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VARIATIONAL SUM OF SUBDIFFERENTIALS OF CONVEX FUNCTIONS

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ABSTRACT

Our goal in this note is to give a simple proof of the result by Attouch-Baillon-Théra concerning variational sum of subdifferential of convex functions. We extend their result from Hilbert spaces to reflexive Banach spaces. The obtained results are applied to derive chain rules for subdifferential of convex functions and to estimate the subdifferential of the marginal function without qualification conditions.

1. Introduction

Let X be a Banach space and X^* be its topological dual. We denote by B_X (respectively B_{X^*}) the closed unit ball of X (respectively X^*). The domain of a function $f : X \to \mathbf{R} \cup \{+\infty\}$ is the set

$$\operatorname{dom} f = \{x \in X : f(x) < +\infty\}.$$

and f is said to be propre if $\operatorname{dom} f \neq \emptyset$. The pairing between X and X^* is denoted by $< .,. >$. The regularized function of Moreau-Yoshida of f is the function $f_\lambda : X \to \mathbf{R}$ defined by

$$f_\lambda(x) = \inf\{f(u) + \frac{1}{2\lambda}\|x - u\|^2 : u \in X\}$$

where $\lambda > 0$. The variational sum of two subdifferentials of convex functions f and $g : X \to \mathbf{R} \cup \{+\infty\}$ is the set

$$\partial f \underset{v}{+} \partial g = \|.\| - \liminf_{\substack{\lambda \to 0^+ \\ \mu \to 0^+}} \partial f_\lambda + \partial g_\mu$$

i.e., $(x, x^*) \in \partial f \underset{v}{+} \partial g$ iff for all sequences $\lambda_n \to 0^+, \mu_n \to 0^+$ there exist sequences (x_n) and (x_n^*) such that

$$x_n^* \in \partial f_{\lambda_n}(x_n) + \partial g_{\mu_n}(x_n), \ \|x_n - x\| \to 0, \text{ and } \|x_n^* - x^*\| \to 0.$$

Here $\partial f(x)$ denotes the subdifferential in the sense of convex analysis of f at x.

Our goal in this note is to extend to reflexive Banach spaces and to give a simple proof of the result by Attouch-Baillon-Théra [1] which is established in the case of Hilbert spaces.

Theorem 1.1 *Let $f, g : X \to \mathbf{R} \cup \{+\infty\}$ be two lower semicontinuous convex propre functions such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. *Suppose that X is a Hilbert space. Then*

$$\partial f \underset{v}{+} \partial g = \|.\| - \limsup_{\substack{\lambda \to 0^+ \\ \mu \to 0^+}} \partial f_\lambda + \partial g_\mu = \partial(f+g).$$

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Theorem 2.2 *Suppose in Theorem 1.1 that X is reflexive instead of Hilbert. Then*

$$\partial f \underset{v}{+} \partial g = \|.\| - \limsup_{\substack{\lambda \to 0^+ \\ \mu \to 0^+}} \partial f_\lambda + \partial g_\mu = \|.\| - \lim_{\lambda \to 0^+} \partial f_\lambda + \partial g = \partial(f+g)$$

The proof of this theorem is based on the Ekeland's variational principle [2] and the following lemma whose proof is very simple to obtain. Note that our proof also works in the nonconvex case.

Lemma 2.1 *Let $r > 0$. Under assumptions of Theorem 2.2, the following assertions are equivalent:*

1.

$$x^* \in \partial(f+g)(\bar{x});$$

2.

$$\forall \varepsilon > 0 \quad \exists \eta > 0 \quad \forall \lambda, \mu \in]0, \eta[$$

$$f_\lambda(x) + g_\mu(x) - f_\lambda(\bar{x}) - g_\mu(\bar{x}) - \langle x^*, x - \bar{x} \rangle + \varepsilon \geq 0 \quad \forall x \in \bar{x} + rB_X;$$

3.

$$\forall \varepsilon > 0 \, \exists \lambda, \mu \in]0, \varepsilon[\text{ such that}$$

$$f_\lambda(x) + g_\mu(x) - f_\lambda(\bar{x}) - g_\mu(\bar{x}) - \langle x^*, x - \bar{x} \rangle + \varepsilon \geq 0, \forall x \in \bar{x} + rB_X;$$

4.

$$\forall \varepsilon > 0 \, \exists \eta > 0 \, \forall \lambda \in]0, \eta[$$

$$f_\lambda(x) + g(x) - f_\lambda(\bar{x}) - g(\bar{x}) - < x^*, x - \bar{x} > + \varepsilon \geq 0 \, \forall x \in \bar{x} + rB_X$$

5.

$$\forall \varepsilon > 0 \, \exists \lambda \in]0, \varepsilon[\text{ such that}$$

$$f_\lambda(x) + g(x) - f_\lambda(\bar{x}) - g(\bar{x}) - \langle x^*, x - \bar{x} \rangle + \varepsilon \geq 0, \forall x \in \bar{x} + rB_X.$$

Proof. It suffices to show that (1) is equivalent to (2). The other equivalences may be obtained similary.

(1) $\Longrightarrow$ (2) : Suppose that (2) is false. Then there exist $\varepsilon > 0$ and sequences $\lambda_n \to 0^+$, $\mu_n \to 0^+$ and $(x_n)_n \subset \bar{x} + rB_X$ such that

$$f_{\lambda_n}(x_n) + g_{\mu_n}(x_n) - f_{\lambda_n}(\bar{x}) - g_{\mu_n}(\bar{x}) - \langle x^*, x_n - \bar{x} \rangle + \varepsilon < 0.$$

Since X is reflexif, B_X is sequentially weakly compact, and hence, extracting subsequence, we may assume that the sequence $(x_n)_n$ weakly converges to some x in $\bar{x} + rB_X$. So using the Mosco-convergence of (f_{λ_n}) to f and (g_{μ_n}) to g we get

$$f(x) + g(x) - f(\bar{x}) - g(\bar{x}) - \langle x^*, x - \bar{x} \rangle + \varepsilon \leq 0$$

and this contradicts (1).

(2) $\Longrightarrow$ (1) : This implications comes from the fact that $(f_\lambda(x))$ goes to $f(x)$ as λ goes to 0^+. ∎

Proof of Theorem 2.2 It suffices to show that

$$\|.\| - \limsup_{\substack{\lambda \to 0^+ \\ \mu \to 0^+}} \partial f_\lambda + \partial g_\mu \subset \partial(f+g) \subset \partial f \underset{v}{+} \partial g.$$

The other inclusions may be obtained in a similar way.

Let $(x, x^*) \in \|.\| - \limsup_{\substack{\lambda \to 0^+ \\ \mu \to 0^+}} \partial f_\lambda + \partial g_\mu$. Then there are sequences $\lambda_n \to 0^+$, $\mu_n \to 0^+$, $x_n \to x$ and $x_n^* \to x^*$ (in norm) such that, for n sufficiently large

$$x_n^* \in \partial f_{\lambda_n}(x_n) + \partial g_{\mu_n}(x_n).$$

So for all $x \in X$

$$\langle x_n^*, x - x_n \rangle \le f_{\lambda_n}(x) + g_{\mu_n}(x) - f_{\lambda_n}(x_n) - g_{\mu_n}(x_n)$$

and using the fact that (f_{λ_n}) and (g_{μ_n}) converge and epi-converge to f and g repectively, and the norm-convergence of (x_n^*) and (x_n) to x^* and x, we get $x^* \in \partial(f+g)(x)$, whence the first inclusion.

For the second one, we let $r > 0$ and $(x, x^*) \in \partial(f+g)$. Then for all $u \in x + rB_X$

$$f(u) + g(u) - f(x) - g(x) - \langle x^*, u - x \rangle \ge 0.$$

By Lemma 2.1, we have for all $\varepsilon \in]0, \frac{r^2}{2}[$ there exists $\eta > 0$ such that for all $\lambda, \mu \in]0, \eta[$ and $u \in x + rB_X$

$$f_\lambda(u) + g_\mu(u) - f_\lambda(x) - g_\mu(x) - \langle x^*, u - x \rangle + \varepsilon \ge 0,$$

or equivalently

$$f_\lambda(x) + g_\mu(x) \le \inf_{u \in x + rB_X} (f_\lambda(u) + g_\mu(u) - \langle x^*, u - x \rangle) + \varepsilon.$$

Thus, by Ekeland variational principle[2], there exists $x_\varepsilon := x_{\varepsilon,\lambda,\mu}$ such that $x_\varepsilon \in x + rB_X$

$$\|x - x_{\|}\varepsilon < \sqrt{\varepsilon}$$

and

$$f_\lambda(x_\varepsilon) + g_\mu(x_\varepsilon) + \langle x^*, u - x_\varepsilon \rangle \le f_\lambda(u) + g_\mu(u) + \sqrt{\varepsilon}\|u - x_\varepsilon\|, \quad \forall u \in x + rB_X.$$

As f_λ and g_λ are locally Lipschitz around x_ε which is an internal point to $x + rB_X$, we obtain

$$x^* \in \partial f_\lambda(x_\varepsilon) + \partial g_\mu(x_\varepsilon) + \sqrt{\varepsilon} B_X.$$

and the proof is complete. ∎

3. Subdifferential calculus without qualification conditions

Our result may be applied to the subdifferential calculus of convex functions without qualification conditions. The following result is an extension of that of Attouch-Baillon-Théra[1] to reflexive Banach spaces. Note that other results concerning subdifferential calculus of convex functions without qualification conditions have been obtained by Hiriart-Urruty-Moussaoui-Seeger-Volle[3], Hiriart-Urruty-Phelps[4], Thibault[7, 8, 9, 10] and Penot[6].

Theorem 3.3 *Under the assumptions of Theorem 2.2 we have for all $\bar{x} \in \operatorname{dom} f \cap \operatorname{dom} g$*

$$\partial(f+g)(\bar{x}) = \{x^* \in X^* : \exists u_n \to \bar{x}, v_n \to \bar{x}, u_n^* \in \partial f(u_n), v_n^* \in \partial g(v_n),$$

$$\|x^* - u_n^* - v_n^*\| \to 0,\ \liminf_{n \to +\infty} \langle v_n^*, u_n - v_n \rangle \ge 0\}.$$

Proof. Let $x^* \in \partial(f+g)(\bar{x})$. Then, by Theorem 2.2, there are $\lambda_n \to 0^+$, $x_n \to \bar{x}$ and $x_n^* \to x^*$ (in norm) such that $x_n^* \in \partial f_{\lambda_n}(x_n) + \partial g_{\lambda_n}(x_n)$. Note that, extracting some subsequences, we may assume that $f_{\lambda_n}(x_n) \to f(\bar{x})$ and $g_{\lambda_n}(x_n) \to g(\bar{x})$. Let $u_n^* \in \partial f_{\lambda_n}(x_n)$ and $v_n^* \in \partial g_{\lambda_n}(x_n)$ with $x_n^* = u_n^* + v_n^*$. By the definitions of f_{λ_n} and g_{λ_n} there are sequences $u_n', v_n' \in X$ such that

$$f(u_n') + \frac{1}{2\lambda_n}\|u_n' - x_n\|^2 \le f_{\lambda_n}(x_n) + \lambda_n^3 \tag{6}$$

and

$$g(v_n') + \frac{1}{2\lambda_n}\|v_n' - x_n\|^2 \le g_{\lambda_n}(x_n) + \lambda_n^3 \tag{7}$$

and hence

$$\langle u_n^*, u - u_n' \rangle \le f(u) - f(u_n') + \lambda_n^3, \quad \forall u \in X$$

and

$$\langle v_n^*, v - v_n' \rangle \le g(v) - g(v_n') + \lambda_n^3, \quad \forall v \in X.$$

By Ekeland variational principle[2] , there exist $u_n, v_n \in X$ such that

$$\|u_n - u_n'\| \le \lambda_n^2, \quad \|v_n - v_n'\| \le \lambda_n^2 \tag{8}$$

$$f(u_n) \le f(u) - \langle u_n^*, u - u_n \rangle + \lambda_n\|u - u_n\|, \quad \forall u \in X$$

$$g(v_n) \le g(v) - \langle v_n^*, v - v_n \rangle + \lambda_n\|v - v_n\|, \quad \forall v \in X.$$

The proof of the first inclusion is then terminated if we show that

$$\liminf_{n \to +\infty} \langle v_n^*, u_n - v_n \rangle \ge 0.$$

As

$$\langle v_n^*, u_n - v_n \rangle = \langle v_n^*, u_n - u_n' \rangle + \langle v_n^*, u_n' - v_n' \rangle + \langle v_n^*, v_n' - v_n \rangle$$

and for some $K > 0$ not depending on n

$$\|v_n^*\| \le \frac{K}{\lambda_n}$$

we get, from (8)

$$|\langle v_n^*, u_n - u_n'\rangle| \le K\lambda_n \quad \text{and} \quad |\langle v_n^*, v_n - v_n'\rangle| \le K\lambda_n$$

it suffices to show that

$$\liminf_{n\to+\infty} \langle v_n^*, u_n' - v_n'\rangle \ge 0.$$

Using (6), (7) and $u_n^* \in \partial f_{\lambda_n}(x_n)$ and $v_n^* \in \partial g_{\lambda_n}(x_n)$, we get

$$\begin{aligned} \|u_n' - v_n'\|^2 &\le 2\|u_n' - x_n\|^2 + 2\|v_n' - x_n\|^2 \\ &\le 4\lambda_n[f_{\lambda_n}(x_n) - f(u_n') + g_{\lambda_n}(x_n) - g(v_n') + 2\lambda_n^2] \\ &\le 4\lambda_n[\langle u_n^*, x_n - u_n'\rangle + \langle v_n^*, x_n - v_n'\rangle + 2\lambda_n^2] \\ &= 4\lambda_n[\langle u_n^* + v_n^*, x_n - u_n'\rangle + \langle v_n^*, u_n' - v_n'\rangle + 2\lambda_n^2] \end{aligned}$$

and hence

$$-\langle u_n^* + v_n^*, x_n - u_n'\rangle - 2\lambda_n^2 \le \langle v_n^*, u_n' - v_n'\rangle.$$

As f and g are convex and lower semicontinuous, we get from (6) and (7) that

$$\|x_n - u_n'\| \to 0 \quad \text{and} \quad \|x_n - v_n'\| \to 0$$

(because f and g are bounded from bellow by functions of the form $\langle p^*, \cdot\rangle + b$ and $\langle q^*, \cdot\rangle + a$). Then, since $\|x^* - (v_n^* + u_n^*)\| \to 0$ and $\|x_n - u_n'\| \to 0$

$$\liminf_{n\to+\infty} \langle v_n^*, u_n' - v_n'\rangle \ge 0$$

and this completes the proof. ∎

Many corollaries may be deduced from this result, for example the result concerning exact chain rules for the sum of two convex functions when qualification conditions hold. But we let this to curious readers.

4. Marginal function and composition of convex functions

Calculus rules may be applied to produce characterization of the subdifferential of the marginal function

$$v(x) = \inf_{y\in F(x)} f(x, y)$$

where $f : X \times Y \to \mathbf{R} \cup \{+\infty\}$ is a proper lower semicontinuous convex function and $F : X \to 2^Y$ a multivalued mapping with closed convex graph GrF, and Y is a Banach space. Note that v is convex.

Theorem 4.4 *Let X and Y be two reflexive Banach spaces, and let $\hat{y} \in F(\hat{x})$ with $v(\hat{x}) = f(\hat{x}, \hat{y})$. Then the following assertions are equivalent :*

1) $x^* \in \partial v(\hat{x})$;

2) *there exist* $(x_n, y_n) \to (\hat{x}, \hat{y})$, $(u_n, v_n) \to (\hat{x}, \hat{y})$, $(x_n^*, y_n^*) \in \partial f(x_n, y_n)$, $(u_n^*, v_n^*) \in N(GrF, (u_n, v_n))$ *such that*

$$\|x_n^* + u_n^* - x^*\| + \|y_n^* + v_n^*\| \to 0 \quad \textit{and} \quad \liminf_{n\to+\infty} [\langle x_n^*, u_n - x_n\rangle + \langle y_n^*, v_n - y_n\rangle] \ge 0.$$

Here $N(GrF, (x, y)) = \partial\Psi_{GrF}(x, y)$ and Ψ_{GrF} denotes the indicator function of GrF.

Proof. It suffices to see that $x^* \in \partial v(\hat{x})$ iff $(x^*, 0) \in \partial(f + \Psi_{GrF})(\hat{x}, \hat{y})$ and to apply Theorem 3.3. ∎

We have the following corollary.

Corollary 4.1 *Let the hypothesis of Theorem 3.3 be satisfied with $F(x) = Ax$, where A is a linear continuous mapping. Then the following are equivalent :*

1) $x^* \in \partial v(\hat{x})$;

2) *there exist* $(x_n, y_n) \to (\hat{x}, \hat{y})$, $u_n \to \hat{x}$, $x_n^* \in X^*$, *and* $v_n^*, e_n^* \in Y^*$ *such that* $(x_n^*, v_n^* + e_n^*) \in \partial f(x_n, y_n)$ *and*

$$\|x_n^* - v_n^* \circ A - x^*\| + \|e_n^*\| \to 0 \quad \textit{and} \quad \liminf_{n\to+\infty} \langle v_n^*, y_n - Ax_n\rangle \ge 0.$$

As in the paper by Thibault[9] we may derive chain rules for composite convex functions. For this we need some notations. Let a closed convex cone P inducing a preorder $\le_Y$ on Y defined by $y_1 \le_Y y_2$ iff $y_2 - y_1 \in P$. Let $+\infty$ be an abstract maximal element adjoined to Y.

Recall that a mapping $g : X \to Y \cup \{+\infty\}$ is convex if for all $x, x' \in X$, and $t \in]0, 1[$

$$g(tx + (1-t)x') \le_Y tg(x) + (1-t)g(x').$$

We set $\mathrm{dom}g := \{x \in X : g(x) \in Y\}$ the effective domain of g, $\mathrm{Im}g := g(X)$ the effective image of g and $\mathrm{epi}g := \{(x, y) : g(x) \le_Y y\}$ the epigraph of g.

A function $f : X \to \mathbf{R} \cup \{+\infty\}$ is $P-$nondecreasing on a subset S of Y if $f(y_1) \le f(y_2)$, for all $y_1, y_2 \in S$ satisfying $y_1 \le_Y y_2$. By convention one puts $f(+\infty) = +\infty$. We easily see that if f is convex and $P-$nondecreasing over $\mathrm{Im}g + P$, then $f \circ g$ is convex.

Corollary 4.2 *Let X and Y be two reflexive Banach spaces. Suppose that $g : X \to Y \cup \{+\infty\}$ is a convex mapping with closed epigraph and that $f : X \to \mathbf{R} \cup \{+\infty\}$ is a proper convex lower semicontinuous function which is $P-$nondecreasing on $\mathrm{Im}g + P$. Then for $\hat{y} = g(\hat{x})$ the following are equivalent :*

1. *$x^* \in \partial(f \circ g)(\hat{x})$;*
2. *There exist $y_n \to \hat{y}$, $(u_n, v_n) \to (\hat{x}, \hat{y})$, $y_n^* \in \partial f(y_n)$ $u_n^* \in X^*$ and $v_n^* \in -P^0$ such that*
   - $\langle v_n^*, v_n - g(u_n)\rangle = 0$;
   - $\|u_n^* - x^*\| + \|y_n^* - v_n^*\| \to 0$;
   - $u_n^* \in \partial(v_n^* \circ g)(u_n)$;
   - $\liminf_{n\to+\infty}\langle v_n^*, y_n - g(u_n)\rangle \geq 0$.

Here P^0 denotes the negative polar of P.

Proof. Consider the multivalued mapping F defined by $F(x) = g(x) + P$. Then its graph is exactly the epigraph of g and hence GrF is closed and convex. Since

$$(f \circ g)(x) = \inf_{y \in F(x)} f(y)$$

and, by Proposition 3.2 in Jourani[5],

$$\begin{aligned} N(GrF, (x,y)) &= \{(u^*, -v^*) \in X^* \times Y^* : \ u^* \in \partial(v^* \circ g)(u), \\ &\qquad v^* \in -P^0, \langle v^*, v - g(u)\rangle = 0\} \end{aligned}$$

the result follows from Theorem 4.4. ∎

5. References

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