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Tangency Conditions for Multivalued Mappings

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Abstract. We prove that interiority conditions imply tangency conditions for two multivalued mappings from a topological space into a normed vector space. As a consequence, we obtain the lower semicontinuity of the intersection of two multivalued mappings. An application to the epi-upper semicontinuity of the sum of convex vector-valued mappings is given.

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1. Introduction

The important role that the lower semicontinuity properties of the intersection of some multivalued mappings plays in connection with optimization theory, is well known and widely recognized, for example in the study of stability and differentiability of parametrized problems where the feasible sets are given as intersections of multivalued mappings.

Several conditions have been proposed for guaranteeing the lower semicontinuity of the intersection of two multivalued mappings (see [2–5, 7, 9, 10, 16–21, 24, 26, 27] for instance).

In this paper, we discuss two such conditions called tangency conditions for multivalued mappings. We show that interiority conditions ensure tangency conditions for two multivalued mappings and, hence, the lower semicontinuity of their intersection. Our results improve some results of Lechicki–Spakowski [17] and Penot [24]. Our results are applied to produce epi-upper semicontinuity of the sum of convex vector-valued mappings.

2. Preliminaries

In this paper, X denotes a topological space and Y denotes a normed vector space equipped with the norm $\|\cdot\|$. We let Y^* denote the topological dual space of Y endowed with the weak-star topology. $\langle\cdot,\cdot\rangle$ denotes the pairing between

Y and Y^* and B_Y and B_Y^* denote the closed unit balls of Y and Y^*. $d(y,S)$ denotes the distance function of S, that is,

$$d(y,S) = \inf_{z\in S} \|y - z\|.$$

For a multivalued mapping F, its graph, $\operatorname{Gr} F$ is the set

$$\operatorname{Gr} F = \{(x,y)\colon\ y \in F(x)\}$$

and $\overline{F}(x) = \overline{F(x)}$, where $\overline{F(x)}$ is the closure of $F(x)$.

LEMMA 2.1. *If F: $X \rightrightarrows Y$ is a multivalued mapping, then*

$$\liminf_{x\to x_0} F(x) = \liminf_{x\to x_0} \overline{F}(x).$$

A multivalued mapping F: $X \rightrightarrows Y$ is said to be *convex-valued* near $x_0 \in X$ if $F(x)$ is convex for x near x_0.

LEMMA 2.2. *Let F, G: $X \rightrightarrows Y$ be two multivalued mappings which are convex-valued near x_0. Suppose that*

$$\begin{array}{l} \textit{there exist } \alpha > 0,\ s > 0 \textit{ and a neighbourhood } X_0 \textit{ of } x_0 \textit{ such that} \\ sB_Y \subset F(x) \cap \alpha B_Y - G(x) \cap \alpha B_Y \textit{ for all } x \in X_0. \end{array} \tag{2.1}$$

Then

$$\liminf_{x\to x_0} F(x) \cap G(x) = \liminf_{x\to x_0} \overline{F}(x) \cap \overline{G}(x).$$

Proof. Let $y \in \liminf_{x\to x_0} \overline{F}(x) \cap \overline{G}(x)$. Then, for all $\varepsilon > 0$, there exists a neighbourhood $X_1 \subset X_0$ of x_0 such that, for all $x \in X_1$, there exists $y_x \in \overline{F}(x) \cap \overline{G}(x)$ such that $\|y_x - y\| \leqslant \varepsilon/2$. So there exist $z_x \in F(x)$ and $v_x \in G(x)$ such that

$$\|z_x - y_x\| \leqslant \varepsilon/2 \quad \text{and} \quad \|v_x - y_x\| \leqslant \varepsilon/2.$$

Set $t_x = \|z_x - v_x\|$. By (2.1) there are $p_x \in F(x) \cap \alpha B_Y$ and $q_x \in G(x) \cap \alpha B_Y$ such that $s(v_x - z_x) = t_x(p_x - q_x)$. Set

$$w_x = (s + t_x)^{-1}(t_x p_x + s z_x) = (s + t_x)^{-1}(s v_x + t_x q_x).$$

Then, by convexity, $w_x \in F(x) \cap G(x)$. As $\lim_{x\to x_0} w_x = y$, one gets $y \in \liminf_{x\to x_0} F(x) \cap G(x)$. □

Condition (2.1) was considered in Penot [24] in order to show that

$$\liminf_{x\to x_0} F(x) \cap G(x) = \liminf_{x\to x_0} F(x) \cap \liminf_{x\to x_0} G(x).$$

If F and G are constant, i.e., $F(x) = C$ and $G(x) = D$ for all $x \in X$, then the result of this lemma can be formulated as follow: If C and D are convex and if there exist $\alpha > 0$ and $s > 0$ such that

$$sB_Y \subset C \cap \alpha B_Y - D \cap \alpha B_Y,$$

then $\overline{C \cap D} = \overline{C} \cap \overline{D}$.

In the sequel, the following lemma will be used.

LEMMA 2.3. *Let C be a nonempty convex subset of a finite-dimensional space Z. Suppose that there exists $s > 0$ such that for some $\varepsilon \in]0, s[$*

$$sB_Z \subset C + \varepsilon B_Z. \tag{2.2}$$

Then $(s - \varepsilon) \operatorname{int} B_Z \subset C$.

Proof. Suppose the contrary and let $z \in (s - \varepsilon) \operatorname{int} B_Z$ with $z \notin C$. Since Z is a finite-dimensional space, by a separation theorem, there exists $z^* \in Z^*$ with $\|z^*\| = 1$ such that

$$\sup_{u \in C} \langle z^*, u\rangle \leqslant \langle z^*, z\rangle.$$

By (2.2) we have, for all $v \in sB_Z$, the existence of $b \in B_Z$ such that $v - \varepsilon b \in C$ and so

$$\langle z^*, v - \varepsilon b\rangle \leqslant \langle z^*, z\rangle$$

which implies that

$$s - \varepsilon \leqslant \langle z^*, z\rangle < s - \varepsilon$$

and this contradiction completes the proof. □

Recall (see, for example, [2–4, 16–21, 24]) that a multivalued mapping F: $X \rightrightarrows Y$ is said to be *lower semicontinuous* (l.s.c.) at x_0 if, for each open subset V in Y with $F(x_0) \cap V \neq \emptyset$, $F^{-1}(V)$ is a neighbourhood of x_0 in X. Equivalently, F is l.s.c. at x_0 iff

$$F(x_0) \subset \liminf_{x \to x_0} F(x) \quad .$$

or, equivalently, for all $y_0 \in F(x_0)$, $\lim_{x \to x_0} d(y_0, F(x)) = 0$.

F is said to be *lower hemicontinuous* (l.h.c.) at x_0 if, for any $\varepsilon > 0$, there exists a neighbourhood W of x_0 such that, for each $x \in W$,

$$F(x_0) \subset F(x) + \varepsilon B_Y.$$

Following Penot [24], F is said to be *boundedly lower hemicontinuous* (b.l.h.c.) at x_0 if, for each $\beta > 0$ and $\varepsilon > 0$, there exists a neighbourhood W of x_0 such that for all $x \in W$

$$F(x_0) \cap \beta B_Y \subset F(x) + \varepsilon B_Y.$$

We refer the reader to the paper of Penot [24] for additional information concerning the connection between these notions of continuity.

According to Borwein and Théra [7], a multivalued mapping $F\colon X \rightrightarrows Y$ will be said to be *strongly lower semicontinuous* (s.l.s.c.) at x_0 if its values locally have a nonempty interior at x_0 and if for each $y_0 \in \operatorname{int} F(x_0)$, there exists some neighbourhood W of x_0 such that $y_0 \in \operatorname{int} F(x)$ for each $x \in W$.

PROPOSITION 2.4 ([7, 24]). *Let $F\colon X \rightrightarrows Y$ be a multivalued mapping which is l.s.c. and convex-valued near x_0. Suppose that $F(x_0)$ has a nonempty interior and that Y is finite-dimensional. Then F is s.l.s.c. at x_0.*

Proof. Let $y_0 \in \operatorname{int} F(x_0)$. Then there exists $s > 0$ such that

$$y_0 + sB_Y \subset F(x_0) \cap \alpha B_Y,$$

where $\alpha \geqslant \|y_0\| + s$. From Proposition 1.3 in [24], F is b.l.h.c. at x_0. So we have, for all $\varepsilon > 0$, the existence of a neighbourhood X_0 of x_0 such that, for all $x \in X_0$,

$$F(x_0) \cap \alpha B_Y \subset F(x) + \varepsilon B_Y.$$

So, for all $x \in X_0$,

$$sB_Y \subset F(x) - \{y_0\} + \varepsilon B_Y$$

and taking $\varepsilon = s/2$ Lemma 2.3 yields

$$\operatorname{int} \frac{s}{2} B_Y \subset F(x) - \{y_0\}. \qquad \square$$

To close this section, we give the following lemma.

LEMMA 2.5. *Let $S \subset Y$ be a closed convex set. Then for all $y \notin S$ and $y^* \in \partial d(y, S)$ $\|y^*\| = 1$, where $\partial f(x)$ denotes the subdifferential of f at x in the sense of convex analysis.*

Proof. See the proof of Proposition 1.5 in [14]. $\square$

3. Tangency Conditions under Interiority Conditions

Let C and D be two subsets of Y and let $y_0 \in C \cap D$. In his paper [9] (see also [23] and [25]), Dolecki has introduced the following tangency condition:

$$\begin{aligned}&\text{There exist } a > 0 \text{ and a neighbourhood } Y_0 \text{ of } y_0\\ &\text{such that, for all } y \in Y_0 \cap C,\ d(y, C \cap D) \leqslant \operatorname{ad}(y, D)\end{aligned}$$

to show that

$$T(C \cap D, y_0) = T(C, y_0) \cap T(D, y_0).$$

Here $T(C, y_0)$ denotes the lower tangent cone to C at y_0, i.e., $T(C, y_0) := \liminf_{t \to 0^+} t^{-1}(C - y_0)$.

Using his condition as a point of departure, we introduce similar concepts for multivalued mappings.

DEFINITION 3.1. Let F, G: $X \rightrightarrows Y$ be two multivalued mappings. We say that F and G satisfy the tangency condition at x_0 if, for all $y_0 \in \overline{F}(x_0) \cap \overline{G}(x_0)$, there exist $r > 0$, $a > 0$ and a neighbourhood W of x_0 such that

$$d(y, F(x) \cap G(x)) \leqslant ad(y, F(x))$$

for all $x \in W$ and $y \in y_0 + rB_Y$ with $(x, y) \in \operatorname{Gr} G$ and $d(y, F(x)) < r$.

We say that F and G satisfy the uniform tangency condition at x_0 if r, a and W are not depending on y_0.

We immediately have the following fact.

LEMMA 3.2. *Let F, G: $X \rightrightarrows Y$ be two multivalued mappings which satisfy the tangency condition at x_0. Then*

$$\liminf_{x \to x_0} F(x) \cap G(x) = \liminf_{x \to x_0} F(x) \cap \liminf_{x \to x_0} G(x).$$

Proof. The inclusion

$$\liminf_{x \to x_0} F(x) \cap G(x) \subset \liminf_{x \to x_0} F(x) \cap \liminf_{x \to x_0} G(x)$$

is obvious. Conversely, let $y_0 \in \liminf_{x \to x_0} F(x) \cap \liminf_{x \to x_0} G(x)$. On the one hand, $y_0 \in \overline{F}(x_0) \cap \overline{G}(x_0)$ and, by assumption, we have the existence of $r > 0$, $a > 0$ and a neighbourhood W of x_0 such that

$$d(y, F(x) \cap G(x)) \leqslant ad(y, F(x))$$

for all $x \in W$ and $y \in y_0 + rB_Y$ with $(x, y) \in \operatorname{Gr} G$ and $d(y, F(x)) < r$. On the other hand, one can find a neighbourhood $W_1 \subset W$ of x_0 and selections $x \to y_x$ and $x \to z_x$ of F and G, respectively, such that $y_0 = \lim_{x \to x_0} y_x = \lim_{x \to x_0} z_x$. Thus

$$d(z_x, F(x) \cap G(x)) \leqslant ad(z_x, F(x)) \leqslant a\|z_x - y_x\|$$

and, hence, there exists a selection $x \to w_x$ of $F(\cdot) \cap G(\cdot)$ such that $y_0 = \lim_{x \to x_0} w_x$ which implies that $y_0 \in \liminf_{x \to x_0} F(x) \cap G(x)$. □

THEOREM 3.3. *Let F and G be two multivalued mappings from X into Y with convex values near x_0. Suppose that* (2.1) *holds. Then F and G satisfy the tangency condition at x_0.*

If, in addition, $\overline{F}(x_0) \cap \overline{G}(x_0)$ is bounded, then F and G satisfy the uniform tangency condition at x_0.

Proof. Let $r > 0$ and $X_1 \subset X_0$ be a neighbourhood of x_0. Then, for all $y_0 \in \overline{F}(x_0) \cap \overline{G}(x_0)$, the set

$$A := \{(x, y) \in \operatorname{Gr} G \cap (X_1 \times (y_0 + rB_Y)) \colon d(y, F(x)) < r\}$$

is nonempty. So let $(x, y) \in A$, then, for all $\varepsilon \in]0, r - d(y, F(x))[$, there exists $u \in F(x)$ such that

$$\|y - u\| \leqslant d(y, F(x)) + \varepsilon.$$

If $y = u$, then there is nothing to prove. So suppose $y \neq u$ and set $t = \|y - u\|$. By assumption, there exists $v \in F(x) \cap \alpha B_Y$ and $w \in G(x) \cap \alpha B_Y$ such that

$$s(y - u) = t(v - w)$$

or equivalently

$$(s + t)^{-1}(sy + tw) = (s + t)^{-1}(su + tv).$$

Set $q = (s + t)^{-1}(su + tv)$. Then $q \in F(x) \cap G(x)$ and

$$\begin{aligned} \|y - q\| &= (s + t)^{-1}\|s(y - u) + t(y - v)\| \\ &\leqslant \|y - u\| + s^{-1}t\|y - v\| \\ &\leqslant d(y, F(x)) + \varepsilon + s^{-1}t(\|y_0\| + r + \alpha) \\ &\leqslant d(y, F(x)) + \varepsilon + s^{-1}(\|y_0\| + r + \alpha)(d(y, F(x)) + \varepsilon) \end{aligned}$$

and, hence, for all $\varepsilon \in]0, r - d(y, F(x))[$

$$d(y, F(x) \cap G(x)) \leqslant a(d(y, F(x)) + \varepsilon),$$

where $a = 1 + s^{-1}(\|y_0\| + r + \alpha)$ and, hence,

$$d(y, F(x) \cap G(x)) \leqslant \mathrm{ad}(y, F(x)).$$ □

As a first consequence of this theorem, we obtain the following corollary which plays an important role in subdifferential calculus rules.

COROLLARY 3.4. *Let C and D be two nonempty convex subsets of Y. Then the following statements are equivalent:*

(i) *there exist $s > 0$ and $\alpha > 0$ such that*

$$sB_Y \subset C \cap \alpha B_Y - D \cap \alpha B_Y,$$

(ii) *for all $y_0 \in \overline{C} \cap \overline{D}$, there exist $r > 0$ and $a > 0$ such that*

$$d(y, (C - z) \cap D) \leqslant \mathrm{ad}(y + z, C)$$

for all $y \in (y_0 + rB_X) \cap D$ and $z \in rB_Y$.

Proof. (ii) $\Rightarrow$ (i): let $z \in rB_Y$ that

$$d(y_0, (C - z) \cap D) \leqslant \text{a}d(y_0 + z, C).$$

Then, since $d(y_0+z, C) \leqslant \|z\|$, there exists $u \in (C-z)\cap D$ such that $\|y_0 - u\| \leqslant 2a\|z\| \leqslant 2ar$. So $z \in C - u$ with $u \in D \cap (y_0 + 2arB_Y)$. Set $s = r$ and $\alpha = \|y_0\| + r(2a+1)$. Then $sB_Y \subset C \cap \alpha B_Y - D \cap \alpha B_Y$.

For the inverse implication, it suffices to apply Theorem 3.3 by setting $F(z) = C - z$ and $G(z) = D$. □

Note that in the literature, Corollary 3.4 is established in the case where C and D are closed and Y is a Banach space.

Other conditions ensuring (ii) in the nonconvex case are given in, for example, [6, 12–15, 23]. More generally, we have

COROLLARY 3.5. *Let all the hypotheses of Theorem* 3.3 *be satisfied. Then, for all $y_0 \in F(x_0) \cap G(x_0)$, there exist a neighbourhood X_1 of x_0, $a > 0$ and $r > 0$ such that*

$$d(y, (F(x) - z) \cap G(x)) \leqslant \text{a}d(y + z, F(x)) \tag{3.1}$$

for all $x \in X_1$, $y \in y_0 + rB_Y$ and $z \in rB_Y$ with $(x, y) \in \operatorname{Gr} G$ and $d(y, F(x)) < r$. Conversely, if F and G are l.h.c. at x_0, then (3.1) *implies* (2.1).

Proof. Endow $X \times Y$ with the product topology and set $F_1(x, y) = F(x) - y$ and $G_1(x, y) = G(x)$. We easily show that (2.1) is fulfilled for F_1 and G_1 and Theorem 3.3 completes the first part of the corollary. For the second part we have, by (3.1) and the l.h.c. of F and G, that for all $\varepsilon \in]0, r/2[$ the existence of a neighbourhood $W \subset X_1$ of x_0 such that for each $x \in W$ and $z \in r/2B_Y$ there exist $y \in G(x)$ and $u \in F(x)$ such that $\|y - y_0\| < \varepsilon$, $\|u - y_0\| < \varepsilon$ and

$$d(y, (F(x) - z) \cap G(x)) \leqslant a\|y + z - u\|.$$

Thus, for each $x \in W$ and $z \in r/2B_Y$, there exists $w \in (F(x) - z) \cap G(x)$ such that $w \in y_0 + (1 + 2a)rB_Y$ and, hence,

$$z \in F(x) \cap \alpha B_Y - G(x) \cap \alpha B_Y,$$

where $\alpha = 2(\|y_0\| + (1 + 2a)r)$. □

The second consequence of Theorem 3.3 and Lemma 3.2 is the following result.

COROLLARY 3.6 ([24]). *Let F, G: $X \rightrightarrows Y$ be two multivalued mappings with convex values near x_0. Suppose that* (2.1) *holds. Then*

$$\liminf_{x \to x_0} F(x) \cap G(x) = \liminf_{x \to x_0} F(x) \cap \liminf_{x \to x_0} G(x).$$

4. Tangency Conditions under Approximate Interiority Conditions

The approximate interiority condition that we introduce here is the following:

$$\begin{aligned}&\text{there exists } s > 0 \text{ such that for all } \varepsilon > 0 \text{ there exist } \alpha > 0\\ &\text{and a neighbourhood } X_0 \text{ of } x_0 \text{ such that}\\ &sB_Y \subset F(x) \cap \alpha B_Y - G(x) \cap \alpha B_Y + \varepsilon B_Y\\ &\text{for each } x \in X_0.\end{aligned} \tag{4.1}$$

It is clear that (2.1) implies (4.1) and this implication can be strict. For example, let C be a proper subspace which is dense in Y and take $F(x) = C$ and $G(x) = \{0\}$ for all $x \in X$. In general, (4.1) (with local convexity of the values of F and G) does not ensure the tangency conditions. To see this, take $F(x) = C$ and $G(x) = \{0\}$ for all $x \in X$.

When Y is a Banach space, using the Baire lemma, we can show that (2.1) and (4.1) are equivalent whenever the values of F and G are closed and convex. In our case, Y is a normed vector space and so we cannot apply the Baire lemma to prove this equivalence. But in this section we show that the completeness of Y can be replaced by the completeness of the values of G.

LEMMA 4.1. *Let C and D be two closed and convex subsets of Y with D complete and bounded (or D is complete and C is bounded). Then the following conditions are equivalent:*

(i) $0 \in \operatorname{int} \overline{(C - D)}$,
(ii) *for all $y_0 \in C \cap D$, there exist $r > 0$ and $a > 0$ such that*

$$d(y, (C - z) \cap D) \leqslant a d(y + z, C)$$

for all $y \in (y_0 + rB_X) \cap D$ and $z \in rB_Y$, and
(iii) $0 \in \operatorname{int}(C - D)$.

Proof. The implication (iii) $\Rightarrow$ (i) is obvious and by Corollary 3.4 we have the equivalence between (ii) and (iii). So we have to show that (i) $\Rightarrow$ (ii). Suppose the contrapositive. Then there exists $y_0 \in C \cap D$ such that for all integer n, there exists y_n and z_n such that

$$d(y_n, (C - z_n) \cap D) > n d(y_n + z_n, C), \tag{4.2}$$

$$\|y_n - y_0\| \leqslant 1/n \quad \text{and} \quad \|z_n\| \leqslant 1/n \quad \text{with}$$
$$y_n \in D \quad \text{and} \quad d(y_n + z_n, C) \leqslant 1/n.$$

Consider the function $f_n(y) = d(y + z_n, C)$ and set $\varepsilon_n^2 = f_n(y_n)$ and $\lambda_n = \min(n\varepsilon_n^2, \varepsilon_n)$. As C is closed and convex and, by (4.2), $y_n + z_n \notin C$, it follows that f_n is convex and $\varepsilon_n^2 > 0$ with $\varepsilon_n \to 0^+$. Since

$$f_n(y_n) \leqslant \inf_{y \in D} f_n(y) + \varepsilon_n^2,$$

then, by Ekeland's variational principle [11], there exists $y'_n \in D$ such that

$$\|y_n - y'_n\| < \lambda_n \tag{4.3}$$

and

$$f_n(y'_n) \leqslant f_n(y) + s_n\|y - y'_n\|, \quad \text{for all } y \in D,$$

where $s_n = \varepsilon_n^2/\lambda_n = \max(1/n, \varepsilon_n) = 1/n$. Thus, since the function $y \to f_n(y) + \|y - y_n\|$ is Lipschitz, it follows that

$$d(y'_n + z_n, C) \leqslant d(y + z_n, C) + 2d(y, D) + s_n\|y - y'_n\|$$

for all y in some neighbourhood of y'_n. So

$$0 \in \partial d(\cdot + z_n, C)(y'_n) + 2\partial d(\cdot, D)(y'_n) + s_n B_Y^*$$

and, hence, there exist $y_n^* \in \partial d(\cdot + z_n, C)(y'_n)$, $z_n^* \in 2\partial d(\cdot, D)(y'_n)$ and $b_n^* \in B_Y^*$ such that $y_n^* + z_n^* = s_n b_n^*$. From (i) there exists $s > 0$ such that for all $\varepsilon > 0$ for all $y \in sB_Y$ we have the existence of $v_n \in C$, $w_n \in D$ and $b_n \in B_Y$ such that $y = v_n - w_n + \varepsilon b_n$ and hence

$$\begin{aligned} &s_n\langle b_n^*, v_n - y - y'_n + \varepsilon b_n\rangle \\ &\quad = \langle y_n^* + z_n^*, v_n - y - y'_n + \varepsilon b_n\rangle \\ &\quad = \langle y_n^*, v_n - y'_n\rangle - \langle y_n^*, y\rangle + \varepsilon\langle y_n^*, b_n\rangle + \langle z_n^*, v_n - y'_n - y + \varepsilon b_n\rangle. \end{aligned}$$

Since D is bounded there exists $\alpha > 0$ such that

$$\|w_n\| \leqslant \alpha, \quad \forall n,$$

$\langle y_n^*, v_n - y'_n\rangle \leqslant 0$ and $\langle z_n^*, v_n - y'_n - y + \varepsilon b_n\rangle \leqslant 0$, we get

$$\langle y_n^*, y\rangle \leqslant 2s_n(\alpha + s + \|y_0\| + 1 + \varepsilon) + \varepsilon$$

for all $y \in sB_Y$ and, hence, for all integer n

$$\|v_n\| = \|y + w_n - \varepsilon b_n\| \leqslant s + \alpha + \varepsilon.$$

Thus,

$$s\|y_n^*\| \leqslant 2s_n(\alpha + s + \|y_0\| + 1 + \varepsilon) + \varepsilon.$$

Note that by (4.2) and (4.3), $y'_n + z_n \notin C$ and by Lemma 2.5 $1 = \|y_n^*\|$. So

$$s \leqslant 2s_n(\alpha + s + \|y_0\| + 1 + \varepsilon) + \varepsilon$$

and since $s_n \to 0^+$ we have, for all $\varepsilon > 0$, $s \leqslant 2\varepsilon$ and this contradiction completes the proof. □

The lemma states in particular that

$$\operatorname{int}\overline{(C - D)} = \operatorname{int}(C - D).$$

The following example shows that the assumption on the boundedness of D (or C) cannot be relaxed in the above lemma.

EXAMPLE 4.2 ([8]). Let $Y = l^2$ be the Hilbert space of square summable sequences and let $v \in Y$, $v_k > 0$ for all integer k. Set

$$C = \mathbb{R}v \quad \text{and} \quad D = \{y \in Y\colon\ y_k \geqslant 0 \text{ for all } k\}.$$

Then $0 \in \operatorname{int}\overline{(C - D)}$ and $\operatorname{int}(C - D) = \emptyset$.

Using Lemma 4.1, we obtain

LEMMA 4.3. *Let $F, G\colon\ X \rightrightarrows Y$ be two closed convex-valued multivalued mappings near x_0. Suppose that the values of G are complete. Then (2.1) and (4.1) are equivalent.*

Proof. By (4.1) there exists $s > 0$ such that for all $\varepsilon \in]0, s[$ there exist $\alpha > 0$ and a neighbourhood X_0 of x_0 such that

$$sB_Y \subset \overline{F(x) \cap \alpha B_Y - G(x) \cap \alpha B_Y} + \varepsilon B_Y$$

for all $x \in X_0$ and by the cancellation law theorem [28], we have

$$(s - \varepsilon)B_Y \subset \overline{F(x) \cap \alpha B_Y - G(x) \cap \alpha B_Y}.$$

Using Lemma 4.1, we deduce that

$$\operatorname{int}\overline{F(x) \cap \alpha B_Y - G(x) \cap \alpha B_Y} = \operatorname{int}(F(x) \cap \alpha B_Y - G(x) \cap \alpha B_Y).$$

Therefore

$$\frac{(s - \varepsilon)}{2} B_Y \subset F(x) \cap \alpha B_Y - G(x) \cap \alpha B_Y. \qquad \square$$

With the help of this lemma and Theorem 3.1, we obtain the following theorem.

THEOREM 4.4. *Let $F, G\colon\ X \rightrightarrows Y$ be two multivalued mappings with closed-convex values near x_0. Suppose that the values of G are complete. Suppose also that (4.1) holds. Then F and G satisfy the tangency condition at x_0.*

If, in addition, $F(x_0) \cap G(x_0)$ is bounded, then F and G satisfy the uniform tangency condition at x_0.

COROLLARY 4.5. *Let $F, G\colon\ X \rightrightarrows Y$ be two multivalued mappings with closed convex values near x_0. Suppose that the values of G are complete. Suppose also that F and G are l.h.c. at x_0 and there exist $s > 0$ and $\alpha > 0$ such that*

$$sB_Y \subset F(x_0) \cap \alpha B_Y - G(x_0) \cap \alpha B_Y.$$

Then $F(\cdot) \cap G(\cdot)$ is l.s.c. at x_0.

Proof. This is immediate from Theorem 4.4 and Lemma 2.3. $\square$

As a consequence of this corollary, we obtain the following.

COROLLARY 4.6. *Let F and G be as in Theorem 4.4. Suppose that F and G are l.h.c. at x_0 and*

$$G(x_0) \cap \operatorname{int} F(x_0) \neq \emptyset.$$

Then $F(\cdot) \cap G(\cdot)$ is l.s.c. at x_0.

The assumptions of Corollary 4.6 are not sufficient to guarantee the l.h.c. of $F(\cdot) \cap G(\cdot)$ at x_0.

EXAMPLE 4.7 ([17]). Let $Y = \mathrm{L}^\infty$ and define F, G: $[0,1] \rightrightarrows Y$ as follows:

$$F(x) = \{(t_k) \in Y\colon\ t_1 \geqslant x \text{ and } t_k \leqslant k - x \text{ for } k \geqslant 2\}$$

and

$$\begin{aligned} G(x) \;=\; &\{(t_k) \in Y\colon\ t_1 \leqslant 1 - x \text{ and } t_k \leqslant k(1 - t_1 - x), \\ &t_k \leqslant k + t_1/k - x/k \text{ for } k \geqslant 2\}. \end{aligned}$$

Then F and G are closed convex-valued. Moreover, they are l.h.c. at 0 and $G(0) \cap \operatorname{int} F(0) \neq \emptyset$. However, $F(\cdot) \cap G(\cdot)$ is not l.h.c. at 0.

In their paper [17], Lechicki and Spakowski have shown that, in addition to the assumptions of Corollary 4.6, the following assumptions:

$$\operatorname{int}[F(x_0) \cap G(x_0)] \neq \emptyset \quad \text{and} \quad F(x_0) \cap G(x_0) \quad \text{is bounded} \tag{4.4}$$

ensure the l.h.c. of $F(\cdot) \cap G(\cdot)$ at x_0.

The purpose of the following corollary is to relax their interiority assumption (4.4).

COROLLARY 4.8. *Let F, G: $X \rightrightarrows Y$ be two multivalued mappings with closed convex values near x_0. Suppose that the values of G are complete and $F(x_0) \cap G(x_0)$ is bounded. Suppose also that F and G are l.h.c. at x_0 and there exist $s > 0$ and $\alpha > 0$ such that*

$$sB_Y \subset F(x_0) \cap \alpha B_Y - G(x_0) \cap \alpha B_Y.$$

Then $F(\cdot) \cap G(\cdot)$ is l.h.c.at x_0.

Proof. We can easily show that all the assumptions of Theorem 4.4 are satisfied. So there exist $r > 0$, $a > 0$ and a neighbourhood W_1 of x_0 such that for all $y_0 \in F(x_0) \cap G(x_0)$

$$d(y, F(x) \cap G(x)) \leqslant \mathrm{a}d(y, F(x)) \tag{4.5}$$

for all $x \in W_1$, $y \in y_0 + rB_Y$ with $(x, y) \in \operatorname{Gr} G$ and $d(y, F(x)) < r$. Let $\varepsilon \in]0, r/2[$ and let $W \subset W_1$ be a neighbourhood of x_0 such that for all $x \in W$

$$F(x_0) \subset F(x) + \varepsilon B_Y \quad \text{and} \quad G(x_0) \subset G(x) + \varepsilon B_Y.$$

So, for all $x \in W$, there exists $y \in G(x)$ such that $y \in y_0 + \varepsilon B_Y$. For this y, we have $d(y, F(x)) < 2\varepsilon$ and hence, by (4.5), there exists $z \in F(x) \cap G(x)$ such that

$$\|y - z\| \leqslant 2a\varepsilon.$$

Thus $y_0 \in F(x) \cap G(x) + (2a + 1)\varepsilon B_Y$ and, hence, for all $\varepsilon > 0$ there exists a neighbourhood W of x_0 such that for all $x \in W$

$$F(x_0) \cap G(x_0) \subset F(x) \cap G(x) + (2a + 1)\varepsilon B_Y. \qquad \square$$

We may also state the following corollary.

COROLLARY 4.9 ([24]). *Let F, G: $X \rightrightarrows Y$ be two multivalued mappings with closed convex values near x_0. Suppose that the values of G are complete. Suppose also that F and G are b.l.h.c. at x_0 and there exist $s > 0$ and $\alpha > 0$ such that*

$$sB_Y \subset F(x_0) \cap \alpha B_Y - G(x_0) \cap \alpha B_Y.$$

Then $F(\cdot) \cap G(\cdot)$ is b.l.h.c. at x_0.

Proof. Let $\beta \geqslant \alpha$. It suffices to apply Corollary 4.8, by considering the following multivalued mappings

$$F_\beta(x) = \begin{cases} F(x_0) \cap \beta B_Y & \text{if } x = x_0, \\ F(x) & \text{otherwise} \end{cases}$$

and

$$G_\beta(x) = \begin{cases} G(x_0) \cap \beta B_Y & \text{if } x = x_0, \\ G(x) & \text{otherwise.} \end{cases} \qquad \square$$

5. Application to the Epi-Upper Semicontinuity of a Sum

Let Z be a normed vector space ordered by a closed convex cone Z_+. We denote by $Z^{\cdot} = Z \cup \{+\infty\}$ the set obtained by adding to Z a greatest element $+\infty$. Given a mapping f: $Y \to Z^{\cdot}$, we denote [24] by

$$E(f) := \{(y, z) \in Y \times Z\colon\ z \in f(y) + Z_+\}$$

its epigraph; f is said to be convex if $E(f)$ is convex. The extended level set [24] of f associated to $\alpha \in \mathbb{R}_+$ is

$$T(f, \alpha) = f^{-1}(\alpha B_Z - Z_+).$$

In the following definition, Penot [24] extended a well-known notion of epi-upper semicontinuity (see [1] and [26] and their references) to a vectorial framework.

DEFINITION 5.1 ([24]). A family $(f_x)_{x \in X}$ of mappings from Y into $Z^{\cdot}$ parametrized by X is said to be epi-upper semicontinuous (e.-u.s.c.) at $x_0 \in X$ if

$$E(f_{x_0}) \subset \liminf_{x \to x_0} E(f_x).$$

Let $(f_x)_{x \in X}$ and $(g_x)_{x \in X}$ be two families of convex mappings from Y into $Z^{\cdot}$ and let $(h_x)_{x \in X}$ be given by $h_x = f_x + g_x$.

Using the assumption

$$\begin{array}{l} \text{there exist } \alpha > 0,\ s > 0 \text{ and a neighbourhood } X_0 \text{ of } x_0 \text{ such that} \\ sB_Y \subset T(f_x, \alpha) \cap \alpha B_Y - T(g_x, \alpha) \cap \alpha B_Y, \text{ for all } x \in X_0 \end{array} \tag{5.1}$$

Penot [24] showed that $(h_x)_{x \in X}$ is e.-u.s.c. at $x_0 \in X$ whenever $(f_x)_{x \in X}$ and $(g_x)_{x \in X}$ are e.-u.s.c. at x_0. Here we use his assumption (5.1) to give an analytic content in the form of an inequality between the families of mappings $(f_x)_{x \in X}$, $(g_x)_{x \in X}$ and $(h_x)_{x \in X}$.

THEOREM 5.2. *Let $(f_x)_{x \in X}$ and $(g_x)_{x \in X}$ be two families of convex mappings from Y into $Z^{\cdot}$. Suppose that (5.1) holds. Then, for all $(y_0, z_0) \in E(h_{x_0})$, there exist $a > 0$, $r > 0$ and a neighbourhood X_0 of x_0 such that*

$$d((y, z + v), E(h_x)) \leqslant \mathrm{a}d((y, z), E(f_x))$$

for all $x \in X_0$, $y \in y_0 + rB_Y$, $z \in z_0 - g_{x_0}(y_0) + rB_Z$ and $v \in g_{x_0}(y_0) + rB_Z$ with $(y, v) \in E(g_x)$ and $d(y, z, E(f_x)) < r$.

Consider the multivalued mappings F: $X \rightrightarrows Y \times Z \times Z$ and G: $X \rightrightarrows Y \times Z \times Z$ defined by

$$F(x) = \{(y, z, v) \in Y \times Z \times Z \colon\ z \in f_x(y) + Z_+\}$$

and

$$G(x) = \{(y, z, v) \in Y \times Z \times Z \colon\ v \in g_x(y) + Z_+\}.$$

For the rest, we endow $Y \times Z \times Z$ with the sum norm.

We have the following criterion which shows that F and G satisfy (2.1).

LEMMA 5.3. *If (5.1) holds then there exists $\beta > 0$ and a neighbourhood X_0 of x_0 such that*

$$sB_{Y \times Z \times Z} \subset F(x) \cap \beta B_{Y \times Z \times Z} - G(x) \cap \beta B_{Y \times Z \times Z}$$

for all $x \in X_0$.

Proof. Here we use the arguments by Penot [24]. Let $(y,z,v) \in sB_{Y\times Z\times Z}$ and let $x \in X_0$. By (5.1) there exist $y_x \in T(f_x,\alpha)\cap\alpha B_Y$, $y'_x \in T(g_x,\alpha)\cap\alpha B_Y$ with $y = y_x - y'_x$. By definition of level sets, there exist $u_x, u'_x \in \alpha B_Z$ such that $u_x \in f_x(y_x) + Z_+$ and $u'_x \in g_x(y'_x) + Z_+$. Set

$$z_x = u_x + (z + u'_x - u_x)^+, \qquad z'_x = u'_x + (z + u'_x - u_x)^-$$

and

$$v_x = u_x + (v + u'_x - u_x)^+, \qquad v'_x = u'_x + (v + u'_x - u_x)^-,$$

where $w^+ = \max(0,w)$, $w^- = (-w)^+$ so $w = w^+ - w^-$. Then

$$(y,z,v) = (y_x, z_x, v_x) - (y'_x, z'_x, v'_x)$$

with

$$(y_x, z_x, v_x) \in F(x) \cap (3\alpha + s)B_{Y\times Z\times Z}$$

and

$$(y'_x, z'_x, v'_x) \in G(x) \cap (3\alpha + s)B_{Y\times Z\times Z}. \qquad \square$$

Proof of Theorem 5.2. First note that $(y_0, z_0) \in E(h_{x_0})$ iff $(y_0, z_0 - g_{x_0}(y_0), g_{x_0}(y_0)) \in F(x_0) \cap G(x_0)$. Then by Lemma 5.3 and Theorem 3.3, there exist $a > 0$, $r > 0$ and a neighbourhood X_0 of x_0 such that

$$d(y,z,v,F(x)\cap G(x)) \leqslant a d(y,z,v,F(x))$$

for all $x \in X_0$, $y \in y_0 + rB_Y$, $z \in z_0 - g_{x_0}(y_0) + rB_Z$ and $v \in g_{x_0}(y_0) + rB_Z$ with $(x,y,z,v) \in \operatorname{Gr} G$ and $d(y,z,v,F(x)) < r$. So the proof is complete if we see that $d(y,z,v,F(x)) = d(y,z,E(f_x))$ and

$$\begin{aligned}
&d(y,z,v,F(x)\cap G(x))\\
&\quad= \inf_{\substack{f_x(y')\leqslant z'\\ g_x(y')\leqslant v'}} \|y-y'\| + \|z-z'\| + \|v-v'\|\\
&\quad\geqslant \inf_{\substack{f_x(y')\leqslant z'\\ g_x(y')\leqslant v'}} \|y-y'\| + \|z+v-z'-v'\|\\
&\quad\geqslant \inf_{f_x(y')+g_x(y')\leqslant z'+v'} \|y-y'\| + \|z+v-z'-v'\|\\
&\quad\geqslant \inf_{f_x(y')+g_x(y')\leqslant w'} \|y-y'\| + \|z+v-w'\|\\
&\quad= d(y, z+v, E(h_x)).
\end{aligned} \qquad \square$$

Remark. We can also give an analytic content in the form of inequality between the families of mappings $(f_x)_{x\in X}$, $(g_x)_{x\in X}$ and $(k_x)_{x\in X}$ with $k_x = \max(f_x, g_x)$

as follows: Let $(f_x)_{x\in X}$ and $(g_x)_{x\in X}$ be two families of convex mappings from Y into $Z^\cdot$. Suppose that (5.1) holds. Then for all $(y_0, z_0) \in E(h_{x_0})$, there exist $a > 0$, $r > 0$ and a neighbourhood X_0 of x_0 such that

$$d(y, z, E(k_x)) \leqslant \mathrm{a}d(y, z, E(f_x))$$

for all $x \in X_0$, $y \in y_0 + rB_Y$ and $z \in z_0 + rB_Z$ with $(y, z) \in E(g_x)$ and $d(y, z, E(f_x)) < r$.

COROLLARY 5.4. *Let $(f_x)_{x\in X}$ and $(g_x)_{x\in X}$ be as in Theorem* 5.2. *Suppose that* (5.1) *holds. Then $(h_x)_{x\in X}$ is e.-u.s.c. at x_0 whenever $(f_x)_{x\in X}$ and $(g_x)_{x\in X}$ are e.-u.s.c. at x_0.*

Proof. Let $(y_0, z_0) \in E(h_{x_0})$. Then $(y_0, z_0 - g_{x_0}(y_0)) \in E(f_{x_0})$ and $(y_0, g_{x_0}(y_0)) \in E(g_{x_0})$. By Theorem 5.2, there exist $a > 0$, $r > 0$ and a neighbourhood X_0 of x_0 such that

$$d(y, z + v, E(h_x)) \leqslant \mathrm{a}d(y, z, E(f_x)) \tag{5.2}$$

for all $x \in X_0$, $y \in y_0 + rB_Y$, $z \in z_0 - g_{x_0}(y_0) + rB_Z$ and $v \in g_{x_0}(y_0) + rB_Z$ with $(y, v) \in E(g_x)$ and $d(y, z, E(f_x)) < r$. So, by the e.-u.s.c. of $(f_x)_{x\in X}$ and $(g_x)_{x\in X}$ we can find a neighbourhood $X_1 \subset X_0$ of x_0 and selections $x \to (y_x, z_x)$ and $x \to (u_x, v_x)$ with $\lim_{x\to x_0} y_x = \lim_{x\to x_0} u_x = y_0$, $\lim_{x\to x_0} z_x = z_0 - g_{x_0}(y_0)$ and $\lim_{x\to x_0} v_x = g_{x_0}(y_0)$ such that $(y_x, z_x) \in E(f_x)$ and $(u_x, v_x) \in E(g_x)$. Then, by (5.2), there exists $(u'_x, v'_x) \in E(h_x)$ such that

$$\|u_x - u'_x\| + \|v_x + z_x - v'_x\| \leqslant a\|u_x - y_x\|$$

and, hence, $\lim_{x\to x_0} u'_x = y_0$ and $\lim_{x\to x_0} v'_x = z_0$. Whence $(y_0, z_0) \in \liminf_{x\to x_0} E(h_{x_0})$. □

Remark. Note that this result may be obtained by using Corollary 3.6.

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