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Open mapping theorem and inversion theorem for γ-paraconvex multivalued mappings and applications

by

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Abstract. We extend the open mapping theorem and inversion theorem of Robinson for convex multivalued mappings to γ-paraconvex multivalued mappings. Some questions posed by Rolewicz are also investigated. Our results are applied to obtain a generalization of the Farkas lemma for γ-paraconvex multivalued mappings.

1. Introduction. The open mapping theorem and inversion theorem play a key role in analysis and optimization theory, namely in the study of necessary optimality conditions and the stability of parametric optimization problems (see for example [4], [8–13], and references therein). It also lies at the heart of some particularly effective methods for establishing calculus rules for directional derivatives and subgradients of nondifferentiable mappings (see [10], [20] and references therein).

In his paper [12] (see also [19]), Robinson established the open mapping theorem and inversion theorem for convex multivalued mappings. Rolewicz [16] studied Lipschitz properties of γ-paraconvex multivalued mappings. A multivalued mapping F from a normed vector space X into a normed vector space Y is called *γ-paraconvex* ($\gamma > 0$) if there is a constant $C > 0$ such that for all $x, u \in X$ and all $\alpha \in [0,1]$,

$$\alpha F(x) + (1-\alpha)F(u) \subset F(\alpha x + (1-\alpha)u) + C\|x-u\|^\gamma B_Y.$$

Here B_Y denotes the closed unit ball of Y.

In the case $\gamma > 1$, Rolewicz [14–15] showed that F is γ-paraconvex iff there exists $C > 0$ such that for all $x, u \in X$ and all $\alpha \in [0,1]$,

$$\alpha F(x)+(1-\alpha)F(u) \subset F(\alpha x+(1-\alpha)u)+C\min(\alpha,1-\alpha)\|x-u\|^\gamma B_Y. \tag{1.1}$$

Note that any *convex* multivalued mapping, i.e. satisfying for all $x, u \in X$ and all $\alpha \in [0,1]$,

$$\alpha F(x) + (1-\alpha)F(u) \subset F(\alpha x + (1-\alpha)u),$$

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is γ-paraconvex, but the converse may be false. Let, for example, f be a function from $\mathbb{R}$ into $\mathbb{R}$ given by $f(x) = ||x| - 1|$ and define the multivalued mapping F by

$$F(x) = f(x) + \mathbb{R}_+.$$

Then F is 1-paraconvex but not convex.

For this class of multivalued mappings, Rolewicz [16] established the following theorem.

THEOREM 1.1. *Let X be a real Hilbert space, and let $F : X \rightrightarrows Y$ be a closed-valued 2-paraconvex multivalued mapping. Let $x_0 \in \operatorname{int} F^{-1}(Y)$. Suppose that $\operatorname{int} F(x_0) \neq \emptyset$. Then for all $y_0 \in F(x_0)$ there exist $r, a > 0$ such that*

$$F(x) \cap (y_0 + rB_Y) \subset F(x_0) + a\|x - x_0\| B_Y$$

for all $x \in x_0 + rB_X$.

In this paper, we extend the results of [12, Theorems 1 and 2] to γ-paraconvex multivalued mappings, for any $\gamma > 1$, and we give affirmative answers to the following questions posed by Rolewicz [16]:

- Is Theorem 1.1 true without the hypothesis that $F(x_0)$ has nonempty interior?
- Is it true in Banach spaces?
- Is it true for $1 < \gamma \le 2$?

We apply our results to obtain a generalization of the Farkas lemma for γ-paraconvex multivalued mappings.

2. Open mapping theorem, inversion theorem and Lipschitz property of γ-paraconvex multivalued mappings. To present our open mapping theorem for γ-paraconvex multivalued mappings, we establish the following technical lemma which is in the spirit of that of Robinson [12, Lemma 1].

LEMMA 2.1. *Let X and Y be normed linear spaces, and let $F : X \rightrightarrows Y$ be a multivalued mapping with closed graph. Suppose that:*

(a_1) *there exists $\alpha > 0$ such that the set $C_\alpha := \operatorname{Gr} F \cap (\alpha B_X \times Y)$ is nonempty;*

(a_2) *F^{-1} is γ-paraconvex with $\gamma > 1$, and*

(a_3) *X is complete.*

Then for all $\beta > \alpha$,

$$\operatorname{int} \operatorname{cl} P_Y(C_\alpha) \subset \operatorname{int} P_Y(C_\beta).$$

Here P_Y denotes the projection from $X \times Y$ onto Y and $\operatorname{Gr} F$ is the graph of F, that is,

$$\operatorname{Gr} F = \{(x, y) \in X \times Y : y \in F(x)\}$$

and F^{-1} denotes the inverse multivalued mapping of F.

Proof. Since F^{-1} is γ-paraconvex there exists $C > 0$ such that for all $y, z \in Y$ and all $\alpha \in [0, 1]$,

$$\alpha F^{-1}(y) + (1-\alpha)F^{-1}(z) \subset F^{-1}(\alpha y + (1-\alpha)z) + C\min(\alpha, 1-\alpha)\|y - z\|^\gamma B_X$$

(see (1.1)). Let $y' \in \operatorname{int} \operatorname{cl} P_Y(C_\alpha)$ and let $\varepsilon \in]0, 1/2[$ be such that

$$K := 2C\varepsilon < \beta - \alpha \quad \text{and} \quad y' + \varepsilon B_Y \subset \operatorname{cl} P_Y(C_\alpha).$$

Choose $(x_0, y_0) \in C_\alpha$ so that $y_0 = y' + r_0$ and $\|r_0\| \le \frac{1}{2}\varepsilon$; define $x_{-1} = x_0$. For $k = 0$ we have

$$\text{(2.1)} \qquad \begin{cases} \|r_k\| \le \frac{1}{2}\varepsilon, \\ y_k = y' + 2^{-k} r_k, \\ \|x_k - x_{k-1}\| \le 2^{1-k}(K + \alpha), \\ (x_k, y_k) \in \operatorname{Gr} F, \\ \|x_k\| \le \alpha + K(1 - 2^{-k-1}). \end{cases}$$

Suppose that (2.1) holds for $k = n \ge 0$. Let $r_n \in \frac{1}{2}\varepsilon B_Y$. Then we have $\|(2 - 2^{-n})r_n\| < \varepsilon$ and hence $y' - (2 - 2^{-n})r_n \in y' + \varepsilon B_Y$. Thus there exists $r_{n+1} \in \frac{1}{2}\varepsilon B_Y$ such that $v := y' - (2 - 2^{-n})r_n + r_{n+1} \in P_Y(C_\alpha)$ and hence there exists $u \in X$ with $(u, v) \in C_\alpha$. Define

$$y_{n+1} = (1 - 2^{-n-1})y_n + 2^{-n-1}v.$$

By γ-paraconvexity of F^{-1} there exists $b_{n+1} \in B_Y$ such that

$$x_{n+1} := (1 - 2^{-n-1})x_n + 2^{-n-1}u + C2^{-n-1}\|y_n - v\|^\gamma b_{n+1} \in F^{-1}(y_{n+1}).$$

So

$$\begin{aligned} y_{n+1} &= (1 - 2^{-n-1})y_n + 2^{-n-1}v \\ &= (1 - 2^{-n-1})(y' + 2^{-n}r_n) + 2^{-n-1}(y' - (2 - 2^{-n})r_n + r_{n+1}) \\ &= y' + 2^{-n-1}r_{n+1}, \end{aligned}$$

and consequently,

$$\|y_n - v\| \le \|y_n - y'\| + (2 - 2^{-n})\|r_n\| + \|r_{n+1}\| \le 2\varepsilon.$$

Now,

$$\begin{aligned} \|x_{n+1} - x_n\| &\le 2^{-n-1}\|x_n - u\| + C2^{-n-1}\|y_n - v\|^\gamma \\ &\le 2^{-n-1}[\|x_n\| + \|u\| + K] \\ &\le 2^{-n-1}[\alpha + K(1 - 2^{-n-1}) + \alpha + K] \\ &\le 2^{-n}(\alpha + K), \end{aligned}$$

and thus

$$\begin{aligned}\|x_{n+1}\| &\le (1-2^{-n-1})\|x_n\| + 2^{-n-1}\|u\| + C2^{-n-1}\|y_n - v\|^\gamma \\ &\le (1-2^{-n-1})(\alpha + K(1-2^{-n-1})) + 2^{-n-1}(\alpha+K) \\ &= \alpha + K((1-2^{-n-1})^2 + 2^{-n-1}) \le \alpha + K(1-2^{-n-2}).\end{aligned}$$

So (2.1) holds for $k = n+1$ and hence, by induction, for all k. For any k and any $n \ge 1$ we have

$$\begin{aligned}\|x_{k+n} - x_k\| &\le \sum_{i=0}^{n-1} \|x_{k+i+1} - x_{k+i}\| \le (K+\alpha)\sum_{i=0}^{n-1} 2^{-k-i} \\ &\le (K+\alpha)2^{1-k}.\end{aligned}$$

Thus (x_k) is a Cauchy sequence and so, by the completeness of X, (x_k) converges to some $x' \in X$. However, (y_k) converges to y', so by the closedness of $\operatorname{Gr} F$, $(x', y') \in \operatorname{Gr} F$. As

$$\|x_n\| \le \alpha + K < \beta$$

we have $x' \in \beta B_X$ and hence $y' \in P_Y(C_\beta)$. ∎

Now we may state our open mapping theorem which extends Theorem 1 of Robinson [12] to γ-paraconvex multivalued mappings.

THEOREM 2.2 (Open mapping theorem). *Let X and Y be Banach spaces, and let $F : X \rightrightarrows Y$ be a multivalued mapping with closed graph. Suppose that F^{-1} is γ-paraconvex, with $\gamma > 1$. Then the following are equivalent*:

(i) $y_0 \in \operatorname{int} F(X)$;

(ii) *for all $x_0 \in F^{-1}(y_0)$ there are $\alpha, \beta > 0$ such that $y_0 + \alpha B_Y \subset F(x_0 + \beta B_X)$.*

Proof. ⇒ Without loss of generality we may suppose that $x_0 = 0$ and $y_0 = 0$, this simply translates the origins in X and Y. Set $D = \operatorname{cl} \operatorname{co}(F(B_X) \cap B_Y)$.

Step 1: *D is absorbing*. Let $y \in Y$. Since $0 \in \operatorname{int} F(X)$ there exists $\mu \in \mathbb{R}$ such that $\mu y \in F(x) \cap B_Y$ for some $x \in X$. If $x \in B_X$ then $\mu y \in D$; otherwise let $\lambda = 1/(\|x\| + C\|\mu y\|) \in [0,1]$, where C is as in the definition of γ-paraconvexity for F^{-1}. Then

$$\lambda x + (1-\lambda)0 \in F^{-1}(\lambda\mu y) + C\lambda\|\mu y\|B_X$$

and hence there exists $b \in B_X$ such that

$$u := \lambda x + C\lambda\|\mu y\|b \in F^{-1}(\lambda\mu y).$$

Thus $\lambda\mu y \in F(u) \cap B_Y$ and $\|u\| \le 1$.

Step 2: *For all $k \in \mathbb{N}^*$ and $\lambda_1 > 0, \ldots, \lambda_k > 0$ with $\sum_{i=1}^k \lambda_i = 1$ and all $y_1, \ldots, y_k \in B_Y$,*

$$\sum_{i=1}^k \lambda_i F^{-1}(y_i) \subset F^{-1}\Big(\sum_{i=1}^k \lambda_i y_i\Big) + CB_X.$$

For $k = 2$,

$$\begin{aligned}\lambda_1 F^{-1}(y_1) + \lambda_2 F^{-1}(y_2) &\subset F^{-1}(\lambda_1 y_1 + \lambda_2 y_2) + C\min(\lambda_1,\lambda_2)\|y_1 - y_2\|B_X \\ &\subset F^{-1}(\lambda_1 y_1 + \lambda_2 y_2) + CB_X.\end{aligned} \tag{2.2}$$

Suppose that (2.2) holds for $k = 2, \ldots, n$ and let us show it for $k = n+1$. Let $\lambda_1 > 0, \ldots, \lambda_{n+1} > 0$ with $\sum_{i=1}^{n+1} \lambda_i = 1$ and let $y_1, \ldots, y_{n+1} \in B_Y$. Then

$$\begin{aligned}&\sum_{i=1}^n \lambda_i F^{-1}(y_i) + \lambda_{n+1}F^{-1}(y_{n+1}) \\ &\quad= \sum_{i=1}^n \lambda_i \Big(\sum_{i=1}^n \frac{\lambda_i}{\sum_{i=1}^n \lambda_i} F^{-1}(y_i)\Big) + \lambda_{n+1}F^{-1}(y_{n+1}) \\ &\quad\subset \sum_{i=1}^n \lambda_i \Big[F^{-1}\Big(\sum_{i=1}^n \frac{\lambda_i}{\sum_{i=1}^n \lambda_i} y_i\Big) + CB_X\Big] + \lambda_{n+1}F^{-1}(y_{n+1}) \\ &\quad\subset F^{-1}\Big(\sum_{i=1}^{n+1} \lambda_i y_i\Big) + C\Big(\sum_{i=1}^n \lambda_i\Big)B_X + C\lambda_{n+1}B_X \\ &\quad= F^{-1}\Big(\sum_{i=1}^{n+1} \lambda_i y_i\Big) + CB_X.\end{aligned}$$

Step 3: *$D \subset \operatorname{cl}(F((C+1)B_X) \cap B_Y)$*. Let $y \in \operatorname{co}(F(B_X) \cap B_Y)$. Then there exist $k \in \mathbb{N}^*$, $\lambda_1 > 0, \ldots, \lambda_k > 0$ with $\sum_{i=1}^k \lambda_i = 1$, $y_1, \ldots, y_k \in B_Y$ and $x_1, \ldots, x_k \in B_X$ such that $x_i \in F^{-1}(y_i)$ and $y = \sum_{i=1}^k \lambda_i y_i$. By Step 2,

$$\sum_{i=1}^k \lambda_i x_i \in F^{-1}\Big(\sum_{i=1}^k \lambda_i y_i\Big) + CB_X$$

and so there exists $b \in B_X$ such that

$$\sum_{i=1}^k \lambda_i x_i + Cb \in F^{-1}(y)$$

and hence $y \in F(\sum_{i=1}^k \lambda_i x_i + Cb) \subset F((1+C)B_X)$.

Step 4: By Step 1, D is a neighbourhood of 0 and hence there exists $\alpha > 0$ such that $2\alpha B_Y \subset D$ and, by Step 3,

$$2\alpha B_Y \subset \operatorname{cl}(F((C+1)B_X) \cap B_Y).$$

Set $C_{C+1} = \operatorname{Gr} F \cap ((C+1)B_X \times Y)$. Then, by Lemma 2.1,

$$\alpha B_Y \subset \operatorname{int}\operatorname{cl} P_Y(C_{C+1}) \subset \operatorname{int} P_Y(C_{C+2})$$

and hence $\alpha B_Y \subset F((C+2)B_X)$.

$\Leftarrow$ This is immediate. ■

Note that, by γ-paraconvexity of F^{-1}, (ii) is equivalent to the following condition: for all $x_0 \in F^{-1}(y_0)$ there are $\alpha, \beta > 0$ such that for all $\lambda \in [0,1]$,

$$y_0 + \lambda\alpha B_Y \subset F(x_0 + \lambda\beta B_X).$$

As a consequence of Lemma 2.1 and the previous steps in the proof of Theorem 2.2, we may obtain the following characterization:

$$0 \in \operatorname{int} F(X) \quad \text{iff} \quad 0 \in \operatorname{core} F(X)$$

provided that F^{-1} is γ-paraconvex with closed graph, and $\gamma > 1$. Indeed, the implication $\Rightarrow$ is obvious, so let us prove the other one. As in Step 1, we show that the set $\operatorname{cl}\operatorname{co}(F(B_X) \cap B_Y)$ is absorbing and so it is a neighbourhood of 0. By Step 3, $\operatorname{cl}\operatorname{co}(F(B_X) \cap B_Y) \subset \operatorname{cl}(F((C+1)B_X) \cap B_Y)$, and hence, by Lemma 2.1, there exists $s > 0$ such that $sB_Y \subset \operatorname{int}\operatorname{cl}\operatorname{co}(F(B_X) \cap B_Y) \subset \operatorname{int}\operatorname{cl}(F((C+1)B_X) \cap B_Y) \subset F((C+2)B_X) \cap B_Y$. ■

The following theorem is an extension of that of Robinson [12, Theorem 2] to γ-paraconvex multivalued mappings.

THEOREM 2.3 (Inversion theorem). *Let X, Y and F be as in Theorem 2.2. Then the following are equivalent*:

(i) $y_0 \in \operatorname{int} F(X)$;

(ii′) *for all $x_0 \in F^{-1}(y_0)$ there are $a, r > 0$ such that*

$$d(x, F^{-1}(y)) \leq ad(y, F(x)) \tag{2.3}$$

for all $x \in x_0 + rB_X$ and $y \in y_0 + rB_Y$. Here $d(x,S)$ denotes the distance function from x to a set S.

Proof. The proof is similar to that given by Robinson [12].

$\Leftarrow$ This is immediate.

$\Rightarrow$ By Theorem 2.2 there exist $\alpha, \beta > 0$ such that $y_0 + \alpha B_Y \subset F(x_0 + \beta B_X)$. Let $r = \frac{1}{2}\min(\alpha, \beta)$. Let $y \in y_0 + rB_Y$ and $x \in x_0 + rB_X$. Inequality (2.3) follows at once if either $x \in \operatorname{dom} F$ or $y \in F(x)$, so suppose neither is true. Choose some $\theta > 0$ and find $y_\theta \in F(x)$ with $0 < \|y - y_\theta\| < d(y, F(x)) + \theta$; define $s := \alpha - \|y - y_0\| > 0$ and take $\varepsilon \in]0, s[$. Let

$$y_\varepsilon := y + (s-\varepsilon)\|y - y_\theta\|^{-1}(y - y_\theta);$$

then $\|y_\varepsilon - y_0\| \leq \|y - y_0\| + s - \varepsilon < \alpha$, so $y_\varepsilon \in y_0 + \alpha B_Y$. Thus there exists $x_\varepsilon \in x_0 + \beta B_X$ with $y_\varepsilon \in F(x_\varepsilon)$. Define $\lambda = (1 + (s-\varepsilon)\|y - y_\theta\|^{-1})^{-1}$. By the γ-paraconvexity of F^{-1} there exists $C > 0$ such that

$$(1-\lambda)x + \lambda x_\varepsilon \in F^{-1}((1-\lambda)y_\theta + \lambda y_\varepsilon) + C\lambda\|y_\theta - y_\varepsilon\|B_X.$$

Thus, since $y = y_\theta + \lambda(y_\varepsilon - y_\theta)$,

$$\begin{aligned} d(x, F^{-1}(y)) &\leq \|x - ((1-\lambda)x + \lambda x_\varepsilon)\| + C\lambda\|y_\varepsilon - y_\theta\| \\ &\leq \lambda\|x - x_\varepsilon\| + C\|y - y_\theta\|. \end{aligned}$$

However, $\|x - x_\varepsilon\| \leq \|x - x_0\| + \|x_0 - x_\varepsilon\| \leq r + \beta$ and

$$\begin{aligned} \lambda = (1 + (s-\varepsilon)\|y - y_\theta\|^{-1})^{-1} &< (s-\varepsilon)^{-1}\|y - y_\theta\| \\ &= \frac{1}{\alpha - \|y - y_0\| - \varepsilon}\|y - y_\theta\|. \end{aligned}$$

Therefore

$$\begin{aligned} d(x, F^{-1}(y)) &\leq \left[\frac{r+\beta}{\alpha - \|y - y_0\| - \varepsilon} + C\right]\|y - y_\theta\| \\ &\leq \left[\frac{r+\beta}{\alpha - \|y - y_0\| - \varepsilon} + C\right](d(y, F(x)) + \theta). \end{aligned}$$

The proof is completed by letting $\theta, \varepsilon \to 0$ because $(\alpha - \|y - y_0\|)^{-1} \leq (\alpha - r)^{-1}$. ■

The aim of this theorem is to give answers to the questions posed by Rolewicz [16].

THEOREM 2.4 (Lipschitz property). *Let X and Y be Banach spaces, and let $F : X \rightrightarrows Y$ be a multivalued mapping with closed graph. Suppose that F is γ-paraconvex with $\gamma > 1$. Then the following are equivalent* :

(i′) $x_0 \in \operatorname{int} F^{-1}(Y)$;

(ii′) *for all $y_0 \in F(x_0)$ there are $a, r > 0$ such that*

$$F(x) \cap (y_0 + rB_Y) \subset F(x') + a\|x - x'\|B_Y$$

for all $x, x' \in x_0 + rB_X$.

Proof. (i′)$\Rightarrow$(ii″). By Theorem 2.3 there exist $a, r > 0$ such that

$$d(y, F(x)) \leq ad(x, F^{-1}(y))$$

for all $x \in x_0 + rB_X$ and $y \in y_0 + rB_Y$. So let $x, x' \in x_0 + rB_X$ and let $y \in (y_0 + rB_Y) \cap F(x)$. Then

$$d(y, F(x')) \leq ad(x', F^{-1}(y)) \leq 2a\|x - x'\|$$

and hence there exists $y' \in F(x')$ such that $\|y - y'\| \leq 2a\|x - x'\|$.

The other implication is immediate. ■

Remarks. 1) Note that in Theorem 2.4 we replaced the closedness of the values of F (which is assumed in Theorem 1.1) by that of $\operatorname{Gr} F$.

2) We can use the relation of openness, metric regularity and pseudo-Lipschitzian property of multivalued mappings (see [1], [11]) to obtain Theorems 2.3 and 2.4 from Theorem 2.2.

As an application, we extend some results concerning the Lipschitz property of convex functions to γ-paraconvex functions.

An extended real-valued function f on X is said to be *γ-paraconvex* if there exists $a > 0$ such that for all $x, u \in X$ and $\alpha \in [0,1]$,

$$f(\alpha x + (1-\alpha)u) \le \alpha f(x) + (1-\alpha) f(u) + a\|x - u\|^\gamma$$

or equivalently, if the multivalued mapping F defined by $F(x) = f(x) + \mathbb{R}_+$ is γ-paraconvex.

Using (1.1) we may also show that f is γ-paraconvex iff there exists $a > 0$ such that for all $x, u \in X$ and $\alpha \in [0,1]$,

$$f(\alpha x + (1-\alpha)u) \le \alpha f(x) + (1-\alpha) f(u) + a\min(\alpha, 1-\alpha)\|x - u\|^\gamma$$

provided that $\gamma > 1$.

As in the convex case we have the following lemma.

LEMMA 2.5. *Let f be γ-paraconvex with $\gamma > 0$. If there exist $b, s > 0$ such that $f(x) \le b$ for all $x \in x_0 + sB_X$, then f is continuous at x_0.*

Proof. Without loss of generality we assume that $x_0 = 0$ and $f(x_0) = 0$. Let $\varepsilon \in \left]0, s^\gamma\right[$ and let $x \in \varepsilon^{1+\gamma^{-1}} B_X$. Then $\varepsilon^{-1}x \in sB_X$ and hence by assumptions there exists $C > 0$ such that

$$\begin{aligned} f(x) &= f(\varepsilon(\varepsilon^{-1}x) + (1-\varepsilon)0) \le \varepsilon f(\varepsilon^{-1}x) + (1-\varepsilon)f(0) + C\|\varepsilon^{-1}x\|^\gamma \\ &\le \varepsilon b + C\varepsilon. \end{aligned}$$

On the other hand,

$$f(0) = f\left(\frac{x}{1+\varepsilon} + \frac{\varepsilon}{1+\varepsilon}(-\varepsilon^{-1}x)\right) \le \frac{1}{1+\varepsilon} f(x) + \frac{\varepsilon}{1+\varepsilon} f(-\varepsilon^{-1}x) + C(1+\varepsilon)^\gamma \varepsilon$$

and so $f(x) \ge -\varepsilon b - C(1+\varepsilon)^{\gamma+1}\varepsilon$. Thus for all $x \in \varepsilon^{1+\gamma^{-1}} B_X$,

$$|f(x)| \le \varepsilon(b + C(1+\varepsilon)^{\gamma+1}). \ \blacksquare$$

Using the previous results we obtain the following proposition about the Lipschitz property of γ-paraconvex functions (see [9] for another proof in normed vector spaces).

PROPOSITION 2.6. *Let f be γ-paraconvex with $\gamma > 1$. If there exist $b > 0$ and a nonempty open set O such that $f(x) \le b$ for all $x \in O$, then f is locally Lipschitzian on O.*

Proof. Note that, by Lemma 2.5, f is continuous at every element of O. So let $x_0 \in O$ and $s > 0$ be such that $x_0 + sB_X \subset O$. Then the multivalued mapping

$$F(x) = \begin{cases} f(x) + \mathbb{R}_+ & \text{if } x \in x_0 + sB_X, \\ 0 & \text{otherwise}, \end{cases}$$

is γ-paraconvex. Since f is continuous on the closed set $x_0 + sB_X$, $\operatorname{Gr} F$ is closed. Then, by Theorem 2.4, there exists $r \in \left]0, s\right[$ such that for all $x, x' \in x_0 + rB_X$,

$$f(x) \in f(x') + \mathbb{R}_+ + C\|x - x'\|[-1, 1]$$

and hence $|f(x) - f(x')| \le C\|x - x'\|$. ■

To close this section, let us give the following characterization of γ-paraconvex multivalued mappings.

PROPOSITION 2.7. 1) *A multivalued mapping $F : X \rightrightarrows Y$ is γ-paraconvex iff there exists $C > 0$ such that for all $x, u \in X$, $y, z \in Y$ and $\alpha \in [0,1]$,*

$$\begin{aligned} &d(\alpha y + (1-\alpha)z, F(\alpha x + (1-\alpha)u)) \\ &\qquad \le \alpha d(y, F(x)) + (1-\alpha) d(z, F(u)) + C\|x - u\|^\gamma. \end{aligned}$$

2) *If F is γ-paraconvex then the function $(x, y) \to d(y, F(x))$ is γ-paraconvex.*

3. Farkas lemma for γ-paraconvex multivalued mappings.

The original Farkas lemma states that, for $x_1^*, \ldots, x_n^* \in X^*$, the following conditions are equivalent:

(a_1) $\langle x_i^*, x\rangle \le 0$, $i = 1, \ldots, n \Rightarrow \langle x^*, x\rangle \le 0$,

(a_2) there exist $\lambda_i \ge 0$, $i = 1, \ldots, n$, such that $x^* = \sum_{i=1}^n \lambda_i x_i^*$.

This version is equivalent to the following one: if $x_1^*, \ldots, x_n^* \in X^*$ are linearly independent then (a_1) and (a_2) are equivalent.

This lemma is used in deriving necessary optimality conditions in linear programming problems. Further progress is made in convex programs. In [17], the Farkas lemma is generalized to a system of linear inequalities with max operations. Swartz [18] established this lemma for infinitely many inequalities. In [7], Jeyakumar presented a general Farkas lemma for a convex process and a generalized convex function. A further extension is established in [6] by replacing the usual bilinear coupling by a convex-concave positively homogeneous function. Recently Glover, Jeyakumar and Oettli [5] studied general cone-constrained quasidifferentiable programming problems and established versions of theorems of the alternative for systems of functions bounded above by sublinear functions.

In this contribution a further extension is established. Using the results of Section 2 we present a general Farkas lemma for linear and nonlinear mappings involving γ-paraconvex multivalued mappings. We show that the classical Farkas lemma can be summarized in the following formula:

$$N(F^{-1}(0),0) = \mathbb{R}_+\partial^- d(0,F(\cdot))(0),$$

where $F : X \rightrightarrows Y$ is a multivalued mapping, and X and Y are Banach spaces. $N(S,x)$ is the *normal cone* to a convex set S at $x \in S$, that is,

$$N(S,x) = \{x^* \in X^* : \langle x^*, h - x\rangle \le 0,\ \forall h \in S\}$$

and $\partial^- f(x)$ denotes the *Dini subdifferential* of f at x, that is,

$$\partial^- f(x) = \{x^* \in X^* : \langle x^*, h\rangle \le d^- f(x,h),\ \forall h \in X\},$$

where $d^- f(x,h) = \liminf_{u\to h, t\to 0^+} t^{-1}(f(x+tu) - f(x))$.

When f is convex $\partial^- f(x)$ coincides with the subdifferential $\partial f(x)$ in the sense of convex analysis.

As a consequence we obtain a Farkas lemma for cone-constrained convex programming problems.

Now we present a Farkas lemma for γ-paraconvex multivalued mappings.

THEOREM 3.1 (Generalized Farkas lemma). *Let $F : X \rightrightarrows Y$ be a multivalued mapping with closed graph and let $(0,0) \in \operatorname{Gr} F$. Suppose that F^{-1} is γ-paraconvex, $\gamma > 1$, and $0 \in \operatorname{int} F(X)$. Then*

$$N(F^{-1}(0),0) = \mathbb{R}_+\partial^- d(0,F(\cdot))(0).$$

Proof. Let $x^* \in N(F^{-1}(0),0)$. Then $\langle x^*, x\rangle \le 0$ for all $x \in F^{-1}(0)$, or equivalently,

$$\langle x^*, x\rangle \le \|x^*\| d(x, F^{-1}(0)), \quad \forall x \in X.$$

By Theorem 2.3 there exist $a > 0$ and a neighbourhood V of 0 in X such that

$$d(x,F^{-1}(0)) \le a d(0,F(x)), \quad \forall x \in V.$$

Thus for all $x \in V$,

$$\langle x^*, x\rangle \le a\|x^*\| d(0,F(x))$$

and hence $x^* \in a\|x^*\|\partial^- d(0,F(\cdot))(0)$.

Conversely, let $x^* \in \partial^- d(0,F(\cdot))(0)$. Then since the function $h \to d(0,F(h))$ is γ-paraconvex, it follows, by Theorem 3.4 of [9], that

$$\langle x^*, h\rangle \le d(0,F(h)) + C\|h\|^\gamma, \quad \forall h \in X,$$

for some positive constant C. Thus for all $h \in F^{-1}(0)$ and $t \in]0,1[$, we have $th \in F^{-1}(0)$ and $\langle x^*, h\rangle \le t^{\gamma-1}C\|h\|^\gamma$. Letting t go to 0 we get $\langle x^*, h\rangle \le 0$ for all $h \in F^{-1}(0)$. ■

Remark. Using Theorem 3.4 of [9] we may replace in Theorem 3.1 the Dini subdifferential by the Clarke subdifferential.

In order to deduce a Farkas lemma for general cone-constrained convex programming problems, suppose that the space Y is ordered by a closed convex cone P satisfying $P \cap (-P) = \{0\}$. Then $y \le z$ is equivalent to $y - z \in P$, for all $y, z \in Y$.

A mapping $g : X \to Y$ is called *P-convex* if $\operatorname{dom} g$ is a convex subset of X and if for all $x, u \in \operatorname{dom} g$ and $\lambda \in [0,1]$,

$$g(\lambda x + (1-\lambda)u) \le \lambda g(x) + (1-\lambda) g(u).$$

PROPOSITION 3.2. *Let $g : X \to Y$ be a P-convex mapping, $g(0) \in P$ and let $y^* \in N(P, g(0))$. Then:*

(a) *the function $y \to d(y,P)$ is increasing in the following sense:*

$$y \le z \Rightarrow d(y,P) \le d(z,P),$$

and the functions $x \to d(g(x),P)$ and $x \to (y^* \circ g)(x)$ are convex;

(b) $(x^*, -y^*) \in N(\operatorname{epi} g, 0, g(0)) \Leftrightarrow x^* \in \partial(y^* \circ g)(0)$;

(c) $\partial d(g(\cdot),P)(0) = \bigcup_{y^* \in \partial d(g(0),P)} \partial(y^* \circ g)(0)$.

Here $\operatorname{epi} g$ denotes the epigraph of g, that is,

$$\operatorname{epi} g := \{(x,y) \in X \times Y : g(x) \le y\}.$$

Proof. (a) Since $y \le z$ there exists $p \in P$ such that $y = z + p$. Thus, as $P + P = P$,

$$d(y,P) = d(z, P - p) = d(z, P + P - p) \le d(z,P).$$

(b) Let $(x^*, -y^*) \in N(\operatorname{epi} g, 0, g(0))$. Then $\langle x^*, x\rangle \le \langle y^*, g(x) - g(0)\rangle$ for all $x \in X$, and hence $x^* \in \partial(y^* \circ g)(0)$.

Conversely, if $x^* \in \partial(y^* \circ g)(0)$ then

$$\langle x^*, x\rangle - (y^* \circ g)(x) + (y^* \circ g)(0) \le 0, \quad \forall x \in X.$$

Thus for all $(x,y) \in \operatorname{epi} g$,

$$\langle x^*, x\rangle - \langle y^*, g(x) - y\rangle - \langle y^*, y - g(0)\rangle \le 0.$$

But $y^* \in N(P, g(0))$, therefore $\langle y^*, g(x) - y\rangle \le 0$ and so for all $(x,y) \in \operatorname{epi} g$,

$$\langle x^*, x\rangle - \langle y^*, y - g(0)\rangle \le 0.$$

(c) Using (a) we obtain for all $(x,y) \in X \times Y$,

$$d(g(x),P) \le d(y,P) + \Psi(\operatorname{epi} g, x, y),$$

where $\Psi(S,x)$ is the indicator function of S which is equal to 0 if $x \in S$ and ∞ otherwise. Thus if $x^* \in \partial d(g(\cdot),P)(0)$, we have

$$(x^*, 0) \in \{0\} \times \partial d(g(0),P) + N(\operatorname{epi} g, 0, g(0))$$

and so there exists $y^* \in \partial d(g(0),P)$ such that $(x^*, -y^*) \in N(\operatorname{epi} g, 0, g(0))$. By (b) we deduce that $x^* \in \partial(y^* \circ g)(0)$.

Conversely, if $y^* \in \partial d(g(0), P)$ and $x^* \in \partial(y^* \circ g)(0)$, then for all $x \in X$,

$$\langle x^*, x\rangle \le \langle y^*, g(x) - g(0)\rangle \le d(g(x), P)$$

and hence $x^* \in \partial d(g(\cdot), P)(0)$. ∎

COROLLARY 3.3. *Let $g : X \to Y$ be a P-convex mapping which is Lipschitz continuous, let $x^* \in X^*$ and C be a closed convex subset of X containing $0 \in g^{-1}(P) \cap C$. Suppose that $0 \in \operatorname{int}[g(C) - P]$. Then the following are equivalent*:

(1) $g(x) \in P$, $x \in C \Rightarrow \langle x^*, x\rangle \le 0$;
(2) *there exists $y^* \in N(P, g(0))$ such that $x^* \in \partial(y^* \circ g)(0) + N(C, 0)$.*

Proof. Consider the multivalued mapping $F : X \rightrightarrows Y$ defined by

$$F(x) = \begin{cases} -g(x) + P & \text{if } x \in C, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then (1) is equivalent to $x^* \in N(F^{-1}(0), 0)$. Since F is a closed convex multivalued mapping and $0 \in \operatorname{int} F(X)$, it follows that, by Theorem 3.1,

$$x^* \in N(F^{-1}(0), 0) \quad \text{iff} \quad x^* \in \mathbb{R}_+ \partial d(0, F(\cdot))(0).$$

The corollary is then established if we show that

$$\mathbb{R}_+ \partial d(0, F(\cdot))(0) = \bigcup_{y^* \in N(P, g(0))} \partial(y^* \circ g)(0) + N(C, 0).$$

Indeed, if $x^* \in \partial d(0, F(\cdot))(0)$, then since $d(0, F(x)) \le d(g(x), P) + \Psi(C, x)$, we have

$$\langle x^*, x\rangle \le d(g(x), P) + \Psi(C, x)$$

and so by the subdifferential calculus rules and Proposition 3.2 we obtain

$$x^* \in \bigcup_{y^* \in N(P, g(0))} \partial(y^* \circ g)(0) + N(C, 0).$$

The inverse inclusion is immediate. ∎

These results can be readily applied to produce necessary optimality conditions for γ-paraconvex programs

$$\text{(P)} \qquad \min f(x) \quad \text{subject to } 0 \in F(x),$$

where $f : X \to \mathbb{R}$ is a γ-paraconvex function which is locally Lipschitzian around x_0 and $F : X \rightrightarrows Y$ is a multivalued mapping with closed graph and such that F^{-1} is γ-paraconvex.

THEOREM 3.4. *Let $\gamma > 1$. If x_0 is a local minimum for* (P) *then*

$$0 \in \partial^- f(x_0) + N(F^{-1}(0), x_0).$$

If in addition $0 \in \operatorname{int} F(X)$, then there exists $K > 0$ such that

$$0 \in \partial^- f(x_0) + K\partial^- d(0, F(\cdot))(x_0),$$

or equivalently, there exists $C_1 > 0$ such that for all $h \in X$,

$$f(x_0) \le f(x) + Kd(0, F(x)) + C_1\|h\|^\gamma.$$

Proof. Denote by Ψ_C the indicator function of a set C. Then since $F^{-1}(0)$ is a convex set we have $N(F^{-1}(0), x_0) = \partial^- \Psi_{F^{-1}(0)}(x_0)$. As x_0 is a local minimum for (P) it follows that $0 \in \partial^-(f + \Psi_{F^{-1}(0)})(x_0)$ or equivalently (Theorem 3.4 and Corollary 5.4 of [9]),

$$0 \in \partial^- f(x_0) + N(F^{-1}(0), x_0).$$

For the second part we have, by Theorem 3.1, the existence of $K > 0$ such that

$$0 \in \partial^- f(x_0) + K\partial^- d(0, F(\cdot))(x_0)$$

or equivalently (Theorem 3.4 and Corollary 5.4 of [9]), the existence of $C_1 > 0$ such that for all $h \in X$,

$$f(x_0) \le f(x) + Kd(0, F(x)) + C_1\|h\|^\gamma. \ ∎$$

4. Remarks. 1) If we adopt the following definition of γ-paraconvex multivalued mappings: there exists $C > 0$ such that for all $x, u \in X$ and all $\alpha \in [0, 1]$,

$$\alpha F(x) + (1 - \alpha)F(u) \subset F(\alpha x + (1 - \alpha)u) + C\min(\alpha, 1 - \alpha)\|x - u\|^\gamma B_Y,$$

then the results of Theorems 2.2–2.4 are valid for all $\gamma > 0$.

2) Using the Lipschitz property of γ-paraconvex functions and Lemma 2.1, we may also give another proof to Theorem 2.3 (see for example the proof of Theorem 2.2 in [3]).

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Jordan polynomials can be analytically recognized

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Abstract. We prove that there exists a real or complex central simple associative algebra $\mathcal{M}$ with minimal one-sided ideals such that, for every non-Jordan associative polynomial $\mathbf{p}$, a Jordan-algebra norm can be given on $\mathcal{M}$ in such a way that the action of $\mathbf{p}$ on $\mathcal{M}$ becomes discontinuous.

1. Introduction. Among the associative polynomials (elements in the free associative algebra on a countably infinite set of indeterminates) the so-called "Jordan polynomials" are of special interest in Jordan theory. Jordan polynomials are those associative polynomials that can be expressed through the indeterminates by means of the sum and the Jordan product. Well-known examples of Jordan polynomials are $\mathbf{x}^2$ and $\mathbf{xyx}$, whereas the associative product $\mathbf{xy}$ and the tetrad $\mathbf{xyzt}+\mathbf{tzyx}$ are examples of non-Jordan polynomials.

Let A be an associative algebra over $\mathbb{K}$ ($=\mathbb{R}$ or $\mathbb{C}$), and $|\cdot|$ be a Jordan-algebra norm on A. Then obviously every Jordan polynomial acts $|\cdot|$-continuously on A. If either A is semiprime and $|\cdot|$ is complete or A is simple and has a unit element, then the associative product of A is $|\cdot|$-continuous ([14], [15], [4]), hence every associative polynomial acts $|\cdot|$-continuously on A. An example of $|\cdot|$-discontinuity of the associative product of A with $|\cdot|$ complete (hence A not semiprime) is given in [14]. The first example of $|\cdot|$-discontinuity of the associative product of A with A semiprime (hence $|\cdot|$ not complete) appears in [2] (see also [17]), but the algebra A in this example is very far from being simple (and even prime): it is an infinite direct sum of finite-dimensional simple ideals. Very recently an example of $|\cdot|$-discontinuity of the associative product of A with A simple (hence neither $|\cdot|$ can be complete nor A can have a unit) has been provided in [4],

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