Weak regularity of functions and sets in Asplund spaces

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Abstract. In this paper, we study a new concept of weak regularity of functions and sets in Asplund spaces. We show that this notion includes prox-regular functions, functions whose subdifferential is weakly submonotone and amenable functions in infinite dimension. We establish also that weak regularity is equivalent to Mordukhovich regularity in finite dimension. Finally, we give characterizations of the weak regularity of epi-Lipschitzian sets in terms of their local representations.

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1 Introduction

Motivated by the strong connection in convex analysis between functions and their Moreau envelopes, Poliquin and Rockafellar [25] introduced and studied the concept of prox-regular functions in finite dimension. They proved that this class includes those of lower semicontinuous proper convex functions, lower $\mathcal{C}^2$-functions, primal lower-nice functions and strongly amenable functions. The last ones are composition of lower semicontinuous proper convex functions with $\mathcal{C}^2$-functions. A number of equivalent characterizations of prox-regular functions are obtained in their papers [25] and [27]. Among these characterizations, one can cite the subdifferential characterization, along the line of the well-known one

in convex analysis which states that a lower-semicontinuous function is convex iff its subdifferential is monotone (see Poliquin [24] for the finite dimensional situation and Correa, Jofré and Thibault [7] for the setting of Banach spaces). In Hilbert context, Bernard and Thibault [3] have given a subdifferential characterization of prox-regular functions. Generalization to the Banach setting is explored by the same authors in [2] (see also [4]).
The aim of the present work is to study a new concept of weak regularity related to the notion of prox-regularity by Poliquin and Rockafellar. Our major motivation to introduce this notion was calmness of systems of the form

$$x \in C, \quad f(x) \leq 0$$

where C is a closed set in Asplund space and f is a lower semicontinuous function (see [15]). Conditions for calmness was based on boundaries of subdifferentials and normal cones than of the full objects (see also Henrion and Outrata [12] and Henrion, Jourani and Outrata [11] for the nonconvex finite dimensional case and Henrion and Jourani [10] for the convex infinite dimensional context).
The importance of the class introduced in this paper is readily appreciated from the fact that it includes not only the classes cited above, but also the class of regular functions, p-convex functions [8], submonotone functions and classes introduced here, as that of weakly submonotone functions (whose subdifferential is weakly submonotone).
The plan of the present article is as follows: Sections 2 and 3 contain notations and the requisite background in nonsmooth analysis. Section 4 deals with the definition of weak regularity and two of its characterizations. The first one is functional as for the second is given in terms of the Mordukhovich regularity. Sections 5 and 6 contains, respectively, an extension of the notion of pseudo-convexity, and a notion of paraconvexity which are sufficient for the weak regularity. Section 7 treats the relation between a new definition of weak submonotonicity and the weak regularity. The concept of weak submonotonicity includes the notion of submonotonicity introdued by Spingarn [33] as well as the notion of local monotonicity by Colombo and Goncharov [9]. A number of equivalent characterizations of the weak regularity are given in terms of the weak submonotonicity. In section 8, we extend the definition of amenability by Poliquin and Rockafellar from finite dimension to the infinite one and we show that this definition implies once more weak regularity. Finally, we give in the last section characterization of the weak regularity of epi-Lipschitzian sets in terms of their local representations.

2 Notations

Throughout the paper X will be an *Asplund space* and X^* its topological dual equipped with the weak-star topology w^*. We will denote by $B(x, r)$ the closed ball centred at x and of radius r and by $d(\cdot, S)$ the distance function to a subset S of X

$$d(x, S) = \inf_{u \in S} \|x - u\|.$$

We will write $x \xrightarrow{f} x_0$ and $x \xrightarrow{S} x_0$ to express $x \to x_0$ with $f(x) \to f(x_0)$ and $x \to x_0$ with $x \in S$, respectively and we will denote by GrF the graph of a multivalued mapping $F : X \mapsto Y$, i. e.,

$$GrF = \{(x, y) : \quad y \in F(x)\}.$$

The indicator function of a set $C \subset X$ is the function ψ_C defined by

$$\psi_C(x) = \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{otherwise.} \end{cases}$$

If not specified the norm in the product of two Banach spaces is defined by $\|(a, b)\| = \|a\| + \|b\|$.

3 Tools from nonsmooth analysis

This section contains some background material on nonsmooth analysis and preliminary results which will be used later. We give only concise definitions and results that will be needed in the paper. For more detailed information on the subject our references are Mordukhovich [16, 17, 18] and Mordukhovich and Shao [19]. Note that in finite dimension, the limiting Fréchet subdifferential in the following definition coincides with the limiting proximal subdifferential as in [6] and the approximate subdifferential as in [13].

Let $f : X \mapsto \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function and $x_0 \in X$ be such that $f(x_0) < \infty$. The *Fréchet subdifferential* is the set

$$\partial_F f(x) = \{x^* \in E : \liminf_{h \to 0} \frac{f(x+h) - f(x) - \langle x^*, h\rangle}{\| h \|} \geq 0\}.$$

The *limiting Fréchet subdifferential* of f at x_0 is the set

$$\partial f(x_0) \quad := \quad \{x^* \in E : \exists x_k \to x_0,\ f(x_k) \to f(x_0) \text{ and } x_k^* \to x^* \text{ with } x_k^* \in \partial_F f(x_k)\}.$$

or equivalently

$$\partial f(x_0) = w^* - seq - \limsup_{x \xrightarrow{f} x_0} \partial_F f(x)$$

where $w^* - seq - \limsup$ denotes the weak-star sequential limit superior.

The *singular limiting Fréchet subdifferential* of f at x_0 is the set

$$\partial^\infty f(x_0) = w^* - seq - \limsup_{\substack{x \xrightarrow{f} x_0 \\ \lambda \to 0^+}} \lambda \partial_F f(x).$$

The Fréchet and the limiting Fréchet normal cones to C at $x_0 \in C$ are given by

$$N_F(C, x_0) = \partial_F \psi_C(x_0) \text{ and } N(C, x_0) = \partial \psi_C(x_0).$$

The Clarke's normal cone $N_c(C, x_0)$ to C at $x_0 \in C$ is given, in terms of the limiting Fréchet normal cone $N(C, x_0)$, by

$$N_c(C, x_0) = \bar{co} N(C, x_0).$$

A function f is said to be *Mordukhovich regular* at x_0 if the Fréchet subdifferential and the limiting Fréchet subdifferential coincide at x_0. In other words

$$\partial f(x_0) = \partial_F f(x_0).$$

In the case where f is locally Lipschitzian at x_0, it is easy to show that Mordukhovich regularity implies Clarke's regularity. This is due to the fact that, for these functions, the Clarke subdifferential $\partial_c f(x_0)$ ([5]) coincides with the weak-star closure of the convex hull of the set $\partial f(x_0)$.
More generally, for any lower semicontinuous function $f : X \mapsto \mathbb{R} \cup \{+\infty\}$ we have

$$\partial_c f(x_0) = \bar{co}[\partial f(x_0) + \partial^\infty f(x_0)]. \tag{1}$$

In finite dimension, both concept of regularity are identical for locally Lipschitzian functions.
Following Mordukhovich and Shao [20] (see also [21]), a set C is said to be *normally sequentially compact* at x_0 if for each sequences (x_k) and (x_k^*) satisfying $x_k \xrightarrow{C} x_0$, and for all k, $x_k^* \in N_F(C, x_k)$ we have

$$x_k^* \xrightarrow{w^*} 0 \Longleftrightarrow \|x_k^*\| \to 0.$$

Let $C \subset X$, with $x_0 \in C$. The contingent cone $T(C, x_0)$ to C at x_0 is the set given by the following Kuratowski limit :

$$T(C, x_0) = \limsup_{t \to 0^+} \frac{C - x_0}{t}.$$

The negative polar of a closed cone $K \subset X$ is defined by

$$K^0 = \{x^* \in X^* : \langle x^*, h \rangle \leq 0, \forall h \in K\}.$$

4 Weak regularity: Definition and characterizations

In this section, we define and characterize the classes of weak regular functions and sets in terms of their limiting Fréchet subdifferential and normal cone. To do this, we consider the set

$$\mathcal{F} = \{\varphi : \mathbb{R} \mapsto \mathbb{R}_+ : \quad \varphi(0) = 0, \lim_{t \to 0} \frac{\varphi(t)}{t} = 0\}.$$

Definition 4.1 1) *f is said to be weakly regular (WR) at x_0 relative to $x^* \in \partial f(x_0)$ if there exist a function $\varphi \in \mathcal{F}$ and $\varepsilon > 0$ such that*

$$f(x) - f(x_0) + \varphi(\|x - x_0\|) \geq \langle y^*, x - x_0 \rangle$$

whenever $\|x - x_0\| < \varepsilon$ and $\|y^* - x^*\| < \varepsilon$ with $y^* \in \partial f(x_0)$.
f is said to be WR at x_0 if it is WR at x_0 relative to each $x^* \in \partial f(x_0)$.
2) *C is WR at x_0 relative to $x^* \in N(C, x_0)$ if its indicator function is WR at x_0 relative to $x^* \in N(C, x_0)$.*

Example 1 *Consider the function $f : \mathbb{R} \mapsto \mathbb{R}$ defined by*

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then f is WR at 0 relative to 0.

Our definition was inspired by the concept of "prox-regular" functions introduced by Poliquin and Rockafellar. A function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is said to be prox-regular at x_0 relative to $x^* \in \partial f(x_0)$ if there exist $r > 0$ and $\varepsilon > 0$ such that

$$f(x) - f(x') + \frac{r}{2}\|x - x'\|^2 \geq \langle y^*, x - x' \rangle$$

whenever $\|x - x_0\| < \varepsilon$, $\|x' - x_0\| < \varepsilon$ and $\|y^* - x^*\| < \varepsilon$ with $y^* \in \partial f(x')$.

As we can see that prox-regularity implies weak regularity (the last example shows that the opposite implication is not true). It is also obvious that if f is convex, it is prox-regular at x_0. The same is true for lower $\mathcal{C}^2$ functions and strongly amenable functions; cf. [25]. As an example of strongly amenable functions is the maximum of a finite number of $\mathcal{C}^1$-functions (see Section 8 for an infinite dimensional definition of this concept) .

Before characterizing the WR property, we give the following result concerning the sum of two WR functions.

Proposition 4.1 *Let $f, g : X \mapsto \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous functions. Suppose that*
i) either epif is normally sequentially compact at $(x_0, f(x_0))$ or epig is normally sequentially compact at $(x_0, g(x_0))$;
ii) $\partial^\infty f(x_0) \cap [-\partial^\infty g(x_0)] = \{0\}$;
iii) f and g are WR at x_0.
Then the function $f + g$ is WR at x_0.

When f and g are indicator functions, we obtain :

Corollary 4.1 *Let C and D be closed sets in X and let $x_0 \in C \cap D$. Suppose that*
i) C or D is normally sequentially compact at x_0;
ii) $N(C, x_0) \cap [-N(D, x_0)] = \{0\}$;
iii) C and D are WR at x_0.
Then the set $C \cap D$ is WR at x_0.

The definition of the WR property can be characterized as follows :

Theorem 4.1 *Let $x^* \in \partial f(x_0)$. Then the following assertions are equivalent:*
i) f is WR at x_0 relative to x^*;
ii) there exists $s > 0$ such that

$$\forall 0 < \varepsilon < s, \quad \exists \delta > 0; \quad \langle y^*, x - x_0 \rangle \leq f(x) - f(x_0) + \varepsilon \|x - x_0\|$$

whenever $\|x - x_0\| \leq \delta$ and $\|y^* - x^*\| \leq s$ with $y^* \in \partial f(x_0)$.

Proof. *i)* $\Longrightarrow$ *ii)*. Since f is WR at x_0 relative to x^*, there exist $\varphi \in \mathcal{F}$ and $s > 0$ such that

$$f(x) - f(x_0) + \varphi(\|x - x_0\|) \geq \langle y^*, x - x_0 \rangle$$

whenever $\|x - x_0\| < s$ and $\|y^* - x^*\| < s$ with $y^* \in \partial f(x_0)$. As $\varphi \in \mathcal{F}$, we have

$$\forall \varepsilon \in]0, s[, \ \exists \delta > 0; \quad \varphi(t) \leq t\varepsilon \quad \forall t \in [0, \delta].$$

So that

$$\forall 0 < \varepsilon < s, \quad \exists \delta > 0; \quad \langle y^*, x - x_0 \rangle \leq f(x) - f(x_0) + \varepsilon \|x - x_0\|$$

whenever $\|x - x_0\| \leq \delta$ and $\|y^* - x^*\| \leq s$ with $y^* \in \partial f(x_0)$.
ii) $\Longrightarrow$ *i)*. The proof of this implication was inspired by the proof of [[33], Theorem 3.9] (see also [[33], Proposition 3.8] and [[1], Lemma 4.4]). Assertion *ii)* allows us to construct a function $g : X \times X^* \mapsto \mathbb{R} \cup \{+\infty\}$ defined by

$$g(x, y^*) = \begin{cases} \frac{f(x) - f(x_0) - \langle y^*, x - x_0 \rangle}{\|x - x_0\|} & \text{if } x \neq x_0 \\ 0 & \text{otherwise} \end{cases}$$

Consider the function $g_1 : \mathbb{R}_+ \mapsto \mathbb{R} \cup \{-\infty\}$ defined by

$$g_1(t) = \begin{cases} 0 & \text{if } t = 0 \\ \inf\{g(x, y^*) : x \in B(x_0, t), \ y^* \in B(x^*, s) \cap \partial f(x_0)\} & \text{otherwise.} \end{cases}$$

Assertion *ii)* ensures that

$$\forall \varepsilon > 0, \ \exists \delta > 0; \ g_1(t) \geq -\varepsilon \ \forall t \in [0, \delta].$$

Consider the function $g_2 : \mathbb{R}_+ \mapsto \mathbb{R}_+$ defined by

$$g_2(t) = \max(-g_1(t), 0).$$

Then $g_2(0) = 0$ and g_2 is continuous at 0. Now we may define our function $\varphi : \mathbb{R} \mapsto \mathbb{R}_+$ by

$$\varphi(t) = |t| g_2(|t|).$$

We easily see that $\varphi \in \mathcal{F}$. Finally, let $x \in B(x_0, \delta)$, with $x \neq x_0$, and set $t = \|x - x_0\|$. Then

$$g(x, y^*) \geq g_1(t) \geq -g_2(t) = -\frac{\varphi(t)}{t} = -\frac{\varphi(\|x - x_0\|)}{\|x - x_0\|}$$

whenever $y^* \in B(x^*, s) \cap \partial f(x_0)$. This asserts that f is WR at x_0 relative to x^*.

$\diamondsuit$

As a consequence, we obtain the following corollary.

Corollary 4.2 *Let $x^* \in N(C, x_0)$. Then the following assertions are equivalent:*
i) C is WR at x_0 relative to x^*;
ii) there exists $s > 0$ such that

$$\forall 0 < \varepsilon < s, \quad \exists \delta > 0; \quad \langle y^*, x - x_0 \rangle \leq \varepsilon \|x - x_0\|$$

whenever $x \in C$, $\|x - x_0\| \leq \delta$ and $\|y^* - x^*\| \leq s$ with $y^* \in N(C, x_0)$.

Next, we state a characterization of the WR property of functions in terms of the Mordukhovich regularity in finite dimension.

Theorem 4.2 *Let X be a finite dimensional space and $f : X \mapsto \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Then the following assertions are equivalent:*
i) f is Mordukhovich regular at x_0;
ii) f is WR at x_0.

Proof. *ii)* $\Longrightarrow$ *i)*: It is obvious.
i) $\Longrightarrow$ *ii)*: Let $x^* \in \partial f(x_0)$. We claim that

$$\forall \varepsilon > 0, \exists \delta > 0; \ f(x) - f(x_0) \geq \langle u^*, x - x_0 \rangle - \varepsilon \|x - x_0\| \tag{2}$$

whenever $x \in B(x_0, \delta)$ and $u^* \in \partial f(x_0) \cap B(x^*, 1)$. Suppose that (2) does not hold. Then there are $\varepsilon > 0$, and sequences $x_n \to x_0$ and $x_n^* \in \partial f(x_0) \cap B(x^*, 1)$ such that

$$f(x_n) - f(x_0) < \langle x_n^*, x_n - x_0 \rangle - \varepsilon \|x_n - x_0\|, \ \forall n \geq 1. \tag{3}$$

Consider the vector $d_n = \frac{x_n - x_0}{\|x_n - x_0\|}$ and the scalar $t_n = \|x_n - x_0\|$. Extracting subsequence if necessary, we may assume that $d_n \to d$, with $\|d\| = 1$ and $x_n^* \to u^*$, with $u^* \in \partial f(x_0)$. Then it follows from relation (3) that

$$\liminf_{n \to +\infty} \frac{f(x_0 + t_n d_n) - f(x_0)}{t_n} \leq \langle u^*, d \rangle - \varepsilon. \tag{4}$$

The Mordukhovich regularity of f at x_0 implies that

$$\liminf_{n \to +\infty} \frac{f(x_0 + t_n d_n) - f(x_0)}{t_n} \geq \langle u^*, d \rangle$$

which, together with (4), leads to the contradiction $0 \le -\varepsilon$. To conclude the proof, it suffices now to apply Theorem 4.1.

$\diamondsuit$

As a corollory, we obtain the following characterization of WR sets.

Corollary 4.3 *Let X be a finite dimensional space and $C \subset X$ a closed set containing x_0. Then the following assertions are equivalent:*
i) C is Mordukhovich regular at x_0;
ii) C is WR at x_0.

This corollary is restricted to the finite dimensional spaces. Indeed, as the following example shows, the implication $i) \Longrightarrow ii)$ does not hold in infinite dimensional spaces.

Example 2 *This example was inspired by Counter-example 3.1 in [34] and Example 7.1 in [9]. Let X be a separable Hilbert space, with orthonormal basis written as $\{e_0, e_1, \cdots, e_n \cdots\}$.*
For $n = 1, 2, \cdots$ set

$$A_n = \frac{e_n}{n} \cup \{\frac{e_n}{n} + \bigcup_{m=1}^{\infty} [\frac{1}{m}, +\infty[\{\frac{e_m}{n} + e_0\}\}$$

and

$$C = \mathbb{R}_+ e_0 \cup [\bigcup_{n=1}^{\infty} A_n].$$

Then C is closed and

$$N_F(C, 0) = N(C, 0) = (\mathbb{R}_+ e_0)^0 \text{ and } -e_0 \in intN(C, 0).$$

So that C is Mordukhovich regular at 0 but not WR at 0. Indeed, suppose that C is WR at 0. By Corollary 4.2, there exists $s > 0$ such that

$$\forall 0 < \varepsilon < s, \quad \exists \delta > 0; \quad \langle x^*, x\rangle \le \varepsilon \|x\| \tag{5}$$

whenever $x \in C$, $\|x\| \le \delta$ and $\|x^* + e_0\| \le s$ with $x^* \in N(C, 0)$. As $-e_0 \in intN(C, 0)$, we may assume that $B(-e_0, s) \subset N(C, 0)$. Combining the last inclusion and relation (5), we obtain

$$-\langle e_0, x\rangle \le (\varepsilon - s)\|x\|, \forall x \in B(0, \delta) \cap C.$$

In particular for n large enough we obtain

$$-\langle e_0, \frac{e_n}{n}\rangle \le (\varepsilon - s)\|\frac{e_n}{n}\|$$

and hence $0 \le \varepsilon - s$ and this is a contradiction. So that C is not WR at 0.

5 Weak regularity from near pseudo-convexity

Another interesting particular case of the weak regularity is pseudo-convexity. A closed set C is pseudo-convex at some point $x_0 \in C$ if

$$C - x_0 \subset \bar{co}T(C, x_0).$$

We mention here that this definition is not the original one. This later was given without the convex hull.
As we are interested in a local analysis, we may extend this definition as follows:

Definition 5.1 *1) A set C is nearly pseudo-convex (for short NPS) at x_0 if the following holds*

$$\forall \varepsilon > 0,\ \exists \delta > 0;\ d(x - x_0, \bar{co}T(C, x_0)) \leq \varepsilon \|x - x_0\|,\ \forall x \in C \cap B(x_0, \delta). \qquad (6)$$

2) A function $f : X \mapsto I\!R \cup \{+\infty\}$ is said to be nearly pseudo-convex (for short NPS) at x_0 is its epigraph (epi$f$) is NPS at $(x_0, f(x_0))$.

Proposition 5.1 *i) If C is closed and pseudo-convex at x_0, then*

$$\langle x^*, x - x_0 \rangle \leq 0, \quad \forall x \in C,\ \forall x^* \in N_F(C, x_0).$$

ii) If f is lower semicontinuous and pseudo-convex at x_0, then for all $x^* \in \partial_F f(x_0)$ we have

$$\langle x^*, x - x_0 \rangle \leq f(x) - f(x_0) \quad \forall x \in X.$$

Proposition 5.2 *i) If the set C is NPC and Mordukhovich regular at x_0, then it is weakly regular at x_0.*
ii) If the function f is NPC and Mordukhovich regular at x_0 and upper-Lipschitz at x_0, that is,

$$\exists K > 0,\ \delta_0 > 0;\ |f(x) - f(x_0)| \leq K\|x - x_0\|\ \forall x \in B(x_0, \delta_0)$$

then it is weakly regular at x_0.

Proof. i) Since C is NPC at x_0 and $N_F(C, x_0) \subset T^0(C, x_0)$, we easily obtain

$$\forall \varepsilon > 0,\ \exists \delta > 0;\ \langle x^*, x - x_0 \rangle \leq \varepsilon \|x^*\| \|x - x_0\|,\ \forall x \in C \cap B(x_0, \delta), x^* \in N_F(C, x_0)$$

and the Mordukhovich regularity, together with Corollary 4.2, completes the proof of i).
ii) Since f is NPC at x_0 and $\partial_F f(x_0) \subset \{x^* : (x^*, -1) \in T^0(\text{epi}f, (x_0, f(x_0)))\}$ we have for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle x^*, x - x_0 \rangle \leq f(x) - f(x_0) + \varepsilon(\|x^*\| + 1)(\|x - x_0\| + |f(x) - f(x_0)|)$$

whenever $x \in B(x_0, \delta)$ and $x^* \in \partial_F f(x_0)$. Set $\delta_1 = \min(\delta_0, \delta)$. Now, since f is upper-Lipschitz at x_0, we have

$$\langle x^*, x - x_0 \rangle \leq f(x) - f(x_0) + \varepsilon(\|x^*\| + 1)(K + 1)\|x - x_0\|$$

whenever $x \in B(x_0, \delta_1)$ and $x^* \in \partial_F f(x_0)$. Once more, the Mordukhovich regularity and Theorem 4.1 complete the proof of ii).

$\diamondsuit$

6 Weak regularity from paraconvexity

Another class of functions which are WR is the class of paraconvex functions introduced by Rolewicz [31] and studied in [14] and [32]. Let $\varphi \in \mathcal{F}$. We recall that a function $f : X \mapsto \mathbb{R}$ is φ-paraconvex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + t(1-t)\varphi(\|x-y\|)$$

whenever $t \in]0,1[$ and $x, y \in X$.
For special functions $\varphi \in \mathcal{F}$, it is shown in [14] and [32] that

$$f \text{ is } \varphi\text{-paraconvex} \iff \partial_c f \text{ is } \varphi\text{-monotone.}$$

As we are interested in a local study, we may introduce the following definition.

Definition 6.1 *Let $\varphi \in \mathcal{F}$. We say that f is φ-paraconvex at $x_0 \in domf$ if there exists $\delta > 0$ such that*

$$f(x_0 + t(x - x_0)) \leq tf(x) + (1-t)f(x_0) + t(1-t)\varphi(\|x - x_0\|)$$

whenever $t \in]0,1[$ and $x \in B(x_0, \delta)$.

Example 3 *Consider the function $f : \mathbb{R} \mapsto \mathbb{R}$ defined by*

$$f(x) = \begin{cases} -\sqrt{-x} & \text{if } x \leq 0 \\ x^2 & \text{otherwise} \end{cases}$$

Then f is φ-paraconvex at 0, with $\varphi \equiv 0$.

Proposition 6.1 *Let $\varphi \in \mathcal{F}$. If f is φ-paraconvex and Mordukhovich regular at x_0, then it is WR at x_0.*

Proof. Since f is φ-paraconvex at x_0, there exists $\delta > 0$ such that

$$\liminf_{t \to 0^+} \frac{f(x_0 + t(x - x_0)) - f(x_0)}{t} \leq f(x) - f(x_0) + \varphi(\|x - x_0\|)$$

whenever $x \in B(x_0, \delta)$. We easily show that

$$x^* \in \partial_F f(x_0) \Longrightarrow \langle x^*, x - x_0 \rangle \leq \liminf_{t \to 0^+} \frac{f(x_0 + t(x - x_0)) - f(x_0)}{t}$$

whenever $x \in B(x_0, \delta)$. Hence, each $x^* \in \partial_F f(x_0)$ satisfies

$$\langle x^*, x - x_0 \rangle \leq f(x) - f(x_0) + \varphi(\|x - x_0\|), \quad \forall x \in B(x_0, \delta).$$

$\diamondsuit$

Remark 1 *Example 1 shows that f is Mordukhovich regular at 0, because $\partial_F f(0) = \partial f(0) = [0, +\infty[$, and hence WR at 0. But f is not φ-paraconvex at 0.*

Remark 2 *If we examine the proof of this proposition, we may see that the condition*

$$\exists\delta > 0; \liminf_{t\to 0^+} \frac{f(x_0 + t(x - x_0)) - f(x_0)}{t} \leq f(x) - f(x_0) + \varphi(\|x - x_0\|) \quad (7)$$

whenever $x \in B(x_0, \delta)$, together with the Mordukhovich regularity of f at x_0, implies the WR property of f at x_0.

Remark 3 *The class of ε-convex functions, introduced and studied in Ngai, Luc and Théra [22, 23], satisfies the WR property.*

7 Weak regularity from weak submonotonicity

The notion of submonotonicity (called semi-submonotonicity in [1]) has been introduced and characterized by Spingarn in finite dimensional spaces (see also [1] for infinite dimensional extensions).
A set-valued mapping $T : X \mapsto X^*$ is called *submonotone* at x_0, if for every $\varepsilon > 0$, there exists $\delta > 0$, such that, for all $x \in B(x_0, \delta) \cap \mathrm{dom}T$, all $x^* \in T(x)$, and all $x_0^* \in T(x_0)$ one has

$$\langle x^* - x_0^*, x - x_0\rangle \geq -\varepsilon\|x - x_0\|. \quad (8)$$

For our purpose, we do not need to consider this concept. We consider an other one, weaker than the previous, which implies the WR property. A set-valued mapping $T : X \mapsto X^*$ is called *weakly submonotone* at $x_0 \in \mathrm{dom}T$, if for all sequence $x_n \to x_0$, $x_n \neq x_0$ for all $n \geq 1$, and all bounded sequences $x_n^* \in T(x_n)$, and all $u_n^* \in T(x_0)$ one has

$$\liminf_{n\to+\infty} \langle x_n^* - u_n^*, \frac{x_n - x_0}{\|x_n - x_0\|}\rangle \geq 0. \quad (9)$$

It is easy to see that when T is uniformly bounded in some neighbourhood of x_0, then the concepts of submonotonicity and weak submonotonicity of T at x_0 are equivalent.
The following example shows that the condition of boundedness is essential to obtain the equivalence.

Example 4 *Consider the set $C \subset \mathbb{R}^2$ defined by*

$$C = \{(x, y) \in \mathbb{R}^2 : x \geq 0, \sqrt{x} \leq y \leq 2\sqrt{x}\}.$$

Then the set-valued mapping $x \mapsto N(C, x)$ is weakly submonotone at 0, but not submonotone at 0. Consider the sequence $u_n := (\frac{1}{n^2}, \frac{1}{n})$. Then $u_n \in C$ and

$$N(C, u_n) = \mathbb{R}_+(n, -2) \text{ and } N(C, 0) = \mathbb{R} \times \mathbb{R}_-.$$

Moreover

$$\langle (n, -2) - (n, 0), \frac{u_n}{\|u_n\|}\rangle = -2(1 + \frac{1}{n^2})^{\frac{-1}{2}}$$

which violates the submonotonicity.

We have to mention that a different notion, called *local monotonicity*, had been previously introduced by Colombo and Goncharov [9] in Hilbert spaces. We shall adopt the same definition in Banach spaces. A set-valued mapping $T : X \mapsto X^*$ is called *locally monotone* at $x_0 \in \mathrm{dom}T$, if for all sequences $x_n \to x_0$ and $u_n \to x_0$, $x_n \neq u_n$, and all bounded sequences $x_n^* \in T(x_n)$, and all $u_n^* \in T(u_n)$ one has

$$\liminf_{n\to+\infty} \langle x_n^* - u_n^*, \frac{x_n - u_n}{\|x_n - u_n\|} \rangle \geq 0. \tag{10}$$

Local submonotonicity clearly implies weak submonotonicity. The following example in Spingarn [33] shows that the converse is false.

Example 5 *Consider the function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ defined by*

$$f(x,y) = \begin{cases} |y| & \text{if } x \leq 0 \\ |y| - x^2 & \text{if } x \geq 0,\ |y| \geq x^2 \\ \frac{(x^4 - y^2)}{2x^2} & \text{if } x \geq 0,\ |y| \leq x^2 \end{cases}$$

It is shown in [33] that f is locally Lipschitz and that $x \mapsto \partial_c f(x)$ is submonotone at 0. Consider the sequences $x_n = (\frac{1}{n}, \frac{1}{n^2})$, $y_n = (\frac{1}{n}, \frac{-1}{n^2})$, $x_n^* = (\frac{2}{n}, -1)$ and $y_n^* = (\frac{2}{n}, 1)$. Then $x_n^* \in \partial_c f(x_n)$ and $y_n^* \in \partial_c f(y_n)$. Moreover

$$\frac{\langle x_n - y_n, x_n^* - y_n^* \rangle}{\|x_n - y_n\|} = -2, \quad \forall n$$

so $x \mapsto \partial_c f(x)$ is not locally submonotone at 0.

Due to the relation (1), we have the following result.

Proposition 7.1 *Let $f : X \mapsto \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function which is finite at x_0. Then the following assertions are equivalent:*
i) the set-valued mapping $x \mapsto \partial_c f(x)$ is submonotone at x_0;
ii) the set-valued mappings $x \mapsto \partial f(x)$ and $x \mapsto \partial^\infty f(x)$ are submonotone at x_0.

Since for a locally Lipschitzian function f at x_0,

$$\partial^\infty f(x) = \{0\}, \qquad \text{for } x \text{ near } x_0$$

and in some neighbourhood of x_0 the concepts of submonotonicity and weak submonotonicity are equivalent, we obtain the following corollary as a consequence of the last proposition.

Corollary 7.1 *Let $f : X \mapsto \mathbb{R} \cup \{+\infty\}$ be a function which is locally Lipschitz at x_0. Then the following assertions are equivalent:*
i) the set-valued mapping $x \mapsto \partial_c f(x)$ is weakly submonotone at x_0;
ii) the set-valued mapping $x \mapsto \partial f(x)$ is weakly submonotone at x_0.

In the following result, we prove that weak submonotonicity implies WR property.

Theorem 7.1 *Let $f : X \mapsto I\!R \cup \{+\infty\}$ be a lower semicontinuous function which is locally Lipschitz at x_0. Suppose that the operator $x \mapsto \partial f(x)$ is weakly submonotone at x_0. Then*

$$\forall \varepsilon > 0,\ \exists \delta > 0;\ f(x) - f(x_0) \geq \langle x^*, x - x_0 \rangle - \varepsilon \|x - x_0\| \tag{11}$$

whenever $x \in B(x_0, \delta)$ and $x^* \in \partial f(x_0)$. Moreover f is WR at x_0.

Proof. We claim that

$$\forall \varepsilon > 0,\ \exists \delta > 0;\ f(x) - f(x_0) \geq \langle x^*, x - x_0 \rangle - \varepsilon \|x - x_0\| \tag{12}$$

whenever $x \in B(x_0, \delta)$ and $x^* \in \partial f(x_0)$. Suppose that (12) does not hold. Then there are $\varepsilon > 0$, and sequences $x_n \to x_0$ and $x_n^* \in \partial f(x_0)$ such that

$$f(x_n) - f(x_0) < \langle x_n^*, x_n - x_0 \rangle - \varepsilon \|x_n - x_0\|,\ \forall n \geq 1. \tag{13}$$

Since f is locally Lipschitz at x_0, the Mean Value Theorem produces $t_n \in]0, 1[$ and $y_n^* \in \partial_c f(x_0 + t_n(x_n - x_0))$ such that

$$f(x_n) - f(x_0) = \langle y_n^*, x_n - x_0 \rangle. \tag{14}$$

Put $y_n = x_0 + t_n(x_n - x_0)$. Using relations (13) and (14), we get

$$\langle y_n^*, y_n - x_0 \rangle < \langle x_n^*, y_n - x_0 \rangle - \varepsilon \|y_n - x_0\|$$

and hence

$$\langle y_n^* - x_n^*, y_n - x_0 \rangle < -\varepsilon \|y_n - x_0\|$$

and this contradicts the weak submonotonicity assumption. To complete the proof, it suffices to apply Theorem 4.1.

$\diamondsuit$

The following example, inspired by [33], shows that the subdifferential of WR functions is not necessary weakly submonotone.

Example 6 *For $n \geq 2$, we set*

$$\alpha_n = \frac{n(n+1)}{n^2 - n - 1} \ \text{ and } \ c_n = (n^2 - n - 1)[n(n+1)]^{\alpha_n - 2}.$$

Consider the function $f : I\!R \mapsto I\!R_+$ defined by

$$f(x) = \begin{cases} \frac{1}{4} & \text{if } |x| \geq \frac{1}{2} \\ c_n(|x| - \frac{1}{n+1})^{\alpha_n} + \frac{n}{(n+1)^2} & \text{if } \frac{1}{n+1} \leq |x| \leq \frac{1}{n},\ \forall n \geq 2 \\ 0 & \text{if } x = 0. \end{cases}$$

Then f is locally Lipschitz and $|x| - x^2 \leq f(x) \leq |x|$, for all x. Hence $\partial_F f(0) = [-1, 1] = \partial f(0)$ and relation (11) holds at $x_0 = 0$. However the set-valued mapping $x \mapsto \partial f(x)$ is not weakly submonotone at 0.

A straightforward sufficient condition for WR property of sets is the following.

Theorem 7.2 *Let $C \subset X$ be a closed set containing x_0. Consider the following assertions :*
i) $x \mapsto N(C,x)$ is weakly submonotone at x_0;
ii) the following condition holds: for all sequence $x_n \to x_0$, $x_n \in C$ and $x_n \neq x_0$ for all $n \geq 1$, and all bounded sequence $x_n^* \in N(C,x_0)$

$$\limsup_{n\to\infty} \langle x_n^*, \frac{x_n - x_0}{\|x_n - x_0\|} \rangle \leq 0; \tag{15}$$

iii) C is WR at x_0.
Then i) $\Longrightarrow$ ii) $\Longrightarrow$ iii). Furthermore in finite dimensional spaces ii) $\Longleftrightarrow$ iii).

Proof. *i)* $\Longrightarrow$ *ii)* : This implication is obvious.
ii) $\Longrightarrow$ *iii)* : Let $x^* \in N(C,x_0)$. We claim that

$$\forall \varepsilon > 0, \exists \delta > 0;\ \langle y^*, x - x_0 \rangle \leq \varepsilon \|x - x_0\| \tag{16}$$

whenever $x \in B(x_0,\delta) \cap C$ and $y^* \in N(C,x_0) \cap B(x^*,1)$. Suppose that (16) does not hold. Then there are $\varepsilon > 0$, and sequences $x_n \xrightarrow{C} x_0$ and $x_n^* \in N(C,x_0) \cap B(x^*,1)$ such that

$$\langle x_n^*, x_n - x_0 \rangle > \varepsilon \|x_n - x_0\|,\ \forall n \geq 1. \tag{17}$$

This shows that

$$\limsup_{n\to\infty} \langle x_n^*, \frac{x_n - x_0}{\|x_n - x_0\|} \rangle > \varepsilon$$

and contradicts (15).
Now, suppose that X is of finite dimension. We prove that *iii)* $\Longrightarrow$ *ii)*. Suppose the contrary. Then there are sequence $x_n \xrightarrow{C} x_0$ and a bounded sequence $u_n^* \in N(C,x_0)$ such that

$$\limsup_{n\to+\infty} \langle u_n^*, \frac{x_n - x_0}{\|x_n - x_0\|} \rangle > 0. \tag{18}$$

Extracting subsequences if necessary, we may assume that $u_n^* \to u^*$ in norm, because X is of finite dimension, with $u^* \in N(C,x_0)$ and

$$\limsup_{n\to+\infty} \langle u_n^*, \frac{x_n - x_0}{\|x_n - x_0\|} \rangle = \lim_{n\to+\infty} \langle u_n^*, \frac{x_n - x_0}{\|x_n - x_0\|} \rangle.$$

So that there exist $\varepsilon > 0$ and $n_0 \geq 1$ such that

$$\langle u_n^*, x_n - x_0 \rangle > \varepsilon \|x_n - x_0\|, \quad \forall n \geq n_0.$$

Since C is WR at x_0 relative to u^* and $u_n^* \to u^*$ in norm, there exists $n_1 > n_0$ such that

$$\langle u_n^*, x_n - x_0 \rangle \leq \frac{\varepsilon}{3} \|x_n - x_0\|,\ \forall n \geq n_1.$$

Combining the last two relations we get a contradiction.

$\diamondsuit$

The implication *ii)* $\Longrightarrow$ *i)* does not hold even in finite dimension.

Example 7 *([9]) Consider the decreasing sequences*

$$a_n = \frac{1}{n^2} - \frac{2}{n^4}, \quad b_n = \frac{1}{n^2} - \frac{1}{n^4}, \quad n \geq 2.$$

Observe that $b_{n+1} < a_n$ for all n. Taking into account that all the segments $[a_n, b_n]$ are disjoint, we define a $\mathcal{C}^2$ function $\psi :]0, +\infty[\to \mathbb{R}^+$ such that $\psi(x) = \frac{1}{n}$ for $x \in [a_n, b_n]$ and $\sqrt{x} < \psi(x) < 2\sqrt{x}$ for all $x > 0$. Clearly, ψ can be continuously extended to $\mathbb{R}^+$ by setting $\psi(0) = 0$. Consider the set $C \subset \mathbb{R}^2$ given by

$$C = \{(x, y) \in \mathbb{R}^2 : x \geq 0, \, \psi(x) \leq y \leq 2\sqrt{x}\}.$$

Then C is WR at $(0,0)$. However the set-valued mapping is not weakly submonotone at $(0,0)$. Indeed, let $c_n \in]a_n, b_n[$ and $x_n = (c_n, \frac{1}{n})$. Then $(0,-1) \in N(C, x_n)$ and

$$\langle (0,-1) - 0, \frac{x_n - 0}{\|x_n - 0\|} \rangle \to -1.$$

As a consequence, we obtain the following characterization.

Corollary 7.2 *Let $C \subset X$ be a closed set containing x_0. Then the following assertions are equivalent:*
i) $x \mapsto N(C, x)$ is weakly submonotone at x_0;
ii) $x \mapsto N_c(C, x)$ is weakly submonotone at x_0.

Proof. *ii)* $\Longrightarrow$ *i)*: It is due to the inclusion $N(C, x) \subset N_c(C, x)$, for all $x \in C$.
i) $\Longrightarrow$ *ii)*: Suppose that *i)* holds. Then, Theorem 7.2 implies that C is WR at x_0, and hence it is Mordukhovich regular at x_0 and the result follows.

$\diamondsuit$

8 Weak regularity from amenability

To give an other example of WR functions, we extend the concept of amenability, introduced in Poliquin and Rockafellar [26], from finite dimensional spaces to the (Asplund) Banach spaces.

Definition 8.1 *A function $f : X \mapsto \mathbb{R} \cup \{+\infty\}$ is amenable at x_0, with $f(x_0) < \infty$, if it has the representation $f = g \circ F$ in a neighborhood of x_0 for a mapping $F : X \mapsto Y$ which is strictly differentiable at x_0, an Asplund space Y and a proper lower semicontinuous convex function $g : Y \mapsto \mathbb{R} \cup \{+\infty\}$ satisfying at x_0 with respect to the convex set $D := \operatorname{dom} g$ the basic constraint qualification that*

$$y^* \in N(D, F(x_0)), \quad \nabla^* F(x_0) y^* = 0 \Longrightarrow y^* = 0 \tag{19}$$

and the topological property that

$$\textit{epig is normally sequentially compact at } (F(x_0), g(F(x_0))) \tag{20}$$

Remark 4 *Note that, since g is convex the constraint qualification (19) is equivalent to the following one*

$$y^* \in \partial^\infty g(F(x_0)), \quad \nabla^* F(x_0) y^* = 0 \Longrightarrow y^* = 0 \tag{21}$$

Theorem 8.1 *Every amenable function is weakly regular.*

The proof of this theorem is based on the following lemma.

Lemma 8.1 *Let f be an amenable function at x_0 and has the form $f = g \circ F$, where g and F satisfy relations (19) and (20). Then*
i) $\partial f(x_0) = \bigcup_{y^* \in \partial g(F(x_0))} \nabla^* F(x_0) y^*$

ii) for each $x^* \in \partial f(x_0)$, there exists $r > 0$ such that

$$\partial f(x_0) \cap B(x^*, 1) \subset \nabla^* F(x_0)[\partial g(F(x_0)) \cap B(0, r)].$$

Proof. *i)* This assertion follows from classical chain rules for the limiting Fréchet subdifferential (see for example [18]).
ii) Suppose the contrary. Then for each integer n there exist $u_n^* \in \partial f(x_0) \cap B(x^*, 1)$ and $y_n^* \in \partial g(F(x_0))$, with $\|y_n^*\| > n$, such that

$$u_n^* = \nabla^* F(x_0) y_n^*. \tag{22}$$

Extracting a subsequence if necessary, we may assume that the sequence $(\frac{y_n^*}{\|y_n^*\|})$ weak-star converges to some y^*. Since epi$g$ is normally sequentially compact at $(F(x_0), g(F(x_0)))$ and

$$(\frac{y_n^*}{\|y_n^*\|}, \frac{-1}{\|y_n^*\|}) \in N(\text{epi}g, (F(x_0), g(F(x_0)))$$

it follows that $y* \neq 0$ and $(y^*, 0) \in N(\text{epi}g, (F(x_0), g(F(x_0)))$. Using relation (22) and the last inclusion we get

$$y^* \in N(D, F(x_0)),\ y^* \neq 0 \,\text{and}\, \nabla^* F(x_0) y^* = 0$$

and this contradicts relation (19) and completes the proof of the lemma.

$\diamondsuit$

Proof of Theorem 8.1. Let f be an amenable function at x_0 and of the form $f = g \circ F$, where g and F satisfy relations (19) and (20). Since F is strictly differentiable at x_0, we have

$$\forall \varepsilon > 0,\ \exists \delta > 0;\ \|F(x) - F(x_0) - \nabla F(x_0)(x - x_0)\| \leq \varepsilon \|x - x_0\|$$

whenever $x \in B(x_0, \delta)$. Let $x^* \in \partial f(x_0)$. We will show that f is WR at x_0 relative to x^*. By Lemma 8.1, there exists $r > 0$ such that

$$\partial f(x_0) \cap B(x^*, 1) \subset \nabla^* F(x_0)[\partial g(F(x_0)) \cap B(0, r)].$$

Pick $u^* \in \partial f(x_0) \cap B(x^*, 1)$. Then there exists $y^* \in \partial g(F(x_0)) \cap B(0, r)$ such that $u^* = \nabla^* F(x_0) y^*$. As g is convex, it follows that

$$g(F(x)) - g(F(x_0)) \geq \langle y^*, F(x) - F(x_0)\rangle, \quad \forall x \in X.$$

Thus for all $x \in B(x_0, \delta)$

$$\begin{aligned} f(x) - f(x_0) \quad & \geq \langle y^*, F(x) - F(x_0) - \nabla F(x_0)(x - x_0)\rangle + \langle y^*, \nabla F(x_0)(x - x_0)\rangle \\ & \geq -\|y^*\| \|F(x) - F(x_0) - \nabla F(x_0)(x - x_0)\| + \langle \nabla^* F(x_0) y^*, x - x_0\rangle \\ & \geq -r\varepsilon \|x - x_0\| + \langle u^*, x - x_0\rangle \end{aligned}$$

and hence

$$\forall \varepsilon > 0,\ \exists \delta > 0;\ f(x) - f(x_0) \geq -\varepsilon \|x - x_0\| + \langle u^*, x - x_0\rangle$$

whenever $x \in B(x_0, \delta)$ and $u^* \in \partial f(x_0) \cap B(x^*, 1)$. So Theorem 4.1 shows that f is WR at x_0.

$\diamondsuit$

If we examine the proof of Theorem 8.1, we observe that this theorem may be stated in a more general situation.

Theorem 8.2 *Let $f : X \mapsto I\!R \cup \{+\infty\}$ be a function of the form $f = g \circ F$ in a neighborhood of x_0 for a mapping $F : X \mapsto Y$ which is strictly differentiable at $x_0 \in Domf$, an Asplund space Y and a proper lower semicontinuous function $g : Y \mapsto I\!R \cup \{+\infty\}$ which is WR at $F(x_0)$, satisfying at x_0 the constraint qualification (21) and the topological property (20). Then f is WR at x_0.*

9 Weak-regularity of epi-lipschitz sets

In this section, we give characterizations of weak-regularity of epi-lipschitz sets. First, we recall ([29], [30]) that a set $C \subset X$ is *epi-lipschitz* at x_0 if there exist a direction $d \in X$ and $\varepsilon > 0$ such that

$$C \cap B(x_0, \varepsilon) + tB(d, \varepsilon) \subset C \quad \forall t \in]0, \varepsilon[.$$

Rockafellar showed that when x_0 is in the boundary of C, then C is epi-lipschitz at x_0 iff C can be represented in a neighbourhood of x_0 as the epigraph of a Lipschitz continuous function f or equivalently there an isomorphism A taking values in X such that

$$C \cap B(x_0, r) = A(\text{epi}f) \cap B(x_0, r) \tag{23}$$

where r is a nonnegative real number.
The function f is called a *locally Lipschitz representation* of C at x_0.

Now, we characterize the weak-regularity of epi-lipschitz sets.

Theorem 9.1 *Let $C \subset X$ be an epi-lipschitz set at x_0 belonging to the boundary of C. Then the following assertions are equivalent :*
i) C is weakly regular at x_0 with respect to each $x^* \in N(C, x_0) \setminus \{0\}$;
ii) every locally representation f of C at x_0 is weakly regular at $(u_0, f(u_0))$, where $A(u_0, f(u_0)) = x_0$;
ii) there exists a locally representation f of C at x_0 which is weakly regular at $(u_0, f(u_0))$, where $A(u_0, f(u_0)) = x_0$.

The proof of the theorem is easily obtained from the following lemma.

Lemma 9.1 *Let U be a Banach space and $f : U \mapsto \mathbb{R}$ be a locally lipschitzian function at u_0. Then the following assertions are equivalent :*
i) f is weakly regular at u_0;
ii) epif is weakly regular at $(u_0, f(u_0))$ with respect to each $x^* \in N(epif, (u_0, f(u_0))) \setminus \{0\}$.

Proof of the lemma. *ii)* $\Longrightarrow$ *i)*: It is obvious.
i) $\Longrightarrow$ *ii)*: Let $(u_0^*, -\lambda_0) \in N(\text{epi}f, (u_0, f(u_0))) \setminus \{0\}$. Since f is locally Lipschiz at u_0, then $\lambda_0 > 0$ and hence $\frac{u_0^*}{\lambda_0} \in \partial f(u_0)$. Now the weak regularity of f at u_0 implies that there exists $0 < s < \lambda_0$ such that

$$\forall 0 < \varepsilon < s, \quad \exists \delta > 0; \quad \langle u^*, u - u_0 \rangle \leq f(u) - f(u_0) + \varepsilon \|u - u_0\|$$

whenever $\|u - u_0\| \leq \delta$ and $\|u^* - \frac{u_0^*}{\lambda_0}\| \leq s$ with $u^* \in \partial f(u_0)$.
Choose $s' > 0$ such that

$$\frac{s'}{\lambda_0} \max(\frac{s' + \|u_0^*\|}{\lambda_0}, 1) \leq s.$$

Let $(x, \alpha) \in \text{epi}f \cap B((u_0, f(x_0)), \delta)$ and $(u^*, -\lambda) \in N(\text{epi}f, (u_0, f(u_0))) \cap B((u_0^*, -\lambda_0), s')$. Then $\frac{u^*}{\lambda} \in \partial f(u_0) \cap B(\frac{u_0^*}{\lambda_0}, s)$ and hence

$$\langle u^*, u - u_0 \rangle - \lambda(\alpha - f(u_0)) \leq \varepsilon \|u - u_0\|,$$

which ensures that epi$f$ is weakly regular at $(u_0, f(u_0))$ with respect to $(u_0^*, -\lambda_0)$. $\diamondsuit$

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