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Subdifferentiability and subdifferential monotonicity of γ-paraconvex functions

by

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Abstract: In this paper, we study subdifferentiability and subdifferential monotonicity of γ-paraconvex functions. We introduce a subdifferential and we show that it coincides with Dini and Clarke subdifferentials for any γ-paraconvex function, $\gamma > 1$. We develop sum rules for this subdifferential with and without constraints qualification. Finally, we show that a function is γ-paraconvex whenever its presubdifferential is γ-monotone, $\gamma > 1$.

Keywords: γ-paraconvex functions, subdifferentials, Mean value theorem, γ-monotony.

1. Introduction

Rolewicz (1979a;b) introduced the concept of γ-paraconvex multivalued mappings. A multivalued mapping F between two normed vector spaces X and Y is γ-paraconvex if there exists a positive constant C such that

$$\lambda F(x) + (1-\lambda)F(u) \subset F(\lambda x + (1-\lambda)u) + C\|x-u\|^\gamma B_Y$$

for all $x, u \in X$ and $\lambda \in [0,1]$, where B_Y denotes the closed unit ball of Y. To this concept, he associated a class of functions called γ-paraconvex functions. An extended real-valued function f on X is γ-paraconvex if the multivalued mapping F defined by

$$F(x) = f(x) + \mathbb{R}_+$$

is γ-paraconvex, or equivalently, there exists $C > 0$ such that

$$f(\lambda x + (1-\lambda)u) \le \lambda f(x) + (1-\lambda)f(u) + C\|x-u\|^\gamma$$

for all $x, u \in X$ and $\lambda \in [0,1]$.

The most important properties, given by Rolewicz (1979a;b;1981), are listed in Section 2.

The aim of this paper is to explore further properties of $\gamma-$paraconvex functions, namely the Lipschitz properties and the subdifferentiability. First, we show that the well known classical theorem, which asserts that every continuous convex function defined on an open convex set $O \subset X$ is locally Lipschitz on O, subsists for $\gamma-$paraconvex functions. Second, we introduce a subdifferential to these functions and we show that it coincides with the Dini subdifferential and the Clarke subdifferential. This allows us to show that every $\gamma-$paraconvex function, $\gamma > 1$, is subdifferentially regular. Third, we establish sum rules for this subdifferential with and without constraint qualification conditions. Finally we prove that a lower semicontinuous function is $\gamma-$paraconvex whenever its subdifferential is $\gamma-$monotone.

We use the following notation. X is a normed vector space, X^* is the topological dual of X always considered with the weak-star topology, B_X is the closed unit ball of X, and $\langle \cdot, \cdot \rangle$ the pairing between X and X^*. We write $x \xrightarrow{S} x_0$ and $x \xrightarrow{f} x_0$ to express, respectively, $x \to x_0$ with $x \in S$ and $x \to x_0$ with $f(x) \to f(x_0)$. We denote by epi$f$ the epigraph of a real-valued function f.

2. Some properties of $\gamma-$paraconvex functions

Rolewicz (1979a) considered 2-paraconvex functions and he showed that this class of functions can be characterized as a difference of convex functions. More precisely he proved that if X is a Hilbert space then f is 2$-$paraconvex iff it can be represented in the form

$$f(x) = g(x) - C\|x\|^2$$

where g is a convex function and $C > 0$. Example 1 in Rolewicz (1979b), shows that for $1 < \gamma < 2$ the similar result does not hold even for $X = \mathbb{R}$.

Using Theorem 2 in Rolewicz (1979b), we may easily show that for $\gamma > 1$ we have the following important characterization (see Jourani, 1995).

PROPOSITION 2.1 *Let $\gamma > 1$. Then f is $\gamma-$paraconvex on a nonempty open convex set $O \subset X$ iff there exists $C > 0$ such that*

$$f(\lambda x + (1-\lambda)u) \leq \lambda f(x) + (1-\lambda)f(u) + C\min(\lambda, 1-\lambda)\|x-u\|^\gamma$$

for all $x, u \in O$ and $\lambda \in [0,1]$.

Note that this result is false for $\gamma = 1$. Let, for example, $f(x) = ||x| - 1|$, then f is 1$-$paraconvex with constant 2 but does not satisfy the inequality with the minimum on λ and $1-\lambda$.

As in the convex case we may establish the following proposition about the Lipschitz properties of $\gamma-$paraconvex functions.

PROPOSITION 2.2 *Let $\gamma > 1$. Then every continuous $\gamma-$paraconvex function f defined on a nonempty open convex set $O \subset X$ is locally Lipschitz on O.*

Proof. The same proof is used in the convex case. Let $x_0 \in O$. Then, by assumptions, there exist $r > 0$ and $a > 0$ such that $x_0 + rB_X \subset O$ and for all $x \in x_0 + rB_X$

$$|f(x)| \leq a.$$

Let $x, u \in x_0 + \frac{r}{2}B_X$. For $\varepsilon > 0$, put $\alpha = \varepsilon + \|x - u\|$ and $z = u + \frac{r}{2\alpha}(u - x)$. Then $\|z - x_0\| \leq \|u - x_0\| + \frac{r}{2\alpha}\|u - x\| \leq r$ and so

$$f(z) \leq a.$$

Set $\lambda = \dfrac{2\alpha}{r + 2\alpha}$. By Proposition 2.1

$$f(u) = f(\lambda z + (1 - \lambda)x) \leq \lambda f(z) + (1 - \lambda)f(x) + C\lambda\|z - x\|^\gamma$$

and hence

$$\begin{aligned} f(u) - f(x) &\leq \lambda(f(z) - f(x)) + C\lambda^{1-\gamma}\|u - x\|^\gamma \\ &\leq 2a\lambda + C\lambda^{1-\gamma}\|u - x\|^\gamma \\ &\leq \tfrac{4a}{r}(\varepsilon + \|u - x\|) + \tfrac{2C(\varepsilon+\|u-x\|)(r+2\varepsilon+2\|u-x\|)^\gamma\|u-x\|^\gamma}{r2^\gamma(\varepsilon+\|u-x\|)^\gamma} \end{aligned}$$

Since $\varepsilon > 0$ is arbitrarily choosen, it follows that

$$f(u) - f(x) \leq (\frac{4a}{r} + \frac{2C(3r)^\gamma}{2^\gamma r})\|u - x\|$$

and changing the roles of u and x we obtain

$$|f(u) - f(x)| \leq (\frac{4a}{r} + \frac{2C(3r)^\gamma}{2^\gamma r})\|u - x\|$$

for all $x, u \in x_0 + \frac{r}{2}B_X$. ■

REMARK 2.1 *Relying on Lemma 2.5 in Jourani (1995), we may show that every γ−paraconvex function, $\gamma > 1$, on a nonempty convex set O on which is bounded from above is locally Lipschitz on O.*

3. Subdifferentiability and subdifferential regularity of the γ−paraconvex functions

It is well known that the subdifferential of an extended real-valued convex function is the set

$$\partial f(x_0) = \{x^* \in X^* : \quad \langle x^*, h\rangle \leq f(x_0 + h) - f(x_0), \quad \forall h\}.$$

This set, which has good sum rules, is convex and weak-star closed. In addition it is nonempty whenever f is continuous around x_0. So can we adapt this subdifferential to γ−praconvex functions? Unfortunately no. To see this, let, for example, $f(x) = ||x| - 1|$, so f is 1−paraconvex and $\partial f(0)$ is an empty set.

Our aim is to introduce a subdifferential for this class of functions and to show that this subdifferential inherits some properties of ∂f. We also show that this subdifferential coincides with the Dini subdifferential and the Clarke subdifferential whenever $\gamma > 1$.

DEFINITION 3.1 *Let $\gamma > 0$ and $C > 0$. Let f be an extended real-valued function on X which is finite at x_0. $x^* \in X^*$ is said to be a $(\gamma, C)-$ subgradient of f at x_0 if there exists a neighbourhood V of 0 such that*

$$\langle x^*, h\rangle \leq f(x_0 + h) - f(x_0) + C\|h\|^\gamma, \quad \forall h \in V.$$

The set of $(\gamma, C)-$ subgradients of f at x_0 is denoted by $\partial^{Loc}_{(\gamma,C)} f(x_0)$.

It is clear that $\partial^{Loc}_{(\gamma,C)} f(x_0)$ is convex and weak-star closed. One of the other important properties of this subdifferential is that it verifies the Fermat rule which states that if x_0 is a local minimum of f then it is a critical point, i. e., $0 \in \partial^{Loc}_{(\gamma,C)} f(x_0)$.

For $\gamma-$paraconvex function we have

PROPOSITION 3.1 *Let f be $\gamma-$paraconvex. Suppose that $\gamma > 1$. Then there exists $C > 0$ such that*

$$\partial^{Loc}_{(\gamma,C)} f(x_0) = \partial_{(\gamma,C)} f(x_0)$$

where

$$\partial_{(\gamma,C)} f(x_0) = \{x^* \in X^* : \langle x^*, h\rangle \leq f(x_0 + h) - f(x_0) + C\|h\|^\gamma, \, \forall h \in X\}.$$

Proof. Let $C > 0$ as in Proposition 2.1. Let $x^* \in \partial^{Loc}_{(\gamma,C)} f(x_0)$. Then there exists a neighbourhood V of 0 such that

$$\langle x^*, h\rangle \leq f(x_0 + h) - f(x_0) + C\|h\|^\gamma, \quad \forall h \in V.$$

Let $h \in X$ and $t > 0$ sufficiently small such that $th \in V$. Then

$$\langle x^*, th\rangle \leq f(x_0 + th) - f(x_0) + C\|th\|^\gamma$$

and so, by $\gamma-$paraconvexity of f, it follows that

$$\langle x^*, h\rangle \leq f(x_0 + h) - f(x_0) + C\|h\|^\gamma + Ct^{\gamma-1}\|h\|^\gamma.$$

Thus, as $\gamma > 1$, passing to the limit on t we obtain the result. ∎

More generally, we may show that the assumptions of this proposition imply the following:

$$\partial_{(\gamma,C)} f(x_0) = seq - \limsup_{x \xrightarrow{f} x_0} \partial^{Loc}_{(\gamma,C)} f(x)$$

where

$$seq - \limsup_{x \xrightarrow{f} x_0} F(x) = \{x^* \in X^* :$$

$$\exists \text{ sequences } x_n \xrightarrow{f} x_0 \text{ and } x_n^* \to x^* / x_n^* \in F(x_n) \quad \forall n\}.$$

Before pursuing the connection of this subdifferential with the Dini subdifferential and the Clarke subdifferential and the study of the subdifferential regularity of γ-paraconvex functions we pause to recall some definitions. Let f be an extended real-valued function on X which is finite at x_0. We set

$$d^- f(x_0, h) = \liminf_{\substack{u \to h \\ t \to 0+}} t^{-1}(f(x_0 + tu) - f(x_0))$$

$$d^\uparrow f(x_0, h) = \sup_{\varepsilon > 0} \limsup_{\substack{(x,\alpha) \xrightarrow{epif} (x_0, f(x_0)) \\ t \to 0+}} \inf_{u \in h + \varepsilon B_X} t^{-1}(f(x + tu) - \alpha)$$

$$\partial^- f(x_0) = \{x^* \in X^* : \quad \langle x^*, h\rangle \leq d^- f(x_0, h), \quad \forall h\}$$

$$\partial_c f(x_0) = \{x^* \in X^* : \quad \langle x^*, h\rangle \leq d^\uparrow f(x_0, h), \quad \forall h\}.$$

The functions $h \to d^- f(x_0, h)$ and $h \to d^\uparrow f(x_0, h)$ are called the (lower) Dini directional derivative and the (upper) subderivative of f at x_0 and the sets $\partial^- f(x_0)$ and $\partial_c f(x_0)$ are the Dini subdifferential and the Clarke subdifferential of f at x_0 (see Ioffe, 1983;1984;1989, Clarke, 1983, Rockafellar, 1979;1980, for more details).

The geometrical characterizations of these derivatives are (see Ioffe, 1984, Clarke, 1983, Rockafellar, 1979;1980)

$$d^- f(x_0, h) = \inf\{r : \quad (h, r) \in K(\text{epi}f, x_0, f(x_0))\}$$

$$d^\uparrow f(x_0, h) = \inf\{r : \quad (h, r) \in T_c(\text{epi}f, x_0, f(x_0))\}$$

and consequently

$$\partial^- f(x_0) = \{x^* \in X^* : \quad (x^*, -1) \in K^0(\text{epi}f, x_0, f(x_0))\}$$

$$\partial_c f(x_0) = \{x^* \in X^* : \quad (x^*, -1) \in N_c(\text{epi}f, x_0, f(x_0))\}$$

where

$$H^0 = \{x^* \in X^* : \quad \langle x^*, h\rangle \leq 0, \quad \forall h \in H\}$$

$K(S, x_0)$ is the contingent cone to S at $x_0 \in S$, i.e.,

$$K(S, x_0) = \{h \in X : \quad \exists t_n \to 0^+, \exists h_n \to h \text{ such that } x_0 + t_n h_n \in S, \forall n\}$$

$T_c(S, x_0)$ is the Clarke normal cone to S at x_0, i. e.,

$$\begin{aligned} T_c(S, x_0) &= \{h \in X : \\ &\quad \forall x_n \xrightarrow{S} x_0,\ \forall t_n \to 0^+,\ \exists h_n \to h \text{ such that } x_n + t_n h_n \in S,\ \forall n\} \end{aligned}$$

and $N_C(S, x_0) = (T_c(S, x_0))^0$ is the Clarke normal cone which can be expressed in terms of the subdifferential of the distance function as follows

$$N_c(S, x_0) = \mathrm{cl}^*[\ \mathbb{R}_+ \partial_c d(x_0, S)].$$

It is known that when f is a convex function

$$T_c(\mathrm{epi} f, x_0, f(x_0)) = \mathrm{cl}\ [\ \mathbb{R}_+(\mathrm{epi}\ f - (x_0, f(x_0))].$$

In the $\gamma-$paraconvex case we have

LEMMA 3.1 *Let f be a $\gamma-$paraconvex function with constant $C > 0$, $\gamma > 1$. Then for all $(x, r) \in$ epi f,*

$$(x - x_0, r - f(x_0) + C\|x - x_0\|^\gamma) \in T_c(epi\ f, x_0, f(x_0)).$$

Proof. Let $((x_n, r_n)) \subset$ epi f converges to $(x_0, f(x_0))$ and let $t_n \to 0^+$. Then since f is $\gamma-$paraconvex and $\gamma > 1$ there exists $C > 0$ such that

$$\begin{aligned} &f(x_n + t_n(x - x_n)) - Ct_n\|x - x_n\|^\gamma \leq \\ &t_n f(x) + (1 - t_n) f(x_n) \leq (1 - t_n) r_n + t_n r, \end{aligned}$$

(Proposition 2.1) and hence

$$(x_n, r_n) + t_n(x - x_n, r - r_n + C\|x - x_n\|^\gamma) \in \mathrm{epi}\ f$$

which yields

$$(x - x_0, r - f(x_0) + C\|x - x_0\|^\gamma) \in T_c(\mathrm{epi}\ f, x_0, f(x_0)).$$

■

THEOREM 3.1 *If f is $\gamma-$paraconvex with constant $C > 0$, $\gamma > 1$, then*

$$\partial_{(\gamma,C)} f(x_0) = \partial^- f(x_0) = \partial_c f(x_0).$$

Proof. For all $h \in X$ there exists $t_n \to 0^+$ and $h_n \to h$ such that

$$d^- f(x_0, h) = \lim_{n \to \infty} t_n^{-1}(f(x_0 + t_n h_n) - f(x_0)).$$

Let $x^* \in \partial_{(\gamma,C)} f(x_0)$. Then

$$\langle x^*, t_n h_n \rangle \leq f(x_0 + t_n h_n) - f(x_0) + Ct_n t_n^{\gamma-1}\|h_n\|^\gamma$$

and hence

$$\langle x^*, h \rangle \leq d^- f(x_0, h).$$

Whence the first inclusion. The second inclusion is trivial. So let us prove the last inclusion $\partial_c f(x_0) \subset \partial_{(\gamma,C)} f(x_0)$. Let $x^* \in \partial_c f(x_0)$. Then $(x^*, -1) \in N_c(\text{epi } f, x_0, f(x_0))$, or equivalently

$$\langle x^*, h\rangle - r \leq 0, \quad \forall (h,r) \in T_c(\text{epi } f, x_0, f(x_0)).$$

Using Lemma 3.1 we get for all $x \in X$, $(x - x_0, f(x) - f(x_0) + C\|x - x_0\|^\gamma) \in T_c(\text{epi } f, x_0, f(x_0))$ and so

$$\langle x^*, x - x_0\rangle \leq f(x) - f(x_0) + C\|x - x_0\|^\gamma.$$

Thus $x^* \in \partial_{(\gamma,C)} f(x_0)$. ■

This result is not correct when $\gamma = 1$. Let, for example, $f(x) = ||x| - 1|$, then f is $1-$paraconvex, $\partial^- f(0) = \emptyset$ while $\partial_{(1,2)} f(0) = \partial_c f(0) = [-1, 1]$.

Comparing the Dini directional derivative and the subderivative we always have

$$d^- f(x_0, h) \leq d^\uparrow f(x_0, h), \quad \forall h \in X.$$

DEFINITION 3.2 *Rockafellar (1980). When the equality holds we say that f is subdifferentially regular at x_0.*

When f is the indicator function of a set S (containing x_0), this means that

$$K(S, x_0) = T_c(S, x_0).$$

COROLLARY 3.1 *Let f be a $\gamma-$paraconvex function, with $\gamma > 1$. Suppose that $\partial_c f(x_0) \neq \emptyset$. Then f is subdifferentially regular at x_0.*

Proof. By Theorem 3.1 we have for all $x^* \in \partial_c f(x_0)$

$$\langle x^*, h\rangle \leq d^- f(x_0, h), \quad \forall h \in X$$

and hence

$$\sup\{\langle x^*, h\rangle : \quad x^* \in \partial_c f(x_0)\} \leq d^- f(x_0, h).$$

So, by Theorem 4 in Rockafellar (1980), $d^\uparrow f(x_0, h) \leq d^- f(x_0, h)$. ■

4. Connection with Gateaux-differentiability

Let f be an extended real-valued function on X. It is said to be Gateaux-differentiable at x_0 if there exists $x^* \in X^*$ such that

$$\lim_{t \to 0^+} t^{-1}(f(x_0 + th) - f(x_0)) = \langle x^*, h\rangle, \text{ for all } h \in X.$$

x^* is called the Gateaux-differential of f at x_0, and is denoted $f'(x_0)$.

Proposition 4.1 *Let f be a Gateaux-differentiable function on a nonempty open convex set $O \subset X$ and let $\gamma > 1$. Then the following assertions are equivalent*

i) f *is* $\gamma-$*paraconvex on* O *with constant* $C > 0$,

ii) $\langle f'(u), x-u\rangle \leq f(x) - f(u) + C\|x-u\|^\gamma, \quad \forall x, u \in O.$

Proof. The implication $i) \Rightarrow ii)$ is a consequence of Theorem 3.1, since $\partial^- f(u) = \{f'(u)\}$. So let us prove the inverse implication. By $ii)$ we have for all $t \in [0,1]$ and all $x, u \in O$

$$t\langle f'(x+t(u-x)), x-u\rangle + f(x+t(u-x)) \leq f(x) + Ct^\gamma\|x-u\|^\gamma$$

$$\begin{aligned}(1-t)\langle f'(x+t(u-x)), u-x\rangle + f(x+t(u-x)) \leq \\ f(u) + C(1-t)^\gamma\|x-u\|^\gamma\end{aligned}$$

and so

$$\begin{aligned}t(1-t)\langle f'(x+t(u-x)), x-u\rangle + (1-t)f(x+t(u-x)) \leq \\ (1-t)f(x) + C(1-t)\|x-u\|^\gamma\end{aligned}$$

$$t(1-t)\langle f'(x+t(u-x)), u-x\rangle + tf(x+t(u-x)) \leq tf(u) + Ct\|x-u\|^\gamma.$$

Thus

$$f(x+t(u-x)) \leq (1-t)f(x) + tf(u) + C\|x-u\|^\gamma.$$

■

5. Subdifferential calculus under constraint qualification conditions

The aim of this section is to show how the assumptions guaranteeing sum rules in the convex case ensure sum rules in the $\gamma-$paraconvex case. To begin we introduce (Ioffe, 1989) the following sets associated to the functions f_1 and f_2

$$S_1 = \{(x,r,s) \in X \times \mathbb{R} \times \mathbb{R} : \quad f_1(x) \leq r\}$$

$$S_2 = \{(x,r,s) \in X \times \mathbb{R} \times \mathbb{R} : \quad f_2(x) \leq s\}$$

$$S = \{(x,r,s) \in X \times \mathbb{R} \times \mathbb{R} : \quad (f_1+f_2)(x) \leq r+s\}.$$

We easily show as in Ioffe (1989)

Lemma 5.1

$$\begin{aligned}N_c(S_1, x_0, f_1(x_0), f_2(x_0)) &= \{(x^*, \lambda, 0) \in X^* \times \mathbb{R} \times \mathbb{R} : (x^*, \lambda) \\ &\in N_c(epi\ f_1, x_0, f_1(x_0))\}\end{aligned}$$

$$N_c(S_2, x_0, f_1(x_0), f_2(x_0)) = \{(x^*, 0, \lambda) \in X^* \times I\!R \times I\!R : (x^*, \lambda) \in N_c(epi\ f_2, x_0, f_2(x_0))\}$$

$$N_c(S, x_0, f_1(x_0), f_2(x_0)) = \{(x^*, \lambda, \lambda) \in X^* \times I\!R \times I\!R : (x^*, \lambda) \in N_c(epi\ (f_1 + f_2), x_0, (f_1 + f_2)(x_0))\}$$

$$N_c(S, x_0, f_1(x_0), f_2(x_0)) \subset N_c(S_1 \cap S_2, x_0, f_1(x_0), f_2(x_0)).$$

The following result will be our basic tool.

PROPOSITION 5.1 *Let f_1 and f_2 be extended real-valued $\gamma-$paraconvex functions on X, $\gamma > 1$, which are finite at x_0. Suppose that there exist $\beta_1 > 0$ and $\beta_2 > 0$ such that*

$$\beta_1 B_{X\times I\!R} \subset epi\ f_1 \cap \beta_2 B_{X\times I\!R} - epi\ f_2 \cap \beta_2 B_{X\times I\!R}. \tag{1}$$

Then there exists $a > 0$ such that

$$d(x, r, s, S_1 \cap S_2) \leq a d(x, r, s, S_2)$$

for all $(x, r, s) \in \beta_1 B_{X\times I\!R\times I\!R} \cap S_1$.

Proof. Note that (1) is equivalent to

$$\beta_1 B_{X\times \mathbb{R}\times \mathbb{R}} \subset S_1 \cap \beta_2 B_{X\times \mathbb{R}\times \mathbb{R}} - S_2 \cap \beta_2 B_{X\times \mathbb{R}\times \mathbb{R}}. \tag{2}$$

Let $(x, r, s) \in \beta_1 B_{X\times \mathbb{R}\times \mathbb{R}} \cap S_1$. Choose $\varepsilon \in]0, 1[$ and take $(x_\varepsilon, r_\varepsilon, s_\varepsilon) \in S_2$ such that

$$\|(x - x_\varepsilon, r - r_\varepsilon, s - s_\varepsilon)\| \leq d(x, r, s, S_2) + \varepsilon.$$

Set $t = \|(x - x_\varepsilon, r - r_\varepsilon, s - s_\varepsilon)\|$. If $t = 0$ then there is nothing to prove. If $t \neq 0$ we have by (2) the existence of $(x_i, r_i, s_i) \in S_i \cap \beta_2 B_{X\times \mathbb{R}\times \mathbb{R}}$, $i = 1, 2$, such that

$$\beta_1(x_\varepsilon - x, r_\varepsilon - r, s_\varepsilon - s) = t(x_1 - x_2, r_1 - r_2, s_1 - s_2)$$

and so

$$\begin{aligned} Z &:= (t + \beta_1)^{-1}(\beta_1 x + t x_1, \beta_1 r + t r_1, \beta_1 s + t s_1) = \\ &\quad (t + \beta_1)^{-1}(\beta_1 x_\varepsilon + t x_2, \beta_1 r_\varepsilon + t r_2, \beta_1 s_\varepsilon + t s_2). \end{aligned}$$

Set $\lambda = \dfrac{\beta_1}{\beta_1 + t}$. As $(x_\varepsilon, r_\varepsilon, s_\varepsilon), (x_2, r_2, s_2) \in S_2$ and $(x, r, s), (x_1, r_1, s_1) \in S_1$ we have, by the $\gamma-$paraconvexity of f_1 and f_2 the existence of $C_1 > 0$ and $C_2 > 0$ such that

$$f_1(\lambda x + (1 - \lambda)x_1) \leq \lambda f_1(x) + (1 - \lambda) f_1(x_1) + C_1(1 - \lambda)\|x - x_1\|^\gamma$$

$$f_2(\lambda x_\varepsilon + (1-\lambda)x_2) \leq \lambda f_2(x_\varepsilon) + (1-\lambda)f_2(x_2) + C_2(1-\lambda)\|x_\varepsilon - x_2\|^\gamma$$

and hence

$$Z + (0, C_1(1-\lambda)\|x - x_1\|^\gamma + C_2(1-\lambda)\|x_\varepsilon - x_2\|^\gamma, 0) \in S_1$$

and

$$Z + (0, 0, C_1(1-\lambda)\|x - x_1\|^\gamma + C_2(1-\lambda)\|x_\varepsilon - x_2\|^\gamma) \in S_2$$

or equivalently

$$\begin{aligned} W \; := \; & Z + (1-\lambda) \\ & (0, C_1\|x - x_1\|^\gamma + C_2\|x_\varepsilon - x_2\|^\gamma, C_1\|x - x_1\|^\gamma + C_2\|x_\varepsilon - x_2\|^\gamma) \\ & \in S_1 \cap S_2. \end{aligned}$$

Thus

$$\begin{aligned} & \|(x,r,s) - W\| \leq \\ & \quad \|(x,r,s) - Z\| + 2(C_1 + C_2)(1-\lambda)[\|x - x_1\|^\gamma + \|x_\varepsilon - x_2\|^\gamma] \\ & \leq \; (1-\lambda)[\|(x - x_1, r - r_1, s - s_1)\| + 2(C_1 + C_2)[\|x - x_1\|^\gamma + \\ & \quad \|x_\varepsilon - x_2\|^\gamma]] \\ & \leq \; \frac{t}{\beta_1}[3(\beta_1 + \beta_2) + 2(C_1 + C_2)(\beta_1 + \beta_2)^\gamma + 2(C_1 + C_2)(2\beta_1 + 1)^\gamma] \\ & \leq \; \beta_1^{-1}[3(\beta_1 + \beta_2) + (C_1 + C_2)(\beta_1 + \beta_2)^\gamma + (C_1 + C_2)(2\beta_1 + 1)^\gamma] \\ & \qquad (d(x,r,s,S_2) + \varepsilon). \end{aligned}$$

As ε is arbitrary and $W \in S_1 \cap S_2$ we conclude that

$$\begin{aligned} & d(x,r,s,S_1 \cap S_2) \leq \beta_1^{-1}[3(\beta_1 + \beta_2) + 2(C_1 + C_2) \\ & (\beta_1 + \beta_2)^\gamma + 2(C_1 + C_2)(2\beta_1 + 1)^\gamma] d(x,r,s,S_2). \end{aligned}$$

■

Now we may state the main result of this section.

THEOREM 5.1 *Let f_1 and f_2 be two extended real-valued γ–paraconvex functions on X, $\gamma > 1$, with constants C_1 and C_2 respectively. Suppose that X is complete. Suppose also that f_1 and f_2 are finite at x_0 and that (1) holds. Then*

$$\partial_{(\gamma,(C_1+C_2))}(f_1 + f_2)(x_0) = \partial_{(\gamma,C_1)} f_1(x_0) + \partial_{(\gamma,C_2)} f_2(x_0).$$

Proof. *Step 1*: We show that there exists $a > 0$ such that

$$\begin{aligned} & \partial_c d(x_0, f_1(x_0), f_2(x_0), S_1 \cap S_2) \subset \\ & a[\partial_c d(x_0, f_1(x_0), f_2(x_0), S_1) + \partial_c d(x_0, f_1(x_0), f_2(x_0), S_2)]. \end{aligned}$$

By Proposition 1.3 in Thibault (1991)

$$d^{\uparrow}d(\cdot, S_1 \cap S_2)(x_0, f_1(x_0), f_2(x_0); h, \alpha, \beta) = \limsup_{\substack{(x,r,s) \xrightarrow{S_1 \cap S_2} (x_0, f_1(x_0), f_2(x_0)) \\ t \to 0^+}} t^{-1} d((x,r,s) + t(h,\alpha,\beta), S_1 \cap S_2)$$

and hence, by Proposition 5.1, there exists $a > 0$ such that for $(x,r,s) \in S_1 \cap S_2$ near $(x_0, f_1(x_0), f_2(x_0)$ and t sufficiently small

$$d(x,r,s,S_1 \cap S_2) \leq a[d((x,r,s) + t(h,\alpha,\beta), S_1) + d((x,r,s) + t(h,\alpha,\beta), S_2)]$$

and so

$$d^{\uparrow}d(\cdot, S_1 \cap S_2)(x_0, f_1(x_0), f_2(x_0); h, \alpha, \beta) \leq ad^{\uparrow}[d(\cdot, S_1) + d(\cdot, S_2)](x_0, f_1(x_0), f_2(x_0), h, \alpha, \beta).$$

Thus

$$\partial_c d(x_0, f_1(x_0), f_2(x_0), S_1 \cap S_2) \subset a\partial_c[d(\cdot, S_1) + d(\cdot, S_2)](x_0, f_1(x_0), f_2(x_0))]$$

and by subdifferential calculus we conclude that

$$\partial_c d(x_0, f_1(x_0), f_2(x_0), S_1 \cap S_2) \subset a[\partial_c d(x_0, f_1(x_0), f_2(x_0), S_1) + \partial_c d(x_0, f_1(x_0), f_2(x_0), S_2)].$$

Step 2 : We show that the set

$$N_c(S_1, x_0, f_1(x_0), f_2(x_0)) + N_c(S_2, x_0, f_1(x_0), f_2(x_0))$$

is weak-star closed. Using Lemma 3.1 we easily show that (2) implies

$$T_c(S_1, x_0, f_1(x_0), f_2(x_0)) - T_c(S_2, x_0, f_1(x_0), f_2(x_0)) = X \times \mathbb{R} \times \mathbb{R}$$

and hence by Theorem 6.3 in Borwein (1986), we obtain the weak-star closedness of

$$N_c(S_1, x_0, f_1(x_0), f_2(x_0)) + N_c(S_2, x_0, f_1(x_0), f_2(x_0)).$$

Step 3 : Let $x^* \in \partial_{(\gamma,(C_1+C_2))}(f_1 + f_2)(x_0)$ then by Theorem 3.1 $x^* \in \partial_c(f_1 + f_2)(x_0)$, or equivalently (Lemma 5.1),

$$(x^*, -1, -1) \in N_c(S, x_0, f_1(x_0), f_2(x_0)).$$

So Lemma 5.1 implies that

$$(x^*, -1, -1) \in N_c(S_1 \cap S_2, x_0, f_1(x_0), f_2(x_0)).$$

By Steps 1 and 2

$$(x^*, -1, -1) \in N_c(S_1, x_0, f_1(x_0), f_2(x_0)) + N_c(S_2, x_0, f_1(x_0), f_2(x_0)).$$

By Lemma 5.1, $x^* = u^* + v^*$, with $(u^*, -1) \in N_c(\text{epi } f_1, x_0, f_1(x_0))$ and $(v^*, -1) \in N_c(\text{epi } f_2, x_0, f_2(x_0))$. Thus $u^* \in \partial_c f_1(x_0)$ and $v^* \in \partial_c f_2(x_0)$ and so, by Theorem 3.4, $u^* \in \partial_{(\gamma, C_1)} f_1(x_0)$ and $v^* \in \partial_{(\gamma, C_2)} f_2(x_0)$. ■

COROLLARY 5.1 *Let f_1 and f_2 be as in Theorem 5.1. Suppose that f_1 is continuous around x_0. Then*

$$\partial_{(\gamma,(C_1+C_2))}(f_1 + f_2)(x_0) = \partial_{(\gamma,C_1)} f_1(x_0) + \partial_{(\gamma,C_2)} f_2(x_0).$$

Proof. Without loss of generality we may assume that $x_0 = 0$ and $f_i(x_0) = 0$, $i = 1, 2$. Since f_1 is continuous around 0 there exist $a > 0$ and $r \in]0, a[$ such that

$$f_1(x) \leq a, \quad \forall x \in rB_X.$$

Let $(x, s) \in rB_{X \times \mathbb{R}}$. Then

$$(x, s) = (x, a) - (0, a - s) \in \text{epi } f_1 \cap (r + a)B_{X \times \mathbb{R}} - \text{epi } f_2.$$

and the proof is complete by applying Theorem 5.1. ■

6. Subdifferential calculus without constraint qualification conditions

In this section we assume that X is a Banach space admitting an equivalent norm which is Gateaux-differentiable off zero.

THEOREM 6.1 *Let f_1 and f_2 be two extended real-valued γ–paraconvex functions on X, $\gamma > 1$, with constants C_1 and C_2. Suppose that f_1 and f_2 are lower semicontinuous around x_0. Then*

$$\partial_{(\gamma,(C_1+C_2))}(f_1 + f_2)(x_0) \subset \limsup_{x_i \xrightarrow{f_i} x_0,\, i=1,2} [\partial_{(\gamma,C_1)} f_1(x_1) + \partial_{(\gamma,C_2)} f_2(x_2)].$$

Proof. Let $x^* \in \partial_{(\gamma,(C_1+C_2))}(f_1 + f_2)(x_0)$. Then, by Theorem 3.1, $x^* \in \partial^-(f_1 + f_2)(x_0)$ and hence for all $n \in \mathbb{N} \backslash \{0\}$, $x^* \in \partial^-_{\frac{1}{n}}(f_1 + f_2)(x_0)$, where

$$\partial^-_\varepsilon f(x_0) = \{x^* \in X^* : \quad \langle x^*, h \rangle \leq d^- f(x_0, h) + \varepsilon \|h\|, \quad \forall h\}.$$

So (see Ioffe, 1983)

$$x^* \in \limsup_{x_i \xrightarrow{f_i} x_0} [\partial^-_{\frac{1}{n}} f_1(x_1) + \partial^-_{\frac{1}{n}} f_2(x_2)]$$

and since f_1 and f_2 are γ-paraconvex, $\gamma > 1$, we get (Theorem 3.1)

$$x^* \in \limsup_{x_i \xrightarrow{f_i} x_0} [\partial_{(\gamma,C_1)} f_1(x_1) + \partial_{(\gamma,C_2)} f_2(x_2) + \frac{2}{n} B_{X^*}].$$

Thus there are nets $x_j^n \xrightarrow{f_1} x_0$, $u_j^n \xrightarrow{f_2} x_0$, $x_{n,j}^* \in \partial_{(\gamma,C_1)} f_1(x_j^n)$, $u_{n,j}^* \in \partial_{(\gamma,C_2)} f_2(u_j^n)$ and $b_{n,j}^* \in \frac{2}{n} B_{X^*}$ such that

$$x_{n,j}^* + u_{n,j}^* + b_{n,j}^* \to x^*.$$

As $(b_{n,j}^*)_j$ is bounded we may assume that $b_{n,j}^* \to b_n^* \in \frac{2}{n} B_{X^*}$ and we obtain

$$x_{n,j}^* + u_{n,j}^* \to x^* - b_n^*$$

and hence

$$x^* - b_n^* \in \limsup_{x_i \xrightarrow{f_i} x_0} [\partial_{(\gamma,C_1)} f_1(x_1) + \partial_{(\gamma,C_2)} f_2(x_2)].$$

But $b_n^* \to 0$ and then

$$x^* \in \limsup_{x_i \xrightarrow{f_i} x_0} [\partial_{(\gamma,C_1)} f_1(x_1) + \partial_{(\gamma,C_2)} f_2(x_2)].$$

■

7. Subdifferential monotonicity

Let $\gamma > 0$. A multivalued mapping $A : X \rightrightarrows X^*$ is γ-monotone if there exists $C > 0$ such that for all $x, u \in X$, $x^* \in A(x)$, $u^* \in A(u)$

$$\langle x^* - u^*, u - x \rangle \le C \|u - x\|^\gamma.$$

As $\partial_{(\gamma,C)} f$ is always γ-monotone, it follows that any subset of the graph of the multivalue mapping $\partial_{(\gamma,C)} f$ is also γ-monotone. In fact, as in Correa, Jofré and Thibault (1994), we show that every presubdifferential ∂ (Thibault and Zagrodny, 1995, Thibault, 1994) on X, which is an operator satisfying the following properties:

for any function $f : X \to \mathbb{R} \cup \{\infty\}$, any continuous convex function $g : X \to \mathbb{R}$, and any $x \in X$

i) $\partial f(x) \subset X^*$ and $\partial f(x) = \emptyset$ whenever $f(x) = +\infty$

ii) if f is lower semicontinuous (lsc) and convex then $\partial f(x)$ coincides with the subdifferential in the sense of convex analysis of f at x

iii) $0 \in \partial f(x)$ whenever x is a local minimum of f

iv) $\partial f(x) = \partial h(x)$ whenever f and h coincide around x

v) when f is lsc,

$$\partial(f + \lambda g)(x) \subset \limsup_{u \xrightarrow{f} x} \partial f(u) + \partial g(x)$$

where limsup denotes here the weak-star sequential limit superior and $u \xrightarrow{f} x$ means $u \to x$ and $f(u) \to f(x)$,

verifies

PROPOSITION 7.1 *Let X be a Banach space. If $f : X \to \mathbb{R} \cup \{\infty\}$ is lower semicontinuous and if ∂f is $\gamma-$monotone, $\gamma > 1$, with constant C then*

1) *for all x in the domain of f $(f(x) < +\infty)$ and y in the domain of ∂f $(\partial f(y) \neq \emptyset)$ we have $[x, y]$ is a subset of the domain of f and*

2) $\partial f(x) \subset \partial_{(\gamma,C)} f(x)$.

Before giving our main result of this section let us show that the result of Mc Linden (1982), subsists for this subdifferential.

PROPOSITION 7.2 *Let f be a lower semicontinuous function on X. Then for all x_0, with $f(x_0) < \infty$ there exists sequence $x_n \xrightarrow{f} x_0$ such that $\partial f(x_n) \neq \emptyset$.*

Proof. It is a direct consequence of the main value theorem by Thibault (1994) for this presubdifferential or Zagrodny (1988) and Mc Linden (1982) for Clarke subdifferential (see also Thibault and Zagrodny, 1995). ■

The following theorem is an adaptation of the main theorem in Correa, Jofré and Thibault (1994) (see also Correa, Jofré and Thibault, 1995, where the result is expressed in terms of the presubdifferential) to the case of $\gamma-$paraconvex functions. For the convenience of the reader we include its proof.

THEOREM 7.1 *Let X be a Banach space and let $\gamma > 1$ and let $f : X \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function. Suppose that ∂f is $\gamma-$monotone. Then there exists $C > 0$ such that for all $x, y \in X$ and $\lambda \in [0, 1]$*

$$\begin{aligned} &f(\lambda x + (1 - \lambda)y) \leq \\ &\lambda f(x) + (1 - \lambda) f(y) + C(\lambda(1 - \lambda)^\gamma + (1 - \lambda)\lambda^\gamma)\|x - y\|^\gamma. \end{aligned}$$

In particular f is $\gamma-$paraconvex.

Proof. Let x, y in the domain of f and $z = \lambda x + (1 - \lambda)y$, with $\lambda \in]0, 1[$. By proposition 7.2, there exists a sequence (y_k) in the domain of ∂f such that $y_k \to y$ and $f(y_k) \to f(y)$. From Proposition 7.1 1) z_k is in the domain of f.

Step 1 : If z_k is not a local minimum of f we can choose z'_k such that

$$\|z'_k - z_k\| < \frac{1}{k} \text{ and } f(z'_k) < f(z_k).$$

By applying Theorem 4.3 in Zagrodny (1988) (which remains valid for this subdifferential see Thibault, 1994) on $[z_k, z'_k]$ we obtain sequences $z_{k,n} \to c_k \in]z_k, z'_k[$, $z^*_{k,n} \in \partial f(z_{k,n})$ such that

$$\liminf_n \langle z^*_{k,n}, z_k - z_{k,n}\rangle \geq [f(z_k) - f(z'_k)]\frac{\|c_k - z_k\|}{\|z'_k - z_k\|} \geq 0.$$

By Proposition 7.1 2) we have $z^*_{k,n} \in \partial_{(\gamma,C)} f(z_{k,n})$. Hence

$$f(x) - f(z_{k,n}) + C\|x - z_{k,n}\|^\gamma \geq \langle z^*_{k,n}, x - z_{k,n}\rangle$$

and

$$f(y_k) - f(z_{k,n}) + C\|y_k - z_{k,n}\|^\gamma \geq \langle z^*_{k,n}, y_k - z_{k,n}\rangle$$

which also implies by the lower semicontinuity of f that

$$\begin{aligned} &\lambda f(x) + (1-\lambda) f(y_k) + C[\lambda\|x - c_k\|^\gamma + (1-\lambda)\|y_k - c_k\|^\gamma] \\ &\geq \liminf_n [f(z_{k,n}) + \langle z^*_{k,n}, y_k - z_{k,n}\rangle] \\ &\geq f(c_k). \end{aligned}$$

Letting k go to ∞ we get

$$\lambda f(x) + (1-\lambda) f(y) + C(\lambda(1-\lambda)^\gamma + (1-\lambda)\lambda^\gamma)\|x - y\|^\gamma \geq f(\lambda x + (1-\lambda)y).$$

Step 2 : If z_k is a local minimum of f then $0 \in \partial f(z_k)$ and, by *ii)*, $0 \in \partial_{(\gamma,C)} f(z_k)$ Hence putting $c_k = z_k$ we obtain

$$f(c_k) \leq f(x) + C\|c_k - x\|^\gamma \text{ and } f(c_k) \leq f(y_k) + C\|c_k - y_k\|^\gamma$$

which implies in this case that

$$f(c_k) \leq \lambda f(x) + (1-\lambda) f(y_k) + C\lambda\|c_k - x\|^\gamma + C(1-\lambda)\|c_k - y_k\|^\gamma.$$

As $f(y_k) \to f(y)$ and $c_k \to z$, it follows from the lower semicontinuity of f that

$$f(z) \leq \lambda f(x) + (1-\lambda) f(y) + C(\lambda(1-\lambda)^\gamma + (1-\lambda)\lambda^\gamma)\|x - y\|^\gamma.$$

■

As a consequence of this theorem and Theorem 3.1 we obtain the following characterization of γ–paraconvex functions for $\gamma > 1$.

COROLLARY 7.1 *Let X and f be as in Theorem 7.1 and suppose that $\gamma > 1$. Then the following assertions are equivalent:*

i) f is γ–paraconvex;

ii) $\partial_c f$ is γ–monotone;

iii) there exists $C > 0$ such that for all $x, y \in X$ and $\lambda \in [0, 1]$

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) + C\lambda(1-\lambda)\|x - y\|^\gamma.$$

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