to Banach spaces.

THEOREM 3.3. *Let $(C_j)_{j\in J}$ be a net of nonempty closed subsets of X and C a nonempty closed subset of X such that*
(i) x_0 is a boundary point of C and
(ii) there exists $r > 0$ such that

$$\lim_{j \in J} haus_{x_0 + rB_x}(C, C_j) = 0.$$

Then

$$\partial_A d(x_0, C) \subset \limsup_{\substack{j \in J \\ u \in C_j \to x_0}} 5\partial_A d(u, C_j)$$

and consequently

$$N_G(C, x_0) \subset \limsup_{\substack{j \in J \\ u \in C_j \to x_0}} N_G(C_j, u). \qquad \square$$

COROLLARY 3.4. *Suppose, in addition to the assumptions of Theorem 3.1 that the functions f, f_j, $j \in J$ are convex. Then (3.1) holds as equality.*

THEOREM 3.5. *Theorems 3.1, 3.2 and 3.3 are equivalent.*

Proof. Theorem 3.1 implies Theorem 3.2: evident.

Theorem 3.3 implies Theorem 3.1: We need the following lemma whose proof is based on Theorem 3.3.

LEMMA 3.6. *Let f, f_j, $j \in J$ be lower semicontinuous real valued functions on X such that for some $r > 0$*

$$\lim_{j \in J} haus_{(x_0, f(x_0)) + rB_{X\times\mathbb{R}}}(epi\ f, epi\ f_j) = 0.$$

If $(x^*, \alpha^*) \in \partial_A d(epi\ f, x_0, f(x_0))$, with $\alpha^* \neq 0$, then

$$(x^*, \alpha^*) \in \limsup_{\substack{j \\ u \to x_0 \\ f_j(u) \to f(x_0)}} 5\partial_A d(epi\ f_j, u, f_j(u)).$$

Consequently, if $(x^*, -1) \in N_G(epi\ f, x_0, f(x_0))$*, then*

$$(x^*, -1) \in \limsup_{\substack{u \xrightarrow{j} x_0 \\ f_j(u) \to f(x_0)}} N_G(epi\ f_j, u, f_j(u)).$$

Proof of Lemma 3.6. By Theorem 3.3 there are nets $(u_j, r_j) \to (x_0, f(x_0))$, with $(u_j, r_j) \in epi\ f_j$, for all $j \in J$, and $(x_j^*, \alpha_j^*) \to (x^*, \alpha^*)$ such that

$$(x_j^*, \alpha_j^*) \in 5\partial_A d(epi\ f_j, u_j, r_j), \forall j \in J.$$

Thus for all $j \in J$, $L \in \mathcal{F}(X)$, $\varepsilon > 0$ and weak-star neighbourhood V of 0 there exists $(u, s) := (x_{j,L,\varepsilon,V}, r_{j,L,\varepsilon,V}) \in ((x_j, r_j) + \varepsilon B_{X\times\mathbb{R}}) \cap epi\ f_j$ such that

$$1/5(x_j^*, \alpha_j^*) \in \partial_\varepsilon^- d_{(u,s)+L\times\mathbb{R}}(epi\ f_j, u, s) + V \times \varepsilon B_{\mathbb{R}}.$$

So there exists $(u^*_{j,L,\varepsilon,V}, s^*_{j,L,\varepsilon,V}) \in \partial_\varepsilon^- d_{(u,s)+L\times\mathbb{R}}(epi\ f_j, u, s)$ such that

$$(x_j^*, \alpha_j^*) - 5(u^*_{j,L,\varepsilon,V}, s^*_{j,L,\varepsilon,V}) \in 5(V \times \varepsilon B_{\mathbb{R}}).$$

Since $\alpha^* \neq 0$ we get $s = f_j(u)$. Indeed from Lemma 1 in [8] and Proposition 2.3.4 in [6] there exists $r_j > 0$ such that for all $(x, r) \in (u, s) + r_j B_{X\times\mathbb{R}}$

$$\begin{aligned} &d(x, r, epi\ f) - \langle u^*_{j,L,\varepsilon,V}, x - u\rangle - s^*_{j,L,\varepsilon,V}(r - s) \\ &\quad + 2\varepsilon[\|x - u\| + |r - s|] + 3d(x, u + L) \geq 0. \end{aligned}$$

So if $f_j(u) < s$ we obtain for some $r'_j \in]0, r_j[$ that for all $r \in s + r'_j B_{\mathbb{R}}$, $f_j(u) < r$ and hence

$$-s^*_{j,L,\varepsilon,V}(r - s) + 2\varepsilon|r - s| \geq 0$$

and this implies that $|s^*_{j,L,\varepsilon,V}| \leq 2\varepsilon$. But ε is arbitrary and $(s^*_{j,L,\varepsilon,V})$ goes to α^*, then we obtain a contradiction with $\alpha^* \neq 0$. So $s = f_j(u)$. As $(x_j^*, \alpha_j^*) - 5(u^*_{j,L,\varepsilon,V}, s^*_{j,L,\varepsilon,V}) \to 0$, it follows that

$$(x^*, \alpha^*) \in \limsup_{\substack{u \xrightarrow{j} x_0 \\ f_j(u) \to f(x_0)}} 5\partial_A d(epi\ f_j, u, f_j(u)).$$

Proof of Theorem 3.5 (continued). Since (f_j) converges uniformly to f around x_0 there exists $r > 0$ such that

$$\lim_{j\in J} haus_{(x_0, f(x_0))+rB_{X\times\mathbb{R}}}(epi\ f, epi\ f_j) = 0.$$

Let $x^* \in \partial_G f(x_0)$, then $(x^*, -1) \in N_G(epi\ f, x_0, f(x_0))$ and there exist, by Lemma 3.6, $u_j \to x_0$, $f_j(u_j) \to f(x_0)$ and $(x_j^*, \alpha_j^*) \to (x^*, -1)$ such that

$(x_j^*, \alpha_j^*) \in N_G(epi\ f_j, u_j, f_j(u_j))$. By Proposition 3.5 in [10], $\alpha_j^* \leq 0$, and hence $\frac{x_j^*}{-\alpha_j^*} \in \partial_G f_j(u_j)$. Thus

$$x^* \in \limsup_{\substack{j \in J \\ u_j \to x_0 \\ f_j(u_j) \to f(x_0)}} \partial_G f_j(u_j).$$

Theorem 3.2 implies Theorem 3.3: Set $f(x) = d(C, x)$ and $f_j(x) = d(C_j, x)$. It is easy to see that (f_j) converges uniformly to f around x_0. So Theorem 3.2 ensures that

$$\partial_G f(x_0) \subset \limsup_{\substack{j \in J \\ u_j \to x_0}} \partial_G f_j(u_j).$$

The proof is complete if we show that

$$\limsup_{\substack{j \in J \\ u \to x_0}} \partial_A d(C_j, u) \subset \limsup_{\substack{j \in J \\ u \in C_j \to x_0}} 5\partial_A d(C_j, u).$$

So let x^* in the left hand side. Then there exist $x_j \to x_0$ and $x_j^* \to x^*$ such that $x_j^* \in \partial_A d(C_j, x_j)$.

Suppose that $x_j \notin C_j$ for all j. Then there exist, by Theorem 2.1, a family (L_j) of finite dimensional subspaces of X cofinal with $\mathcal{F}(X)$, $y_j^* \to x^*$, $y_j \to x_0$, with $y_j \notin C_j$, $\varepsilon_j \to 0^+$ such that $y_j^* \in \partial_{\varepsilon_j}^- d_{y_j + L_j}(C, y_j) \cap (1 + \varepsilon_j) B_{X^*}$ for all j (to simplify, our nets are also indexed by the same directed set as (C_j), see the proof of Lemma 3.6 for details). So by Lemma 1 in [8] and Proposition 2.3.4 in [6] there exists $r_j \to 0^+$ such that

$$(y_j + r_j B_X) \cap C_j) = \emptyset \tag{3.2}$$

and for all $y \in y_j + r_j B_X$

$$\begin{aligned} &d(C_j, y) - \langle y_j^*, y - y_j \rangle + \varepsilon_j \|y - y_j\| \\ &\quad + (2 + 2\varepsilon_j) d(y, y_j + L_j) \geq d(C_j, y_j). \end{aligned}$$

Let $y_j' \in C_j$ such that

$$\|y_j' - y_j \leq d(C_j, y_j) + r_j^3.$$

Then for all $x \in C_j$ and $y \in y_j + r_j B_X$

$$\begin{aligned} &r_j^3 + \|x - y\| - \langle y_j^*, y - y_j \rangle + \varepsilon_j \|y - y_j\| \\ &\quad + (2 + 2\varepsilon_j) d(y, y_j + L_j) \geq \|y_j - y_j'\|. \end{aligned}$$

Set $g_j(x, y) = \|x - y\| - \langle y_j^*, y - y_j\rangle + \varepsilon_j \|y - y_j\| + (2 + 2\varepsilon_j)d(y, y_j + L_j)$, so $g_j(y_j', y_j) = \|y_j' - y_j\|$ and hence for all $x \in C_j$ and $y \in y_j + r_j B_X$

$$g_j(y_j', y_j) \leq g_j(x, y) + r_j^3.$$

Thus by Ekeland's variational principle [7], there exists $(u_j, v_j) \in C_j \times (y_j + r_j B_X)$ such that

$$\|u_j - y_j'\| + \|v_j - y_j\| < r_j^2 \tag{3.3}$$

and for all $x \in C_j$ and $y \in y_j + r_j B_X$

$$g_j(u_j, v_j) \leq g_j(x, y) + r_j(\|x - u_j\| + \|y - v_j\|).$$

By Proposition 2.3.4 in [6] we get that (u_j, v_j) is a local solution of the function

$$(u, y) \to g_j(x, y) + r_j(\|x - u_j\| + \|y - v_j\|) + 5d(C_j, x)$$

because, by (3.3), v_j is an internal point to $y_j + r_j B_X$. By Theorem 2.1 there exist $u_j^* \in 5\partial d(C_j, u_j)$ and $v_j^* \in X^*$, with $\|v_j^*\| = 1$, (because by (3.2) $u_j \neq v_j$) such that

$$v_j^* + x_j^* \in L_j^{\perp} + (r_j + \varepsilon_j)B_X.$$

and

$$\|u_j^* + v_j^*\| \leq r_j + \varepsilon_j.$$

Thus $v_j^* + x_j^* \to 0$ which implies that $v_j^* \to -x^*$ and $u_j^* \to x^*$ and hence

$$x^* \in \limsup_{\substack{j \in J \\ u_j \in C_j \to x_0}} 5\partial_A d(C_j, u_j)$$

and the proof is terminated.

4. Proof of Theorem 3.1

Because of Theorem 3.5, it suffices to prove Theorem 3.3.

Let $x^* \in \partial_A d(C, x_0)$. Then, by Theorem 2.1 there exist a family (L_i) of finite dimensional subspaces of X cofinal with $\mathcal{F}(X)$, $x_i^* \to x^*$, $x_i \xrightarrow{C} x_0$, $\varepsilon_i \to 0^+$ such that $x_i^* \in \partial_{\varepsilon_i}^- d_{x_i + L_i}(C, x_i) \cap (1 + \varepsilon_i)B_{X^*}$ for all i. By Lemma 1 in [8] and Proposition 2.3.4 in [6] there exists $r_i \to 0^+$ such that the function $g_i(x) = -\langle x_i^*, x - x_i\rangle + 2\varepsilon_i \|x - x_i\| + (2 + 2\varepsilon_i)d(x, x_i + L_i)$ attains its local minimum at x_i on $C \cap (x_i + r_i B_X)$. As for all i

$$\lim_{j \in J} haus_{x_i + \frac{r_i}{2} B_X}(C, C_j) = 0$$

there exists $j_i \in J$ (we may suppose that the net $(j_i)_i$ is increasing with respect to the preorder on J) such that

$$C \cap \left(x_i + \frac{r_i}{2} B_X\right) \subset C_{j_i} + r_i^3 B_X$$

and

$$C_{j_i} \cap \left(x_i + \frac{r_i}{2} B_X\right) \subset C + r_i^3 B_X.$$

Since $x_i = v_i + b_i$, with $v_i \in C_{j_i}$ and $b_i \in r_i^3 B_X$, then the set $C_{j_i} \cap (x_i + \frac{r_i}{2} B_X)$ is nonempty. Thus for all $x \in C_{j_i} \cap (x_i + \frac{r_i}{2} B_X)$ there exist $w_i \in C$ and $a_i \in r_i^3 B_X$ such that

$$x = w_i + a_i$$

and hence $w_i \in C \cap (x_i + r_i B_X)$ and

$$g_i(w_i) \geq g_i(x_i)$$

or equivalently

$$g_i(x - a_i) \geq g_i(v_i + b_i)$$

and so

$$16r_i^3 - \langle x_i^*, x - v_i \rangle + 2\varepsilon_i \|x - v_i\| + 2(1 + \varepsilon_i) d(x, v_i + L_i) \geq 0.$$

Set $h_i(x) = -\langle x_i^*, x - v_i \rangle + 2\varepsilon_i \|x - v_i\| + 2(1 + \varepsilon_i) d(x, v_i + L_i)$. Then

$$h_i(v_i) \leq h(x) + 16r_i^3, \quad \forall x \in C_{j_i} \cap \left(x_i + \frac{r_i}{2} B_X\right).$$

By Ekeland's variational principle [7] there exists $u_i \in C_{j_i} \cap \left(x_i + \frac{r_i}{2} B_X\right)$ such that

$$\|v_i - u_i\| \leq 16r_i^2$$

and

$$h_i(u_i) \leq h_i(x) + r_i \|x - u_i\|, \forall x \in C_{j_i} \cap \left(x_i + \frac{r_i}{2} B_X\right).$$

Since u_i is an internal point to $x_i + \frac{r_i}{2} B_X$, Proposition 2.3.4 in [6] ensures that u_i is a local minimum of the function

$$x \to h_i(x) + r_i \|x - u_i\| + 5d(x, C_{j_i})$$

and hence by subdifferential calculus rules

$$x_i^* \in 5\partial_A d(C_{j_i}, u_i) + L_i^\perp + (2\varepsilon_i + r_i) B_{X^*}.$$

So

$$x^* \in \limsup_{\substack{i \\ u_i \in C_{j_i} \to x_0}} [5\partial_A d(C_{j_i}, u_i) + L_i^{\perp}].$$

Consequently

$$x^* \in \limsup_{\substack{j \in J \\ u \in C_j \to x_0}} 5\partial_A d(C_j, u).$$

5. An Application to Moreau–Yosida Proximal Approximations

In this section we study the limit superior of subdifferentials of Moreau–Yosida proximal approximations.

DEFINITION 5.1. *Let f be an extended real valued function on X. For every $\lambda > 0$, the Moreau–Yosida proximal approximation of index λ of f is the function $f_\lambda : X \to \mathbb{R}$ defined by*

$$\forall x \in X \quad f_\lambda(x) = \inf \left\{ f(u) + \frac{1}{2\lambda} \|x - u\|^2 : u \in X \right\}.$$

This approximation enjoys nice properties which are summarized in the following proposition which can be found in Attouch [1].

PROPOSITION 5.2. *Let f be an extended real valued lower semicontinuous function on X satisfying*

$$\exists c > 0 \quad \exists \bar{x} \in X \text{ such that } f(x) \geq -c(\|x - \bar{x}\|^2 + 1), \quad \forall x \in X.$$

We say in this case that f is bounded from below by a negative quadratic form. Then

- *(i) for all $x \in X$, $f(x) = \sup_{\lambda > 0} f_\lambda(x)$, and $(f_\lambda(x))$ increases to $f(x)$ as λ decreases to zero.*
- *(ii) f_λ is locally lipschitz.*

The following lemma is included in Lemma 3.6 of [2]. For the convenience of the reader, we give a proof following the one of these authors.

LEMMA 5.3. *Let f be an extended real valued function on X which is lower semicontinuous. Suppose that f is bounded from below by a negative quadratic form. Then the real valued function g_λ defined on $X \times \mathbb{R}$ by*

$$g_\lambda(x, r) = d(epi\ f_\lambda, x, r)$$

converges uniformly, as $\lambda \to 0^+$, to the function

$$g(x, r) = d(epi\ f, x, r)$$

around any point $(x_0, r_0) \in epi\ f$.

Proof. We may suppose $(x_0, r_0) = (0, 0)$. We show that for all $\alpha > 0$

$$\lim_{\lambda \to 0^+} \underset{\alpha B_{X\times\mathbb{R}}}{haus}(epi\ f_\lambda, epi\ f) = 0.$$

Since $f_\lambda(x) \le f(x)$ for all x, we get

$$epi\ f \subset epi\ f_\lambda. \tag{5.1}$$

By assumptions there exist $\bar{x}$ and $c > 0$ such that

$$f(x) \ge -c(\|x - \bar{x}\|^2 + 1) \quad \forall x \in X. \tag{5.2}$$

Let $\varepsilon \in]0, \frac{1}{(2c)}[$ and set $\eta_\varepsilon = \frac{\varepsilon^2(1-2c\varepsilon)}{2(\alpha+\varepsilon^2+2c[(\alpha+\|\bar{x}\|)^2+1)]}$. Then for all $\lambda \in]0, \eta_\varepsilon[$ and $(x, r) \in epi\ f_\lambda \cap \alpha B_{X\times\mathbb{R}}$ there exists $u \in X$ such that

$$f(u) + \frac{1}{2\lambda}\|x - u\|^2 \le f_\lambda(x) + \lambda \le r + \lambda \le \alpha + \lambda.$$

As

$$\|u - \bar{x}\|^2 \le 2\|u - x\|^2 + 2\|x - \bar{x}\|^2 \le 2\|u - x\|^2 + 2(\alpha + \|\bar{x}\|)^2$$

we get, by (5.2), that

$$\|u - x\|^2 \frac{(1 - 2c\varepsilon)}{2\lambda} \le \alpha + \varepsilon^2 + 2c[(\alpha + \|\bar{x}\|)^2 + 1])]$$

and hence

$$\|u - x\|^2 \le \varepsilon^2.$$

Thus $(x, r) \in (u, r + \varepsilon) + \varepsilon B_{X\times\mathbb{R}}$, with $(u, r + \varepsilon) \in epi\ f$, and consequently

$$epi\ f_\lambda \cap \alpha B_{X\times\mathbb{R}} \subset epi\ f + \varepsilon B_{X\times\mathbb{R}}. \tag{5.3}$$

Then (5.1) and (5.3) show that

$$\lim_{\lambda \to 0^+} haus_{\alpha B_{X\times\mathbb{R}}}(epi\ f_\lambda, epi\ f) = 0$$

which is equivalent to (g_λ) converges uniformly, as $\lambda \to 0^+$, to g on any bounded subset of $X \in \mathbb{R}$.

Now we may state the theorem concerning the limit superior of subdifferentials of these functions. This theorem is a direct consequence of Theorem 3.1 and Lemmas 3.6 and 5.3.

THEOREM 5.4. *Let f be a lower semicontinuous real valued extended function on X. Suppose that f is bounded from below by a negative quadratic form. Then for all x_0 such that $f(x_0) < \infty$*

$$\partial_G f(x_0) \subset \mathop{lim\ sup}_{\substack{\lambda \to 0^+ \\ u \to x_0 \\ f_\lambda(u) \to f(x_0)}} \partial_G f_\lambda(u). \qquad \square$$

In the following theorem, we show that the equality holds for the limiting Fréchet subdifferentials in the case where the space is Asplund, i.e., Banach space on which every continuous convex function is Fréchet differentiable at a dense set of points. On these spaces, the limiting Fréchet subdifferentials $\partial_F f(x_0)$ of f at x_0 has good chain rules and important properties of sequential weak-star closedness (see for example [12-13]). The Fréchet ε- subdifferential $\partial_\varepsilon^F f(x)$ of some extended real- valued function f on X at x

$$\partial_\varepsilon^F f(x) = \left\{ x^* \in X^* : \liminf_{h \to 0} \frac{f(x+h) - f(x) - \langle x^*, h\rangle}{\|h\|} \geq -\varepsilon \right\}.$$

if $f(x) < \infty$ and $\partial_\varepsilon^F f(x) = \emptyset$ if $f(x) = +\infty$. The limiting Fréchet subdifferentials of f at x_0 is the set

$$\partial_F f(x_0) = \text{seq-}\limsup_{\substack{x \xrightarrow{f} x_0 \\ \varepsilon \to 0^+}} \partial_\varepsilon^F f(x).$$

As mentioned in [16], the following result can be deduced from [17]

$$\partial_F f(x_0) = \{x^* \in X^* : (x^*, -1) \in \mathbb{R}_+ \partial_F d(epi\ f, x_0, f(x_0))\}.$$

THEOREM 5.5. *Let X be Asplund space and f be a lower semicontinuous real valued extended function on X. Suppose that f is bounded from below by a negative quadratic form. Then for all x_0 such that $f(x_0) < \infty$*

$$\partial_F f(x_0) = seq - \mathop{lim\ sup}_{\substack{\lambda \to 0^+ \\ u \to x_0 \\ f_\lambda(u) \to f(x_0)}} \partial_F f_\lambda(u).$$

Proof. For the first inclusion, it suffices to show that if $(x^*, \alpha^*) \in \partial_F d(epi\ f, x_0, f(x_0))$, with $\alpha^* \neq 0$ then

$$(x^*, \alpha^*) \in seq - \limsup_{\substack{\lambda \to 0^+ \\ u \to x_0 \\ f_\lambda(u) \to f(x_0)}} 3\partial_F d(epi\ f_\lambda, u, f_\lambda(u)).$$

By definition, there are sequences $(x_n, r_n) \to (x_0, f(x_0))$, with $(x_n, r_n) \in epi\ f$, $\varepsilon_n \to 0^+$, $\gamma_n \to 0^+$ and $(x_n^*, \alpha_n^*) \to (x^*, \alpha^*)$ such that the function

$$(x, r) \to d(x, r, epi\ f) - \langle x_n^*, x - x_n\rangle - \alpha_n^*(r - r_n) + \varepsilon_n(\|x - x_n\| + |r - r_n|)$$

attains a local minimum at (x_n, r_n) on $(x_n, r_n)+\gamma_n B_{X\times\mathbb{R}}$. Thus $r_n = f(x_n)$ because $\alpha^* \neq 0$. By Lemma 5.3, for each n there exists $\lambda_n \in]0, 1/n[$ such that

$$epi\ f_{\lambda_n} \subset epi\ f + \gamma_n^3 B_{X\times\mathbb{R}}$$

and

$$epi\ f \subset epi\ f_{\lambda_n} + \gamma_n^3 B_{X\times\mathbb{R}}.$$

So as in the proof of Theorem 3.1 we show that there exists $(u_n, s_n) \in epi\ f_{\lambda_n} \cap [(x_n, f(x_n)) + \gamma_n/2B_{X\times\mathbb{R}}]$ such that

$$\|x_n - u_n\| + |f(x_n) - s_n| < 16\gamma_n^2$$

and for all $(x, r) \in epi\ f_{\lambda_n} \cap [(x_n, f(x_n)) + \gamma_n/2B_{X\times\mathbb{R}}]$

$$h_n(u_n, s_n) \leq h_n(x, r) + \gamma_n[\|x - u_n\| + |r - s_n|]$$

where $h_n(x, r) = -\langle x_n^*, x - x_n\rangle - \alpha_n^*(r - r_n) + \varepsilon_n(\|x - x_n\| + |r - r_n|)$. Thus (u_n, s_n) is an internal point to $(x_n, f(x_n)) + \gamma_n/2B_{X\times\mathbb{R}}$ and (u_n, s_n) is a local minimum of the function

$$(x, r) \to 3d(x, r, epi\ f_{\lambda_n}) + h_n(x, r) + \gamma_n[\|x - u_n\| + |r - s_n|].$$

We also note that since $\alpha^* \neq 0$, we get $s_n = f_{\lambda_n}(u_n)$. Taking into account that $f_{\lambda_n}(u_n) \to f(x_0)$, the subdifferential calculus implies that

$$(x^*, \alpha^*) \in seq - \limsup_{\substack{\lambda\to 0^+\\ u\to x_0\\ f_\lambda(u)\to f(x_0)}} 3\partial_F d(epi\ f_\lambda, u, f_\lambda(u)).$$

This completes the proof of the first inclusion.

Let us prove the second one. So let

$$x^* \in seq - \limsup_{\substack{\lambda\to 0^+\\ u\to x_0\\ f_\lambda(u)\to f(x_0)}} \partial_F f_\lambda(u).$$

Then there are sequences $\lambda_n \to 0^+$, $u_n \to x_0$ with $f_{\lambda_n}(u_n) \to f(x_0)$ and $u_n^* \to x^*$ such that

$$u_n^* \in \partial_F f_{\lambda_n}(u_n), \quad \forall n.$$

Thus there are sequences $x_n \to x_0$, with $f_{\lambda_n}(x_n) \to f(x_0)$, $x_n^* \to x^*$, $\varepsilon_n \to 0^+$ and $r_n \to 0^+$ such that for n sufficiently large

$$f_{\lambda_n}(x_n + h) - f_{\lambda_n}(x_n) - \langle x_n^*, h\rangle + \varepsilon_n\|h\| \geq 0, \quad \forall h \in r_n B_X. \tag{5.4}$$

On the other hand there exists $v_n \in X$ such that

$$f(v_n) + \frac{1}{2\lambda_n}\|x_n - v_n\|^2 \leq f_{\lambda_n}(x_n) + r_n^3. \tag{5.5}$$

As in the proof of Lemma 5.3 we show that $v_n \to x_0$ and hence $\liminf_{n\to\infty} f(v_n) = f(x_0)$. Taking subsequences we may suppose that $f(v_n) \to f(x_0)$. So combining (5.4) and (5.5), we get

$$\begin{aligned} r_n^3 &+ f(v_n + h) + \frac{1}{2\lambda_n}\|x_n - v_n\|^2 - f(v_n) - \frac{1}{2\lambda_n}\|x_n - v_n\|^2 \\ &- \langle x_n^*, h\rangle + \varepsilon_n\|h\| \geq 0, \quad \forall h \in r_n B_X \end{aligned}$$

or equivalently

$$r_n^3 + f(x) - f(v_n) - \langle x_n^*, x - v_n\rangle + \varepsilon_n\|x - \vartheta_n\| \geq 0, \quad \forall x \in v_n + r_n B_X$$

By Ekeland variational principle [7] there exists $w_n \in v_n + r_n B_X$ such that

$$\|w_n - v_n\| \leq r_n^2 \tag{5.6}$$

and

$$\begin{aligned} f(w_n) \leq & f(x) - \langle x_n^*, x - w_n\rangle + \varepsilon_n\|x - w_n\| \\ & + r_n\|x - w_n\|, \quad \forall x \in v_n + r_n B_X \end{aligned} \tag{5.7}$$

and hence w_n is an internal point to $v_n + r_n B_X$ and for all $(x, \alpha) \in epi\ f \cap [(w_n, f(w_n)) + r_n B_{X\times\mathbb{R}}]$

$$f(w_n) \leq \alpha - \langle x_n^*, x - w_n\rangle + \varepsilon_n\|x - w_n\| + r_n\|x - w_n\|.$$

So there exists $K > 0$ (not depending on n) such that the function

$$(x, \alpha) \to \alpha - \langle x_n^*, x - w_n\rangle + \varepsilon_n\|x - w_n\| + r_n\|x - w_n\| + Kd(epi\ f, x, \alpha)$$

attains a local minimum at $(w_n, f(w_n))$. Thus

$$(x_n^*, -1) \in \partial^F_{(r_n+\varepsilon_n)} Kd(epi\ f, w_n, f(w_n)).$$

and hence

$$(x^*, -1) \in \mathbb{R}_+ \partial_F d(epi\ f, x_0, f(x_0))$$

because of (5.6) and (5.7) for some subsequence $(w_{n'})$ of (w_n) we get $f(w_{n'}) \to f(x_0)$. Thus $x^* \in \partial_F f(x_0)$ because

$$\partial_F f(x_0) = \{x^* \in X^* : \quad x^*, -1) \in \mathbb{R}_+ \partial_F d(epi\ f, x_0, f(x_0))\}.$$

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