ENVELOPES FOR SETS AND FUNCTIONS: REGULARIZATION AND GENERALIZED CONJUGACY

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Abstract. Let $X$ be a vector space and let $\varphi : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be an extended real-valued function. For every function $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$, let us define the $\varphi$-envelope of $f$ by

$$f^\varphi(x) = \sup_{y \in X} \varphi(x - y) - f(y),$$

where $\tau$ denotes the lower subtraction in $\mathbb{R} \cup \{-\infty, +\infty\}$. The main purpose of this paper is to study in great details the properties of the important generalized conjugation map $f \mapsto f^\varphi$. When the function $\varphi$ is closed and convex, $\varphi$-envelopes can be expressed as Legendre-Fenchel conjugates. By particularizing with $\varphi = \frac{1}{p\lambda} \| \cdot \|_p$, for $\lambda > 0$ and $p \geq 1$, this allows us to derive new expressions of the Klee envelopes with index $\lambda$ and power $p$. Links between $\varphi$-envelopes and Legendre-Fenchel conjugates are also explored when $-\varphi$ is closed and convex. The case of Moreau envelopes is examined as a particular case.

Besides the $\varphi$-envelopes of functions, a parallel notion of envelope is introduced for subsets of $X$. Given subsets $\Lambda, C \subset X$, we define the $\Lambda$-envelope of $C$ as $C\Lambda = \bigcap_{x \in C} (x + \Lambda)$. Connections between the transform $C \mapsto C\Lambda$ and the aforementioned $\varphi$-conjugation are investigated.

1. Introduction

Given two topological vector spaces $X, Y$ and a function $c : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$, extending the Legendre-Fenchel conjugacy, Moreau [20, Chapter 14, Section 3] defined, for any function $g : Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ its $c$-conjugacy as the function $g^c : X \to \mathbb{R} \cup \{-\infty, +\infty\}$

$$g^c(x) := \sup_{y \in Y} \left(c(x, y) - g(y)\right)$$

for all $x \in X$; see Section 2 for the (extended) lower subtraction $\tau$. We refer to [4, 6, 7, 9, 17, 20, 27, 35] and the references therein, for various duality results in such a context and for several applications. Given a function $\varphi : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ we will focus on the case $c(x, y) := \varphi(x - y)$ and $Y = X$. Otherwise stated, for a function $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ we will be interested in the function $f^\varphi$, that we call the $\varphi$-envelope of $f$, defined by

$$f^\varphi(x) := \sup_{y \in X} \left(\varphi(x - y) - f(y)\right)$$

for all $x \in X$.

Our first aim in this paper is to study in great details the structure of the transform $f \mapsto f^\varphi$ and provide various properties of $\varphi$-envelopes.

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On the other hand, considering the class $B_X$ of closed balls of a Banach space $X$, Mazur [19] studied some Banach spaces $X$ for which every closed bounded convex subset is the intersection of some subclass of $B_X$; we refer to [10] for a rich survey on the subject. Any such Banach space is actually called in the literature a Banach space with the Mazur intersection property. In his 1983 paper [34] Vial defined strongly convex sets of a normed space as convex sets which are intersections of closed balls with a common radius; sets which are intersections, for a fixed real $r > 0$, of closed balls with radius equal to $r$ are called $r$-strongly convex sets in [34]. This class of convex sets is thoroughly studied by Polovinkin [24] (see also [25] and the references therein). Denoting by $B_X$ the closed unit ball of $X$ centered at zero, any $r$-strongly convex set can be represented in the form

$$\bigcap_{x \in S} (x + rB_X)$$

with some subset $S \subset X$.

So, given a subset $\Lambda$ of the space $X$, our second aim in the paper is to analyze properties of the transform which assigns to each subset $C$ of $X$ the set

$$C^\Lambda := \bigcap_{x \in C} (x + \Lambda).$$

We will also provide the connections between the latter transform and the aforestated transform related to $\varphi$-envelopes.

In Section 2 we recall the lower and upper additions (resp. subtractions), and we also recall various concepts and results in Convex Analysis which will be needed in the paper. Section 3 offers a large list of general properties of $\varphi$-envelopes. Section 4 establishes the connections between $\varphi$-envelopes and the aforementioned transform $C \mapsto C^\Lambda$; many properties of sets which can be represented in this form are also provided. In Section 5 we examine the question whether $\psi = \varphi(-a) - \alpha$ (for some $a \in X$ and $\alpha \in \mathbb{R}$) whenever $\psi$ is a $\varphi$-envelope and $\varphi$ is a $\psi$-envelope. A counterexample is constructed and various sufficient conditions are given. The analogous question is also investigated with sets instead of functions. Section 6 considers additional properties in the case when the function $\varphi$ is either superadditive or subadditive. In Section 7, assuming that $\varphi$ is convex and lower semicontinuous, we provide several links between $\varphi$-envelope of a function and Legendre-Fenchel conjugates of other functions related to $f$. Taking $\varphi$ as a power of the norm, we also provide various results concerning the Klee envelope $\kappa_{\lambda,p}f$ (with index $\lambda$ and power $p$) of a function $f$,

$$\kappa_{\lambda,p}f(x) := \sup_{y \in X} \left( \frac{1}{p\lambda} \|x - y\|^p - f(y) \right) \text{ for all } x \in X.$$ 

Finally in Section 8, assuming that $-\varphi$ is convex and lower semicontinuous, we continue to explore the links between $\varphi$-envelopes and Legendre-Fenchel conjugates. By particularizing with $\varphi = -\frac{1}{p\lambda} \| \cdot \|_p^p$, for $\lambda > 0$ and $p \geq 1$, we obtain several properties of Moreau envelopes with index $\lambda$ and power $p$.

2. Preliminaries

Following Moreau [20], we extend the usual addition on $\mathbb{R}$ to $\mathbb{R} = [-\infty, +\infty]$. We define the upper addition $\hat{+}$ and the lower addition $\hat{-}$ as the laws extending the
usual addition via the following conventions

\[
(-\infty) + (+\infty) = (+\infty) + (-\infty) = +\infty \\
(-\infty) + (+\infty) = (+\infty) + (-\infty) = -\infty.
\]

This leads to introduce the upper subtraction \(\dot{-}\) and the lower subtraction \(\dot{+}\), respectively defined by

\[
s \dot{-} t = s + (-t) \quad \text{and} \quad s \dot{+} t = s + (t) \quad \text{for all} \ s, t \in \mathbb{R}.
\]

Let \(X\) be a vector space; all vector spaces will be real vector spaces. Given two extended real-valued functions \(f, g : X \to \mathbb{R}\), the (Moreau) \(\inf\)-convolution (also called \(\infimal\) convolution) of \(f\) and \(g\) is defined as follows: for every \(x \in X\),

\[
(f \nabla g)(x) = \inf_{y+z=x} \left[ f(y) \dot{+} g(z) \right]
= \inf_{y \in X} \left[ f(y) \dot{+} g(x-y) \right]
= \inf_{z \in X} \left[ f(x-z) \dot{+} g(z) \right].
\]

In a symmetric way, the (Moreau) \(\sup\)-convolution (or \(\supremal\) convolution) of \(f\) and \(g\) is defined by

\[
(f \triangle g)(x) = \sup_{y+z=x} \left[ f(y) \dot{+} g(z) \right]
= \sup_{y \in X} \left[ f(y) \dot{+} g(x-y) \right]
= \sup_{z \in X} \left[ f(x-z) \dot{+} g(z) \right].
\]

For the function \(f\) as above, the set \(\text{dom} f = \{ x \in X, f(x) < +\infty \}\) is called the effective domain of \(f\). We call \(f\) a proper function if \(f(x) < +\infty\) for at least one \(x \in X\), and \(f(x) < -\infty\) for all \(x \in X\), or in other words, if \(\text{dom} f\) is a nonempty set on which \(f\) is finite. The function which is constantly equal to \(+\infty\) (resp. \(-\infty\)) on \(X\) is denoted by \(\omega_X\) (resp. \(-\omega_X\)).

Now assume that \(X\) is a locally convex space; all such spaces in the paper will be Hausdorff. We will denote by \(X^\ast\) the topological dual of \(X\). Then, following again [20] we set

\[
\Gamma(X) := \{ f : X \to \mathbb{R}, f \text{ is a pointwise supremum of a family of continuous} \\
\text{affine functions with slopes in } X^\ast \}
\]

and

\[
\Gamma(X^\ast) := \{ g : X^\ast \to \mathbb{R}, g \text{ is a pointwise supremum of a family of continuous} \\
\text{affine functions with slopes in } X \}.
\]

We denote by \(\Gamma_0(X)\) the set of \(f \in \Gamma(X)\) which differ from \(\omega_X\) and \(-\omega_X\). In the same way, \(\Gamma_0(X^\ast)\) is the set \(\Gamma_0(X^\ast) = \Gamma(X^\ast) \setminus \{ \omega_X, -\omega_X \}\). The classes \(\Gamma_0(X)\) and \(\Gamma_0(X^\ast)\) are respectively characterized by

\[
\Gamma_0(X) = \{ f : X \to \mathbb{R}, f \text{ is closed, convex and proper} \}
= \{ f : X \to \mathbb{R}, f \text{ is } w(X,X^\ast) \text{ closed, convex and proper} \},
\]
The function $f : X \to \mathbb{R}$ is associated, in the duality pairing from $X$ to $X^*$, its Legendre-Fenchel conjugate $f^* : X^* \to \mathbb{R}$ defined by
\[
\forall x^* \in X^*, \quad f^*(x^*) = \sup_{\xi \in X} \langle x^*, \xi \rangle - f(\xi).
\]
In the same way, throughout the paper (unless otherwise stated) the Legendre-Fenchel conjugate of a function $g : X^* \to \mathbb{R}$ defined on the dual space $X^*$ will be taken in the duality pairing from $X^*$ to $X$, that is, $g^* : X \to \mathbb{R}$ is defined on $X$ by
\[
\forall x \in X, \quad g^*(x) = \sup_{\xi \in X^*} \{\langle \xi, x \rangle - g(\xi^*)\}.
\]
The Legendre-Fenchel transform $f \mapsto f^*$ (see, for example, [20]) is known to be a one-to-one mapping from $\Gamma_0(X)$ onto $\Gamma_0(X^*)$. For any $f \in \Gamma_0(X)$ one has $f = f^{**}$ and for any $g \in \Gamma_0(X^*)$ one has $g = g^{**}$, see for example [1, 8, 20].

Given a set $C \subset X$, we denote as usual by $\delta_C$ the indicator function of $C$, i.e., $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ if $x \notin C$. The support function $\sigma_C : X^* \to \mathbb{R}$ of $C$ is defined by
\[
\forall x^* \in X^*, \quad \sigma_C(x^*) = \sup_{\xi \in C} \langle x^*, \xi \rangle,
\]
so $\sigma_C$ coincides with the Legendre-Fenchel conjugate of $\delta_C$. For a nonempty cone $K \subset X$, the support function $\sigma_K$ is equal to the indicator function of the polar cone $K^\circ$ of $K$ defined by
\[
K^\circ = \{x^* \in X^*, \langle x^*, x \rangle \leq 0 \text{ for all } x \in K\}.
\]

For a set $C \subset X$, we denote by $\co(C)$ (resp. $\overline{\co}(C)$) the convex hull (resp. closed convex hull) of $C$. The $w(X^*, X)$-closed convex hull of a set $D \subset X^*$ is denoted by $\overline{\co}^{w^*}(D)$. For a function $f : X \to \mathbb{R}$, its convex hull $\co(f)$ (resp. lower semicontinuous convex hull $\overline{\co}(f)$) is the greatest convex (resp. lower semicontinuous convex) function less or equal to $f$. The $w(X^*, X)$-lower semicontinuous convex hull of a function $g : X^* \to \mathbb{R}$ is denoted by $\overline{\co}^{w^*}(g)$.

If $f \in \Gamma_0(X)$ and if $\varpi \in \text{dom } f$, the recession function $f^\infty$ is defined by
\[
\forall u \in X, \quad f^\infty(u) = \lim_{t \to +\infty} \frac{f(\varpi + tu) - f(\varpi)}{t} = \sup_{t \geq 0} \frac{f(\varpi + tu) - f(\varpi)}{t}.
\]
The function $f^\infty : X \to \mathbb{R} \cup \{+\infty\}$ does not depend on the point $\varpi \in \text{dom } f$ since it is also given by
\[
\forall u \in X, \quad f^\infty(u) = \sup_{x \in \text{dom } f} (f(x+u) - f(x)).
\]
The function $f^\infty$ satisfies $f^\infty \in \Gamma_0(X)$, it is positively homogeneous and we have $f^\infty = \sigma_{\text{dom } f}$. Given a closed convex set $C \subset X$ and $\varpi \in C$, the recession cone $C^\circ$ is defined by
\[
C^\circ = \{u \in X, \quad \varpi + tu \in C \text{ for all } t \geq 0\}.
\]
The set $C^\circ$ does not depend on $\varpi \in C$ and is also given by
\[
C^\circ = \{u \in X, u + C \subset C\}.
\]
It follows from the definition that $C^\infty$ is a closed convex cone and we have $\delta_{C^\infty} = (\delta_C)^\infty$.

For more details on recession analysis, see, for example, [1, 2, 13, 28].

Let us end these preliminaries with the subdifferential of convex analysis. We recall that the subdifferential $\partial f(x)$ of a convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ at $x \in \text{dom } f$ is the set

$$\partial f(x) = \{\xi^* \in X^*, f(y) \geq f(x) + \langle \xi^*, y-x \rangle \text{ for every } y \in X\}.$$  

When $x \notin \text{dom } f$, then $\partial f(x) = \emptyset$ by convention. The domain and the range of the operator $\partial f : X \rightrightarrows X^*$ are respectively given by

$$\text{dom}(\partial f) = \{x \in X, \partial f(x) \neq \emptyset\} \text{ and } \text{Rge}(\partial f) = \{x^* \in X^*, \exists x \in X, x^* \in \partial f(x)\}.$$  

If $f \in \Gamma_0(X)$, the subdifferentials of $f$ and $f^*$ are connected through the following relation

$$x^* \in \partial f(x) \iff x \in \partial f^*(x^*),$$  

for all $x \in X$ and $x^* \in X^*$. For further details, the reader is referred to the classical textbooks on convex analysis, see for example [13, 28].

### 3. Definitions. General properties

Let $X$ be a vector space. For functions $\varphi : X \to [0, \infty]$ and $f : X \to [\mathbb{R}]$, the $\varphi$-envelope of $f$ is defined as follows:

$$\forall x \in X, \quad f^\varphi(x) = \sup_{y \in X} \{\varphi(x-y) - f(y)\} = \sup_{z \in X} \{\varphi(z) - f(x-z)\}.$$  

A function $g : X \to \mathbb{R}$ is said to be a $\varphi$-envelope if there exists $f : X \to \mathbb{R}$ such that $g = f^\varphi$. It is immediate to check that for every function $f : X \to \mathbb{R}$,

$$f^{\varphi_X} = -\varphi_X, \quad \text{while } f_{\varphi_X} = \begin{cases} \varphi_X & \text{if } f \neq \varphi_X \\ -\varphi_X & \text{if } f = \varphi_X. \end{cases}$$  

It ensues that the unique $(-\varphi_X)$-envelope is the function $-\varphi_X$, while the $\varphi_X$-envelopes are $\pm \varphi_X$. The function $f^\varphi$ can be expressed via the inf-convolution and sup-convolution operators

$$f^\varphi = \varphi \triangle (-f) = - ((-\varphi) \triangledown f).$$  

The roles played by $\varphi$ and $f$ in the definition of $f^\varphi$ are opposite in the sense that

$$(-\varphi)^{(-f)} = (-f) \triangle (-(-\varphi)) = (-f) \triangle \varphi = f^\varphi.$$  

The definition of $f^\varphi$ is closely connected to the deconvolution operation. For any $g, h : X \to \mathbb{R}$, the deconvolution of $g$ and $h$ is the function $g \otimes h$ defined by

$$(g \otimes h)(x) = \sup_{y \in X} (g(y) - h(z)), \quad \text{for every } x \in X.$$  

Denoting by $h_-$ the function defined by $h_-(x) = h(-x)$ for every $x \in X$, we deduce immediately from the above definition that

$$g \otimes h = g \triangle (-h_-) = (h_-)^g.$$  

It ensues that for any $f, \varphi : X \to [\mathbb{R}]$,

$$f^\varphi = \varphi \otimes f_-.$$  

The deconvolution operation has been studied in details by many authors, see for example [3, 12, 14, 36].
Following the terminology of Moreau [21], we call $\varphi$-elementary function a function of the form $\varphi(\cdot - y) + \lambda$ with $y \in X$ and $\lambda \in \mathbb{R}$. By using a generalized conjugacy argument, one can show that for any $\varphi$, $f : X \to \overline{\mathbb{R}}$

$(f^\varphi)^\varphi$ is the upper envelope of the $\varphi$-elementary functions that minorize $f$, (6) see for example [21, Section 4] and [30, Section 11.L]. It can easily be deduced the following characterization of $\varphi$-envelopes: for any $g : X \to \mathbb{R}$,

$$g \text{ is the upper envelope of a family of } \varphi\text{-elementary functions}$$

(7)

$$g = (g^\varphi)^\varphi$$

(8)

$$g \text{ is a } \varphi\text{-envelope.}$$

(9)

The expression of the double envelope $(g^\varphi)^\varphi$ can be developed as follows

$$(g^\varphi)^\varphi = \varphi \triangle (-g^\varphi) = \varphi \triangle (- (\varphi \triangle (-g))) = \varphi \triangle ((-\varphi) \nabla g).$$

By using the deconvolution operation, we obtain

$$(g^\varphi)^\varphi = \varphi \odot (\varphi \triangle (-g)) = \varphi \odot (\varphi \odot g).$$

From the equivalence (8) $\Leftrightarrow$ (9), we deduce that

$$g \text{ is a } \varphi\text{-envelope}$$

(10)

$$g = \varphi \triangle ((-\varphi) \nabla g)$$

$$g = \varphi \odot (\varphi \odot g).$$

Now let $f, \psi : X \to \overline{\mathbb{R}}$. Following the terminology of Martinez-Legaz & Penot [18], the function $f$ is said to be (exactly) $\psi$-regular if $f = (f \odot \psi) \nabla \psi$. By taking the opposite in each member of the equality (10), we find

$$-g = (-\varphi) \nabla ((-\varphi) \triangle (-g)) = (-\varphi) \nabla (((-g) \odot (-\varphi)).$$

In view of the above equivalences, this implies that

$$g \text{ is a } \varphi\text{-envelope } \iff -g \text{ is } (-\varphi)\text{-regular in the sense of [18].}$$

We denote by $\mathcal{E}^\varphi(X)$, or $\mathcal{E}^\varphi$ if there is no risk of confusion, the set of $\varphi$-envelopes and by $F_\varphi : \mathcal{E}^\varphi \to \mathcal{E}^\varphi$ the map defined by $F_\varphi(f) = f^\varphi$ for every $f \in \mathcal{E}^\varphi$. The equivalence (8) $\Leftrightarrow$ (9) says that $F_\varphi \circ F_\varphi = Id_{\mathcal{E}^\varphi}$ and $F_\varphi \circ F_\varphi = Id_{\mathcal{E}^\varphi}$, otherwise stated we have:

**Proposition 3.1.** The map $F_\varphi : \mathcal{E}^\varphi \to \mathcal{E}^\varphi$ is bijective and $(F_\varphi)^{-1} = F_{-\varphi}$. 

As a consequence of the previous proposition, if $\varphi$ is even the map $F_\varphi : \mathcal{E}^\varphi \to \mathcal{E}^\varphi$ is bijective and $(F_\varphi)^{-1} = F_{-\varphi}$.

Let us now state several general properties of $\varphi$-envelopes.
Proposition 3.2. Let $X$ be a vector space and let $\varphi : X \to \mathbb{R}$.

(i) For every function $f : X \to \overline{\mathbb{R}}$ and every $a \in X$ and $\beta \in \mathbb{R}$, we have $(f(-a) - \beta)\varphi = f\varphi(-a) + \beta$. If $g \in \mathcal{E}^\varphi$, then $g(-a) + \beta \in \mathcal{E}^\varphi$ for every $a \in X$ and $\beta \in \mathbb{R}$.

(ii) Given a family $(f_i)_{i \in I}$ of functions $f_i : X \to \overline{\mathbb{R}}$, we have $(\inf_{i \in I} f_i)\varphi = \sup_{i \in I} f_i\varphi$. If $g = \sup_{i \in I} g_i$ with $g_i \in \mathcal{E}^\varphi$ for every $i \in I$, then $g \in \mathcal{E}^\varphi$.

(iii) For $f_1, f_2 : X \to \overline{\mathbb{R}}$, we have $(f_1 \vee f_2)^\varphi = f_1^\varphi \vee f_2^\varphi$. Let $g, h : X \to \overline{\mathbb{R}}$. If $h \in \mathcal{E}^\varphi$ and $g \in \mathcal{E}^\varphi$, then $h \in \mathcal{E}^\varphi$. Otherwise stated, if $g \in \mathcal{E}^\varphi$, then $\mathcal{E}^\varphi \subset \mathcal{E}^\varphi$.

(iv) For $f : X \to \overline{\mathbb{R}}$, we have $(f\varphi)_- = f_-\varphi$. As a consequence, $g \in \mathcal{E}^\varphi$ if and only if $g_- \in \mathcal{E}^\varphi$.

Proof. (i) Let $a \in X$ and $\beta \in \mathbb{R}$. For every $x \in X$, we have

\[
(f(-a) - \beta)\varphi(x) = \sup_{y \in X} \{\varphi(x - y) - f(y - a) + \beta\} = \sup_{y' \in X} \{\varphi(x - a - y') - f(y') + \beta\} = f\varphi(x - a) + \beta.
\]

For the second assertion of (i), it suffices to apply the first part with $g = f\varphi$.

(ii) By definition, we have

\[
(\inf_{i \in I} f_i)^\varphi = \varphi \bigtriangleup \left(-\inf_{i \in I} f_i\right) = \varphi \bigtriangleup \sup_{i \in I} (-f_i) = \sup_{i \in I} (\varphi \bigtriangleup (-f_i)) = \sup_{i \in I} f_i^\varphi,
\]

see for example [20].

Now assume that $g = \sup_{i \in I} g_i$ with $g_i \in \mathcal{E}^\varphi$ for every $i \in I$. Then, for each $i \in I$, we have $g_i = f_i^\varphi$ for some $f_i$. It ensues that $g = \sup_{i \in I} f_i^\varphi = (\inf_{i \in I} f_i)^\varphi$, hence $g \in \mathcal{E}^\varphi$.

(iii) By definition, we have

\[
f_1^\varphi \vee f_2^\varphi = \varphi \bigtriangleup (-f_1) \vee \varphi \bigtriangleup (-f_2) = \varphi \bigtriangleup ((-f_1) \bigtriangleup (-f_2)) = \varphi \bigtriangleup (-f_2 \bigtriangleup f_1) = (f_2 \bigtriangleup f_1)^\varphi = (f_1 \bigtriangleup f_2)^\varphi.
\]

Now assume that $h \in \mathcal{E}^\varphi$ and $g \in \mathcal{E}^\varphi$. Then there exist $f_1, f_2 : X \to \overline{\mathbb{R}}$ such that $h = f_1^\varphi$ and $g = f_2^\varphi$. It ensues that $h = f_1^\varphi \vee f_2^\varphi = (f_1 \bigtriangleup f_2)^\varphi$, hence $h \in \mathcal{E}^\varphi$.

(iv) For every $x \in X$, we have

\[
(f\varphi)_-(x) = \sup_{y \in X} \{\varphi(-x - y) - f(y)\} = \sup_{\xi \in X} \{\varphi(-x + \xi) - f(-\xi)\} = \sup_{\xi \in X} \{\varphi_-(x - \xi) - f_-(\xi)\} = f_-\varphi_-(x).
\]

If $g \in \mathcal{E}^\varphi$, there exists $f : X \to \overline{\mathbb{R}}$ such that $g = f\varphi$. It ensues that $g_- = (f\varphi)_- = (f_-)\varphi_-$, hence $g_- \in \mathcal{E}^\varphi$. The proof of the reverse assertion is identical. \qed
In the next proposition, we show that the \( \varphi \)-envelope of a continuous linear functional is affine and we characterize the elements of \( \mathcal{E}^\varphi \) that are linear.

**Proposition 3.3.** Let \( X \) be a locally convex space. Let \( \varphi : X \to \mathbb{R} \) and \( \xi^* \in X^* \). Then we have

\[
(i) \quad \langle \xi^*, \cdot \rangle^\varphi = -\langle \xi^*, \cdot \rangle + (-\varphi)^*(\xi^*).
(ii) \quad \text{If } \varphi \neq -\omega_X, \text{ the following equivalence holds}
\]

\[\langle \xi^*, \cdot \rangle \in \mathcal{E}^\varphi \iff \xi^* \in -\text{dom}(-\varphi)^*.\]

**Proof.** (i) For every \( x \in X \), we have

\[
\langle \xi^*, \cdot \rangle^\varphi(x) = \sup_{y \in X} \{ \varphi(y) - \langle \xi^*, x - y \rangle \}
= -\langle \xi^*, x \rangle + (-\varphi)^*(\xi^*). \tag{11}
\]

First assume that \((-\varphi)^*(-\xi^*) = +\infty \). Then we have \( g^\varphi = \omega_X \), thus implying that \( (g^\varphi)^{-}\varphi = -\omega_X \). It ensues that \((g^\varphi)^{-}\varphi \neq g\), which shows that \( g \notin \mathcal{E}^\varphi \) according to the equivalence \((7) \iff (8)\). Now assume that \((-\varphi)^*(-\xi^*) < +\infty \). Observe that

\[
(-\varphi)^*(-\xi^*) = -\infty \implies \sup_{x \in X} \langle -\xi^*, x \rangle + \varphi(x) = -\infty \implies \varphi = -\omega_X,
\]

which is impossible by assumption. Thus \((-\varphi)^*(-\xi^*) \in \mathbb{R}\), we deduce from \((11)\) that

\[
(g^\varphi)^{-}\varphi = \langle \xi^*, \cdot \rangle + (-\varphi)^*(-\xi^*) - (-\varphi)^*(-\xi^*) = \langle \xi^*, \cdot \rangle = g,
\]

and therefore \( g \in \mathcal{E}^\varphi \). \( \square \)

For every set \( C \subset X \), let us set

\[
\Sigma_C = \{ f : X \to \mathbb{R}, \text{dom} f \subset C \} \quad \text{and} \quad \Sigma^*_C = \{ f^*, f \in \Sigma_C \}.
\]

We adopt the same notations \( \Sigma_D \) and \( \Sigma^*_D \) for a subset \( D \subset X^* \).

**Theorem 3.1.** Let \( X \) be a locally convex space and let \( \varphi : X \to \mathbb{R} \) be such that \( \varphi \neq -\omega_X \). For every subset \( D \subset X^* \), the following assertions are equivalent

\[
(i) \quad \Sigma^*_D \subset \mathcal{E}^\varphi;
(ii) \quad \{ f \in \Gamma_0(X), \text{dom} f^* \subset D \} \subset \mathcal{E}^\varphi;
(iii) \quad D \subset -\text{dom}(-\varphi)^*.
\]

**Proof.** (i) \( \Rightarrow \) (ii) Let \( D \subset X^* \). Observe that

\[
\{ f \in \Gamma_0(X), \text{dom} f^* \subset D \} = \{ g^*, \text{dom} g \subset D \ \text{and} \ g \in \Gamma_0(X) \} \subset \{ g^*, \text{dom} g \subset D \} = \Sigma^*_D.
\]

The implication \((i) \Rightarrow (ii)\) follows immediately.

(ii) \( \Rightarrow \) (iii) Assume that

\[
\{ f \in \Gamma_0(X), \text{dom} f^* \subset D \} \subset \mathcal{E}^\varphi. \tag{12}
\]

Let \( \xi^* \in D \). Observe that \( \langle \xi^*, \cdot \rangle \in \Gamma_0(X) \) and that

\[
\text{dom}((\langle \xi^*, \cdot \rangle)^*) = \text{dom} \delta_{\langle \xi^*, \cdot \rangle} = \{ \xi^* \} \subset D,
\]

then we have
hence $\langle \xi^*, \cdot \rangle \in \mathcal{E}^*$ in view of (12). We then deduce from Proposition 3.3 (ii) that $\xi^* \in \overline{\text{dom}}(-\varphi)^\circ$. Since this is true for every $\xi^* \in D$, we conclude that $D \subset \overline{\text{dom}}(-\varphi)^\circ$. 

(iii) $\Rightarrow$ (i) Now assume that $D \subset \overline{\text{dom}}(-\varphi)^\circ$ and let $f \in \Sigma_D$. There exists $g : X^* \to \mathbb{R}$ such that $f = g^*$ and $\text{dom} \, g \subset D$. The definition of the Legendre-Fenchel conjugate yields

$$f = \sup_{\xi^* \in X^*} \left\{ \langle \xi^*, \cdot \rangle - g(\xi^*) \right\}$$
$$= \sup_{\xi^* \in \text{dom} \, g} \left\{ \langle \xi^*, \cdot \rangle - g(\xi^*) \right\}. \quad (13)$$

Recalling that $\text{dom} \, g \subset D \subset \overline{\text{dom}}(-\varphi)^\circ$, we deduce from Proposition 3.3 (ii) that the linear function $\langle \xi^*, \cdot \rangle$ is a $\varphi$-envelope for every $\xi^* \in \text{dom} \, g$. In view of Proposition 3.2 (i), the affine function $\langle \xi^*, \cdot \rangle - g(\xi^*)$ is also a $\varphi$-envelope for every $\xi^* \in \text{dom} \, g$. Coming back to formula (13), we infer from Proposition 3.2 (ii) that $f$ is a $\varphi$-envelope as a supremum of $\varphi$-envelopes. Finally, we have shown that $f \in \mathcal{E}^*$, which proves the inclusion $\Sigma_D \subset \mathcal{E}^*$.

Given a set $D \subset X^*$, the following result explores the links between the class $\Sigma_D$ and the class of functions $f \in \Gamma_0(X)$ satisfying $\text{dom} \, f^* \subset D$. When the set $D$ is $w(X^*, X)$-closed and convex, these classes can be characterized via the support function of $D$.

**Proposition 3.4.** Let $X$ be a locally convex space and let $D$ be a nonempty subset of $X^*$.

(i) We have

$$\{f \in \Gamma_0(X), \text{dom} \, f^* \subset D\} \cup \{\omega_X, -\omega_X\} \subset \Sigma_D, \quad (14)$$

$$\Sigma_D \subset \{f \in \Gamma_0(X), \text{dom} \, f^* \subset \mathcal{W}w^*(D)\} \cup \{\omega_X, -\omega_X\}. \quad (15)$$

As a consequence, if the set $D \subset X^*$ is $w(X^*, X)$-closed and convex, the following equality holds true

$$\Sigma_D = \{f \in \Gamma_0(X), \text{dom} \, f^* \subset D\} \cup \{\omega_X, -\omega_X\}. \quad (16)$$

(ii) If the set $D \subset X^*$ is $w(X^*, X)$-closed and convex, then

$$\{f \in \Gamma_0(X), \text{dom} \, f^* \subset D\} = \{f \in \Gamma_0(X), f^\infty \leq \sigma_D\}$$
$$= \{f \in \Gamma_0(X), f(y) \leq f(x) + \sigma_D(y-x), \forall x, y \in X\}. \quad (17)$$

Proof. (i) We have already shown the inclusion $\{f \in \Gamma_0(X), \text{dom} \, f^* \subset D\} \subset \Sigma_D$, see the proof of Theorem 3.1. On the other hand, we always have $-\omega_X \in \Sigma_D^\circ$. Since $D \neq \emptyset$, we also have $\omega_X \in \Sigma_D^\circ$. This proves the inclusion (14). Let us now establish (15). Assume that $f \in \Sigma_D^\circ$. There exists $g : X^* \to \mathbb{R}$ such that $\text{dom} \, g \subset D$ and $f = g^*$. We distinguish the cases $\mathcal{W}w^*(g)$ proper and $\mathcal{W}w^*(g)$ improper. If $\mathcal{W}w^*(g) = \omega_X^*$, we have $g = \omega_X^*$, hence $f = -\omega_X$. If $\mathcal{W}w^*(g)$ takes the value $-\infty$, we infer that $g^* = (\mathcal{W}w^*(g))^* = \omega_X$, whence $f = \omega_X$. Let us now assume that $\mathcal{W}w^*(g) \in \Gamma_0(X^*)$. It ensues that $f = g^* = (\mathcal{W}w^*(g))^* \in \Gamma_0(X)$. This implies in turn that $f^* = \mathcal{W}w^*(g)$, thus

$$\text{dom} \, f^* = \text{dom} \, (\mathcal{W}w^*(g)) \subset \mathcal{W}w^*(\text{dom} \, g) \subset \mathcal{W}w^*(D),$$

which ends the proof of (15). When the set $D$ is $w(X^*, X)$-closed and convex, equality (16) is an immediate consequence of the inclusions (14)-(15).
(ii) Assuming that the set $D$ is $w(X^*, X)$-closed and convex, we have $\text{dom } f^* \subset D$ if and only if $\sigma_{\text{dom } f^*} \leq \sigma_D$. Recalling that $\sigma_{\text{dom } f^*} = f^\infty$ (see section 2), we derive equality (17). Since $f^\infty = \sup_{x \in \text{dom } f} (f(\cdot + x) - f(x))$, we deduce in turn equality (18).

Remark 3.1. In general, the inclusions (14) and (15) are strict, as will be shown in Example 7.1.

If $X$ is a Banach space and if the set $D \subset X^*$ is closed, the class of functions $f \in \Gamma_0(X)$ satisfying $\text{dom } f^* \subset D$ can be expressed via the subdifferential of $f$.

**Proposition 3.5.** Let $X$ be a Banach space and let $D$ be a closed subset of $X^*$. Then we have

$$
\{ f \in \Gamma_0(X), \text{dom } f^* \subset D \} = \{ f \in \Gamma_0(X), \partial f(x) \subset D \text{ for all } x \in X \}.
$$

**Proof.** Let us first state as a lemma the following direct consequence of the Brønsted-Rockafellar theorem (see [5, Theorem 2]) concerning the conjugate of a function in $\Gamma_0(X)$.

**Lemma 3.1** (See Theorem 2 in [5]). If $X$ is a Banach space and if $f \in \Gamma_0(X)$, then $\text{cl}(\text{dom } f^*) = \text{cl}(\text{Rge}(\partial f))$.

Assume that the set $D \subset X^*$ is closed. From Lemma 3.1, we have for every $f \in \Gamma_0(X)$

$$
\text{dom } f^* \subset D \iff \text{Rge}(\partial f) \subset D \\
\iff \partial f(x) \subset D \text{ for all } x \in X.
$$

The announced equality follows immediately.

Applying Theorem 3.1 with particular sets $D$, we obtain the following corollaries.

**Corollary 3.1.** Let $X$ be a locally convex space and let $\varphi : X \to \mathbb{R}$ be such that $\varphi \neq -\omega_X$. Then the following equivalence holds

$$
\Gamma(X) \subset \mathcal{E}^\varphi \iff \text{dom } (\varphi)^* = X^*.
$$

**Proof.** It suffices to take $D = X^*$ in the equivalence (i) $\iff$ (iii) of Theorem 3.1.

**Remark 3.2.** Under the assumption $\text{dom } (\varphi)^* = X^*$, the function $\varphi$ cannot be convex (see hereafter). Therefore the set $\mathcal{E}^\varphi$ is strictly larger than $\Gamma(X)$, since it contains the nonconvex function $\varphi$.

If $\text{dom } (\varphi)^* = X^*$, we have $(\varphi)^*(0) < +\infty$. Recalling that $(\varphi)^*(0) = \sup \varphi$, we deduce that the function $\varphi$ is bounded from above on the whole space $X$. If moreover the function $\varphi$ is convex, we infer from a classical result that it is constant, say $\varphi \equiv \beta$ for some $\beta \in \mathbb{R}$. It ensues that $(\varphi)^* = \beta + \delta_{\{0\}}$, hence $\text{dom } (\varphi)^* = \{0\}$, a contradiction. This confirms that functions $\varphi$ with $\text{dom } (\varphi)^* = X^*$ cannot be convex.

Given a set $K \subset X$, recall that a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be $K$-nonincreasing (resp. $K$-nondecreasing) if $f(y) \leq f(x)$ (resp. $f(y) \geq f(x)$) for all $x, y \in X$ such that $y - x \in K$.

**Corollary 3.2.** Let $X$ be a locally convex space. Let $K \subset X$ be a closed convex cone and let $\varphi : X \to \mathbb{R}$ be such that $\varphi \neq -\omega_X$. Then the set $\mathcal{E}^\varphi$ contains all the functions of $\Gamma_0(X)$ which are $K$-nonincreasing if and only if $-K^\circ \subset \text{dom } (\varphi)^*$. 
Proof. Take \( D = K^\circ \) in the equivalence \((ii) \Leftrightarrow (iii)\) of Theorem 3.1 to obtain that
\[
\{ f \in \Gamma_0(X), \text{dom } f^* \subset K^\circ \} \subset \mathcal{E}\varphi \quad \Leftrightarrow \quad K^\circ \subset -\text{dom } (-\varphi)^*
\quad \Leftrightarrow \quad -K^\circ \subset \text{dom } (-\varphi)^*.
\] (19)

On the other hand, observe by (18) that for \( f \in \Gamma_0(X), \)
\[
\text{dom } f^* \subset K^\circ \quad \Leftrightarrow \quad f(y) \leq f(x) + \sigma_{K^\circ}(y - x) \quad \text{for all } x, y \in X
\quad \Leftrightarrow \quad f(y) \leq f(x) + \delta_{K}(y - x) \quad \text{for all } x, y \in X
\quad \Leftrightarrow \quad f \text{ is } K\text{-nonincreasing.}
\] (20)
The announced equivalence then follows immediately from (19) and (20). \( \Box \)

In the sequel, when \( X \) is a normed space we will denote by \( B_X \) (resp. \( B_{X^*} \)) the closed unit ball of \( X \) (resp. \( X^* \)).

**Corollary 3.3.** Let \( (X, \| \cdot \|) \) be a normed space. Let a real \( k \geq 0 \) and let \( \varphi : X \to \mathbb{R} \) be such that \( \varphi \neq -\omega_X \). Then the set \( \mathcal{E}\varphi \) contains all the functions of \( \Gamma_0(X) \) which are \( k \)-Lipschitz continuous on \( X \) if and only if \( kB_{X^*} \subset \text{dom } (-\varphi)^* \).

**Proof.** Take \( D = kB_{X^*} \) in the equivalence \((ii) \Leftrightarrow (iii)\) of Theorem 3.1 to obtain that
\[
\{ f \in \Gamma_0(X), \text{dom } f^* \subset kB_{X^*} \} \subset \mathcal{E}\varphi \quad \Leftrightarrow \quad kB_{X^*} \subset -\text{dom } (-\varphi)^*
\quad \Leftrightarrow \quad kB_{X^*} \subset \text{dom } (-\varphi)^*.
\] (21)

Then observe by (18) that for \( f \in \Gamma_0(X), \)
\[
\text{dom } f^* \subset kB_{X^*} \quad \Leftrightarrow \quad f(y) \leq f(x) + k\| y - x \| \quad \text{for all } x, y \in X
\quad \Leftrightarrow \quad f \text{ is } k\text{-Lipschitz on } X,
\] (22)
where the last equivalence is obtained by reversing the roles of \( x \) and \( y \). The announced equivalence then follows immediately from (21) and (22). \( \Box \)

4. Equivalence between functions and sets

Recall that for \( f : X \to \mathbb{R} \), the epigraph (resp. hypograph) of \( f \) is defined by
\[
\text{epi } f = \{ (x, \lambda) \in X \times \mathbb{R}, f(x) \leq \lambda \} \quad \text{(resp. hypo } f = \{ (x, \lambda) \in X \times \mathbb{R}, f(x) \geq \lambda \}).
\]
The strict epigraph and strict hypograph of \( f \) are obtained by replacing the above inequalities with strict inequalities
\[
\text{epi}_s f = \{ (x, \lambda) \in X \times \mathbb{R}, f(x) < \lambda \} \quad \text{(resp. hypo}_s f = \{ (x, \lambda) \in X \times \mathbb{R}, f(x) > \lambda \}).
\]
The following lemma gives a geometrical interpretation for the inf-convolution and sup-convolution operations. Assertion \((i)\) is well known. For completeness and convenience of the reader we provide a proof of \((ii)\).

**Lemma 4.1.** Let \( X \) be a vector space and let \( f, g : X \to \mathbb{R} \). Then we have
\[
\begin{align*}
\text{(i) } & \text{epi}_s (f \triangledown g) = \text{epi}_s f + \text{epi}_s g, \\
\text{(ii) } & \text{hypo}_s (f \triangle g) = \text{hypo}_s f + \text{hypo}_s g.
\end{align*}
\]
Proof. Point (i) is classical, see for example [20, 30]. Point (ii) is deduced easily from (i) by observing that

\[(x, \lambda) \in \text{hypo}_s f \triangle g \iff (x, -\lambda) \in \text{epi}_s [-f \triangle g)]\]

\[\iff (x, -\lambda) \in \text{epi}_s [-f \triangledown (-g)]\]

\[\iff (x, -\lambda) \in \text{epi}_s (-f) + \text{epi}_s (-g)\]

\[\iff (x, \lambda) \in \text{hypo}_s (f) + \text{hypo}_s (g).\]

\[\square\]

Since \(f^\varphi\) is defined via a sup-convolution operation, we derive the following consequence of Lemma 4.1.

**Proposition 4.1.** Let \(X\) be a vector space and let \(\varphi : X \to \mathbb{R}\).

(i) For every \(f : X \to \mathbb{R}\), we have

\[\text{hypo}_s f^\varphi = \text{hypo}_s (-f) + \text{hypo}_s \varphi\]

\[\text{epi} f^\varphi = \bigcap_{u \in \text{hypo}_s (-f)} u + \text{epi} \varphi.\]

(ii) For every \(g : X \to \mathbb{R}\), the following equivalences hold

\[g \in \mathcal{E}^\varphi \iff \text{hypo}_s g = U + \text{hypo}_s \varphi \quad \text{for some } U \subset X \times \mathbb{R}\]

\[\iff \text{epi} g = \bigcap_{u \in U} u + \text{epi} \varphi \quad \text{for some } U \subset X \times \mathbb{R}.\]

**Proof.** (i) Let \(f, \varphi : X \to \mathbb{R}\). Recalling that \(f^\varphi = \varphi \triangle (-f)\), we deduce from Lemma 4.1 (ii) that

\[\text{hypo}_s f^\varphi = \text{hypo}_s (-f) + \text{hypo}_s \varphi\]

\[= \bigcup_{u \in \text{hypo}_s (-f)} u + \text{hypo}_s \varphi.\]

Taking the complement of each member of the above equality, we infer that

\[\text{epi} f^\varphi = \bigcap_{u \in \text{hypo}_s (-f)} u + \text{epi} \varphi.\]

(ii) Let \(g : X \to \mathbb{R}\). If \(g \in \mathcal{E}^\varphi\), there exists \(f : X \to \mathbb{R}\) such that \(g = f^\varphi\). In view of (i), we obtain that \(\text{hypo}_s g = U + \text{hypo}_s \varphi\) with \(U = \text{hypo}_s (-f)\). Conversely, assume that \(\text{hypo}_s g = U + \text{hypo}_s \varphi\) for some \(U \subset X \times \mathbb{R}\). Then we have

\[\text{hypo}_s g = \bigcup_{(x, \lambda) \in U} (x, \lambda) + \text{hypo}_s \varphi\]

\[= \bigcup_{(x, \lambda) \in U} \text{hypo}_s [\varphi(-x) + \lambda]\]

\[= \text{hypo}_s \left[ \sup_{(x, \lambda) \in U} \varphi(-x) + \lambda \right].\]

Hence we deduce that \(g = \sup_{(x, \lambda) \in U}(\varphi(-x) + \lambda)\), which shows that \(g \in \mathcal{E}^\varphi\). This proves the first equivalence of (ii). For the other equivalence, it suffices to take the complement of the sets arising in each member of the equality concerning \(\text{hypo}_s g\). \[\square\]
Given a set $\Lambda \subset X$, the previous result suggests to consider the class $\mathcal{I}^\Lambda$ of subsets of $X$ defined as follows\(^1\)

\[ \mathcal{I}^\Lambda = \{ C^\Lambda, C \subset X \}, \quad \text{where} \quad C^\Lambda = \bigcap_{x \in C} x + \Lambda. \]

By convention\(^2\), we take $\emptyset^\Lambda = \bigcap_{x \in \emptyset} x + \Lambda = X$ for every set $\Lambda \subset X$. This implies that $X \in \mathcal{I}^\Lambda$ for every $\Lambda \subset X$. It is immediate to check that $\mathcal{I}^X = \{ X \}$, while $\mathcal{I}^\emptyset = \{ \emptyset, X \}$. A set $D \subset X$ belongs to the class $\mathcal{I}^\Lambda$ if it is equal to some intersection of translated sets from $\Lambda$. It ensues immediately that the class $\mathcal{I}^\Lambda$ is stable under translation and intersection.

**Example 4.1.** Take $r > 0$ and $\Lambda = r B_X$. The class $\mathcal{I}^r B_X$ corresponds to the class studied by Vial [34] under the terminology of $r$-strongly convex sets. More generally, for a closed convex set $\Lambda \subset X$, the sets of the form $C^\Lambda$ are called $\Lambda$-strongly convex. The $\Lambda$-strongly convex sets are thoroughly studied by Polovinkin [24], under an additional condition on the set $\Lambda$ (which is assumed to be generating, see [24] for more details).

The definition of $C^\Lambda$ is directly linked to the star-difference of sets. For every $C_1, C_2 \subset X$, the star-difference of $C_1$ with $C_2$ is the set $C_1 \ast - C_2$ given by

\[ C_1 \ast - C_2 = \bigcap_{x \in C_2} C_1 - x. \]

We deduce immediately from the above definition that $C^\Lambda = \Lambda \ast - ( - C)$ for every $C$, $\Lambda \subset X$. The star-difference of sets was used in [26] in the context of differential games. See also [12] for the links between the star-difference of sets and the deconvolution operation, also called epigraphical star-difference.

Given $C \subset X$ and $\Lambda \subset X$, the next proposition gives several expressions for the set $C^\Lambda$.

**Proposition 4.2.** Let $X$ be a vector space. For any sets $C \subset X$ and $\Lambda \subset X$, we have

1. $C^\Lambda = \{ x \in X, x - C \subset \Lambda \} = \{ x \in X, C \subset x - \Lambda \}$;
2. $X \setminus C^\Lambda = C + (X \setminus \Lambda)$ or equivalently $C^{X \setminus \Lambda} = X \setminus (C + \Lambda)$.
3. $(X \setminus \Lambda)^{X \setminus C} = C^\Lambda$.

**Proof.** (i) It suffices to observe that

\[ x \in C^\Lambda \iff \forall u \in C, x \in u + \Lambda \]
\[ \iff \forall u \in C, x - u \in \Lambda \]
\[ \iff x - C \subset \Lambda \]
\[ \iff C \subset x - \Lambda. \]

(ii) From the definition of $C^\Lambda$, we deduce immediately that

\[ X \setminus C^\Lambda = \bigcup_{u \in C} u + (X \setminus \Lambda) = C + (X \setminus \Lambda), \]

---

\(^1\)We draw the attention of the reader to the fact that the notation $C^\Lambda$ must not be confused with that of the set of maps from $\Lambda$ into $C$.

\(^2\)In particular, we obtain $\emptyset^\emptyset = X$. 
which is the first equality in (ii). From this equality with \( X \setminus \Lambda \) in place of \( \Lambda \), we obtain that \( X \setminus C^{X \setminus \Lambda} = C + \Lambda \), or equivalently \( C^{X \setminus \Lambda} = X \setminus (C + \Lambda) \).

(iii) We infer from the previous assertion that
\[
X \setminus [(X \setminus \Lambda)^{X \setminus C}] = (X \setminus \Lambda) + C = X \setminus C^{\Lambda},
\]
whence the equality \( (X \setminus \Lambda)^{X \setminus C} = C^{\Lambda} \).

The elements \( D \) of \( I^{\Lambda} \) can be characterized by the equality \( (D \setminus \Lambda)^{\Lambda} = D \). This is the subject of the next proposition.

**Proposition 4.3.** Let \( X \) be a vector space and let \( \Lambda \subset X \). For any set \( D \subset X \), the set \( (D \setminus \Lambda)^{\Lambda} \) is the smallest element of \( I^{\Lambda} \) containing the set \( D \). As a consequence, the following equivalence holds true
\[
D \in I^{\Lambda} \iff (D \setminus \Lambda)^{\Lambda} = D.
\]

**Proof.** Let \( S \) be the subset of \( X \) defined by
\[
S = \bigcap_{x \in X, x + \Lambda \supset D} x + \Lambda.
\]

We clearly have \( S \in I^{\Lambda} \) and \( S \supset D \). Now let any \( S' \in I^{\Lambda} \) with \( S' \supset D \). By definition, there exists some \( C \subset X \) such that \( S' = \bigcap_{x \in C} x + \Lambda \). The inclusion \( S' \supset D \) implies that \( x + \Lambda \supset D \) for every \( x \in C \) and therefore
\[
S' = \bigcap_{x \in C} x + \Lambda \supset \bigcap_{x \in X, x + \Lambda \supset D} x + \Lambda = S.
\]

This proves that the set \( S \) is the smallest element of \( I^{\Lambda} \) containing \( D \). Recall now from Proposition 4.2 (i) that condition \( x + \Lambda \supset D \) is equivalent to \( x \in D^{-\Lambda} \). We deduce that
\[
S = \bigcap_{x \in D^{-\Lambda}} x + \Lambda = (D^{-\Lambda})^{\Lambda}.
\]

This finishes the proof of the first assertion. The second assertion is an immediate consequence of the first one. \( \square \)

Let us write the expression of the double envelope \( (D^{-\Lambda})^{\Lambda} \) by using the star-difference operation
\[
(D^{-\Lambda})^{\Lambda} = \Lambda \ast ((-D)^{\Lambda}) = \Lambda \ast ((\Lambda \ast D)).
\]

In view of Proposition 4.2, the complement of the set \( (D^{-\Lambda})^{\Lambda} \) can be expressed as
\[
X \setminus (D^{-\Lambda})^{\Lambda} = D^{-\Lambda} + X \setminus \Lambda = ((X \setminus \Lambda)^{X \setminus D} + X \setminus \Lambda) + (X \setminus \Lambda)^{\Lambda} \setminus (X \setminus \Lambda) = X \setminus \Lambda.
\]

From equalities (23)-(24) and Proposition 4.3, we deduce that
\[
D \in I^{\Lambda} \quad \Downarrow \quad D = \Lambda \ast ((\Lambda \ast D)).
\]
This shows that the study of \(\varphi\)-envelopes amounts to that of the class \(\mathcal{I}^{\varphi}\). Conversely, given a set \(\Lambda \subset X\), the class \(\mathcal{I}^{\varphi}\) can be fully described via the \(\delta_{\Lambda}\)-envelopes.

**Proposition 4.4.** Let \(X\) be a vector space and let \(\Lambda \subset X\).

(i) For every function \(f : X \to \mathbb{R}\), we have\(^3\)

\[
\delta_{\Lambda} f = -\inf_{x \in X} f + \delta_{(\text{dom } f)^{\Lambda}}.
\]

As a consequence, the equality \((\delta_{C})^{\Lambda} = \delta_{C^{\Lambda}}\) holds for any nonempty set \(C \subset X\).

(ii) For every function \(g : X \to \mathbb{R}\) such that \(g \neq \pm \omega_X\), we have

\[
g \in \mathcal{E}^{\delta_{\Lambda}} \iff g = \beta + \delta_{C^{\Lambda}} \text{ for some } \beta \in \mathbb{R} \text{ and some } C \neq \emptyset.
\]

**Proof.** (i) For every function \(f : X \to \mathbb{R}\) and every \(x \in X\), the definition of \(\delta_{\Lambda} f\) gives

\[
\delta_{\Lambda} f(x) = \sup_{y \in X} \{\delta_{\Lambda}(x - y) - f(y)\} = \sup_{y \in \text{dom } f} \{\delta_{\Lambda}(x - y) - f(y)\}.
\]

First assume that \(x - \text{dom } f \subset \Lambda\). For every \(y \in \text{dom } f\), we then have \(x - y \in \Lambda\), whence \(\delta_{\Lambda}(x - y) = 0\). It ensues that

\[
\delta_{\Lambda} f(x) = \sup_{y \in \text{dom } f} -f(y) = \sup_{x \in X} (-f) = -\inf_{x \in X} f.
\]

Now assume that \(x - \text{dom } f \not\subset \Lambda\). In this case, there exists \(y \in \text{dom } f\) such that \(x - y \notin \Lambda\). We then have \(\delta_{\Lambda}(x - y) = +\infty\), whence \(\delta_{\Lambda} f(x) = +\infty\). Finally, we obtain for every \(x \in X\)

\[
\delta_{\Lambda} f(x) = \begin{cases} -\inf_{x \in X} f & \text{if } x - \text{dom } f \subset \Lambda \\ +\infty & \text{otherwise}. \end{cases}
\]

Condition \(x - \text{dom } f \subset \Lambda\) is equivalent to \(x \in (\text{dom } f)^{\Lambda}\) in view of Proposition 4.2 (i). Formula (25) follows immediately. For the last assertion, it suffices to take \(f = \delta_{C}\).

(ii) Let \(g \in \mathcal{E}^{\delta_{\Lambda}}\) be such that \(g \neq \pm \omega_X\). There exists \(f : X \to \mathbb{R}\) such that \(g = \delta_{\Lambda} f\), hence we deduce from (i) that \(g = -\inf_{x \in X} f + \delta_{(\text{dom } f)^{\Lambda}}\). Since \(g \neq \pm \omega_X\), we have \(\inf_{X} f \in \mathbb{R}\) and \(\text{dom } f \neq \emptyset\). It suffices then to take \(\beta = -\inf_{X} f\) and \(C = \text{dom } f\). Conversely, assume that \(g = \beta + \delta_{C^{\Lambda}}\) for some \(\beta \in \mathbb{R}\) and some \(C \neq \emptyset\). Assertion (i) then shows that \(g = \delta_{\Lambda} f\) for the function \(f\) defined by \(f = -\beta + \delta_{C}\), hence \(g \in \mathcal{E}^{\delta_{\Lambda}}\).

---

\(^3\)If \(\inf_{X} f = +\infty\) we have \(\text{dom } f = \emptyset\), hence \((\text{dom } f)^{\Lambda} = X\) and \(\delta_{(\text{dom } f)^{\Lambda}} = 0\). Therefore the addition in the right-hand side of (25) is well-defined.
Remark 4.1. The previous proposition shows that for every $C$, $\Lambda \subset X$ with $C \neq \emptyset$
\[
(\delta_C)^{\delta_{\Lambda}} = (-\delta_C) \triangle \delta_{\Lambda} = \delta_{C \Lambda}.
\] (26)

It is interesting to compare this formula with the following one
\[
(\delta_C)^{-\delta_{\Lambda}} = (-\delta_C) \triangle (-\delta_{\Lambda}) = -\delta_{C+\Lambda},
\] (27)

that is obtained as a consequence of the equality $\delta_{C+\Lambda} = \delta_C \triangledown \delta_{\Lambda}$.

Corollary 4.1. Let $X$ be a vector space. For every set $\Lambda \subset X$ and every set $D \subset X$ such that $D \neq \emptyset$ and $D \neq X$, the following equivalence holds
\[
\delta_D \in \mathcal{E}^{\delta_{\Lambda}} \iff D \in \mathcal{T}^{\Lambda}.
\]

In fact, the implication from the left to the right is true as soon as $D \neq \emptyset$, while the reverse one is true if $D \neq X$.

Proof. First assume that $\delta_D \in \mathcal{E}^{\delta_{\Lambda}}$ and that $D \neq \emptyset$. There exists $f : X \to \mathbb{R}$ such that $\delta_D = f^{\delta_{\Lambda}}$, hence we deduce from Proposition 4.4 (i) that $\delta_D = -\inf_X f + \delta_{(\text{dom} f)^{\Lambda}}$. Since $D \neq \emptyset$, we have $\inf_X f = 0$ and $D = (\text{dom} f)^{\Lambda} \in \mathcal{T}^{\Lambda}$. Conversely, assume that $D \in \mathcal{T}^{\Lambda}$ and that $D \neq X$. This implies that $D = C^{\Lambda}$ for some $C \neq \emptyset$, and hence by Proposition 4.4 (i) again $\delta_D = \delta_{C^{\Lambda}} = (\delta_C)^{\delta_{\Lambda}} \in \mathcal{E}^{\delta_{\Lambda}}$. \qed

Let us now study the class $\mathcal{E}^{-\delta_{\Lambda}}$. From the generalized conjugation point of view, the case $\varphi = -\delta_{\Lambda}$ is a special instance of a coupling functional $c : X \times Y \to \mathbb{R}$ of the type $c = -\delta_G$, where $G$ is a subset of $X \times Y$. The corresponding conjugation operator, which arises in quasiconvex analysis, has been considered in many papers, see for example [17, 31, 35].

Proposition 4.5. Let $X$ be a vector space. Let $\Lambda$ be a nonempty subset of $X$ and let $f : X \to \mathbb{R}$. Then we have

(i) \[
f \in \mathcal{E}^{-\delta_{\Lambda}} \iff f = \sup_{y \in \Lambda} \inf_{z \in \Lambda} f(\cdot - y + z).
\]

This means equivalently that for every $x \in X$ and every $\lambda < f(x)$, there exists $y \in \Lambda$ such that $f(x - y + z) \geq \lambda$ for every $z \in \Lambda$.

(ii) If $f \in \mathcal{E}^{-\delta_{\Lambda}}$ and if $\Lambda + \Lambda \subset \Lambda$, then $f$ is $\Lambda$-nondecreasing. Conversely, if $f$ is $\Lambda$-nondecreasing and if $0 \in \Lambda$, then $f \in \mathcal{E}^{-\delta_{\Lambda}}$.

Proof. (i) The equivalence (7) $\iff$ (8) yields
\[
f \in \mathcal{E}^{-\delta_{\Lambda}} \iff f = (f^{(-\delta_{\Lambda})})^{-\delta_{\Lambda}}.
\]

On the other hand, we have
\[
f^{(-\delta_{\Lambda})} = \sup_{\xi \in X} -\delta_{\Lambda}(\xi) - f(\cdot - \xi) = \sup_{-\xi \in \Lambda} -f(\cdot - \xi) = -\inf_{z \in \Lambda} f(\cdot + z)
\]

and hence
\[
(f^{(-\delta_{\Lambda})})^{-\delta_{\Lambda}} = \sup_{y \in \Lambda} -f^{(-\delta_{\Lambda})}(-y) = \sup_{y \in \Lambda} \inf_{z \in \Lambda} f(\cdot - y + z).
\]

We deduce immediately the equivalence (28).

Since the inequality $(f^{(-\delta_{\Lambda})})^{-\delta_{\Lambda}} \leq f$ is always satisfied, we infer that $f \in \mathcal{E}^{-\delta_{\Lambda}}$ if and only if for every $x \in X$,
\[
\sup_{y \in \Lambda} \inf_{z \in \Lambda} f(x - y + z) \geq f(x).
\]
The last assertion of (i) follows immediately.

(ii) Assume that \( f \in \mathcal{E}^{-\delta_{\Lambda}} \) and that \( \Lambda + \Lambda \subset \Lambda \). Let \( \xi \in \Lambda \). In view of (28), we have for every \( x \in X \)

\[
 f(x + \xi) = \sup_{y \in \Lambda} \inf_{z \in \Lambda} f(x + \xi - y + z) \\
 = \sup_{y \in \Lambda} \inf_{z \in \xi + \Lambda} f(x - y + z') \\
 \geq \sup_{y \in \Lambda} \inf_{z \in \Lambda} f(x - y + z') \quad \text{since} \quad \xi + \Lambda \subset \Lambda \\
 = f(x).
\]

Since this is true for every \( \xi \in \Lambda \), we infer that \( f \) is \( \Lambda \)-nondecreasing.

Conversely, assume that \( f \) is \( \Lambda \)-nondecreasing and that \( 0 \in \Lambda \). For every \( y, z \in \Lambda \), we have

\[
 f(-y) \leq f(-y + z) \leq f(z).
\]

It ensues immediately that

\[
 \sup_{y \in \Lambda} f(-y) \leq \sup_{y \in \Lambda} \inf_{z \in \Lambda} f(-y + z) \leq \inf_{z \in \Lambda} f(z).
\]

Since \( 0 \in \Lambda \), we obtain \( \sup_{y \in \Lambda} f(-y) = \inf_{z \in \Lambda} f(-y + z) = f \), and hence \( f = \sup_{y \in \Lambda} \inf_{z \in \Lambda} f(-y + z) \). In view of (28), we conclude that \( f \in \mathcal{E}^{-\delta_{\Lambda}} \). \( \square \)

**Remark 4.2.** When \( \Lambda + \Lambda \subset \Lambda \) and \( 0 \in \Lambda \), the equivalence

\[
 f \in \mathcal{E}^{-\delta_{\Lambda}} \iff f \text{ is } \Lambda\text{-nondecreasing}
\]

can be recovered by using the subadditivity of the function \( \delta_{\Lambda} \), see section 6.

**Proposition 4.6.** Let \( X \) be a vector space and let \( \Lambda, D \subset X \).

(i) The following equivalence holds

\[
 -\delta_{D} \in \mathcal{E}^{-\delta_{\Lambda}} \iff X \setminus D \in \mathcal{I}^{X \setminus \Lambda}.
\]

(ii) If moreover \( \Lambda \neq \emptyset \), we have

\[
 \delta_{D} \in \mathcal{E}^{-\delta_{\Lambda}} \iff D \in \mathcal{I}^{X \setminus \Lambda}.
\]

**Proof.** (i) First observe that the equivalence trivially holds if \( \Lambda = \emptyset \). Now assume that \( \Lambda \neq \emptyset \). Recall that

\[
 -\delta_{D} \in \mathcal{E}^{-\delta_{\Lambda}} \iff -\delta_{D} = \left( (-\delta_{D})^{(\delta_{\Lambda})} \right)^{-\delta_{\Lambda}}. \tag{29}
\]

Further, note by (4) that

\[
 (-\delta_{D})^{(\delta_{\Lambda})} = (-\delta_{D})^{(\delta_{\Lambda})} = (\delta_{\Lambda})^{\delta_{D}} = \delta_{\Lambda}^{\delta_{D}} \quad \text{from formula (26)} \quad \text{and the nonvacy of } \Lambda.
\]

In view of formula (27), we deduce that

\[
 \left( (-\delta_{D})^{(\delta_{\Lambda})} \right)^{-\delta_{\Lambda}} = (\delta_{\Lambda})^{\delta_{D} + \delta_{\Lambda}} = -\delta_{D}^{\delta_{\Lambda}}.
\]
Coming back to (29), we infer that

\[-\delta_D \in \mathcal{E}^{-\delta_\Lambda} \iff D = (-\Lambda)^D + \Lambda \]
\[ \iff D = (X \setminus D)^{-\delta(X \setminus \Lambda)} + \Lambda, \text{ see Proposition 4.2 } (iii) \]
\[ \iff X \setminus D = \left( (X \setminus D)^{-\delta(X \setminus \Lambda)} \right)^{X \setminus \Lambda}, \text{ in view of Proposition 4.2 } (ii) \]
\[ \iff X \setminus D \in \mathcal{I}^{X \setminus \Lambda}, \text{ cf. Proposition 4.3.} \]

(ii) Assume that \( \Lambda \neq \emptyset \). By arguing as in (i), we find

\[ (\delta_D)^{(\delta_\Lambda)^{-}} = (\delta_D)^{-\delta_\Lambda} = -\delta_D^{-\Lambda}, \]
and hence by (4) and (26)

\[ (\delta_D)^{(\delta_\Lambda)^{-}})^{-\delta_\Lambda} = (-\delta_D^{-\Lambda})^{-\delta_\Lambda} = (\delta_\Lambda)^{\delta_D^{-\Lambda}} = \delta_D^{-\Lambda}. \]

Recalling that

\[ \delta_D \in \mathcal{E}^{-\delta_\Lambda} \iff \delta_D = (\delta_D)^{(\delta_\Lambda)^{-}})^{-\delta_\Lambda}, \]
we deduce that

\[ \delta_D \in \mathcal{E}^{-\delta_\Lambda} \iff D = \Lambda^{D^{-\Lambda}} \]
\[ \iff D = (X \setminus (D - \Lambda))^{X \setminus \Lambda} \text{ by Proposition 4.2 } (iii) \]
\[ \iff D = (D^{-\delta(X \setminus \Lambda)})^{X \setminus \Lambda} \text{ by Proposition 4.2 } (ii) \]
\[ \iff D \in \mathcal{I}^{X \setminus \Lambda}, \text{ see Proposition 4.3.} \]

Combining Corollary 4.1 and Proposition 4.6, we derive the following corollary giving various characterizations of \( \mathcal{I}^{\Lambda} \) via the classes \( \mathcal{E}^{\delta_\Lambda} \) and \( \mathcal{E}^{-\delta_X \setminus \Lambda} \).

**Corollary 4.2.** For every set \( \Lambda \subset X \) and every set \( D \subset X \) such that \( D \neq \emptyset \) and \( D \neq X \), the following equivalences hold

\[ D \in \mathcal{I}^{\Lambda} \iff \delta_D \in \mathcal{E}^{\delta_\Lambda} \iff -\delta_X \setminus D \in \mathcal{E}^{-\delta_X \setminus \Lambda} \iff \delta_D \in \mathcal{E}^{-\delta_X \setminus \Lambda}. \]

**Proof.** The first equivalence is a consequence of Corollary 4.1, under the assumptions \( D \neq \emptyset \) and \( D \neq X \). The equivalence \( D \in \mathcal{I}^{\Lambda} \iff -\delta_X \setminus D \in \mathcal{E}^{-\delta_X \setminus \Lambda} \) follows from Proposition 4.6 (i) applied with \( X \setminus D \) (resp. \( X \setminus \Lambda \)) in place of \( D \) (resp. \( \Lambda \)). If \( \Lambda \neq X \), the equivalence \( D \in \mathcal{I}^{\Lambda} \iff \delta_D \in \mathcal{E}^{-\delta_X \setminus \Lambda} \) is a consequence of Proposition 4.6 (ii) applied with \( X \setminus \Lambda \) in place of \( \Lambda \). If \( \Lambda = X \), the equivalence becomes \( D \in \mathcal{I}^{X} \iff \delta_D \in \mathcal{E}^{-\omega_X} \). Since \( \mathcal{I}^{X} = \{X\} \) and \( \mathcal{E}^{-\omega_X} = \{\omega_X\} \), the equivalence amounts to \( D = X \iff \delta_D = -\omega_X \). The condition \( D = X \) is not realized by assumption, while the condition \( \delta_D = -\omega_X \) is never realized. It ensues that the equivalence trivially holds true if \( \Lambda = X \). \( \square \)

For a function \( f : X \to \mathbb{R} \) and \( r \in \mathbb{R} \), the notation \([f \geq r]\) (resp. \([f > r]\)) denotes the set \( \{x \in X, f(x) \geq r\} \) (resp. \( \{x \in X, f(x) > r\} \)). We adopt the corresponding notations for the sublevel sets. Adapting Proposition 3.3 to the framework of sets, we obtain the following statement.

**Proposition 4.7.** Let \( X \) be a locally convex space. Let \( \Lambda \subset X \) and \( \xi^* \in X^* \). Then we have

\[ \text{(i) } [\langle \xi^*, \cdot \rangle > 0]^\Lambda = [\langle \xi^*, \cdot \rangle \leq -\sigma_{X \setminus \Lambda}(-\xi^*)]. \]
(ii) If \( \Lambda \neq X \), the following equivalence holds
\[
[\langle \xi^*, \cdot \rangle \leq 0] \in \mathcal{I}^\Lambda \iff \xi^* \in -\text{dom} \sigma_{X \setminus \Lambda}.
\]

**Proof.** (i) Set \( C = [\langle \xi^*, \cdot \rangle > 0] \) and observe that
\[
x \in C^\Lambda \iff x - C \subset \Lambda
\]
\[
\iff X \setminus \Lambda \subset x - X \setminus C
\]
\[
\iff X \setminus \Lambda \subset \{y \in X, \langle \xi^*, y \rangle \geq \langle \xi^*, x \rangle\}
\]
\[
\iff \forall y \in X \setminus \Lambda, \langle \xi^*, y \rangle \geq \langle \xi^*, x \rangle
\]
\[
\iff \inf_{X \setminus \Lambda} \langle \xi^*, \cdot \rangle \geq \langle \xi^*, x \rangle
\]
\[
\iff -\sigma_{X \setminus \Lambda}(-\xi^*) \geq \langle \xi^*, x \rangle.
\]

Item (i) follows immediately.

(ii) First assume that \( \sigma_{X \setminus \Lambda}(-\xi^*) \in \mathbb{R} \). Recall from (i) that the set \( [\langle \xi^*, \cdot \rangle \leq -\sigma_{X \setminus \Lambda}(-\xi^*)] \) belongs to \( \mathcal{I}^\Lambda \). Let \( \xi \in X \) satisfying\(^4\) the equality \( \langle \xi^*, \xi \rangle = -\sigma_{X \setminus \Lambda}(-\xi^*) \). We then have
\[
[\langle \xi^*, \cdot \rangle \leq 0] = [\langle \xi^*, \cdot \rangle \leq -\sigma_{X \setminus \Lambda}(-\xi^*)] \in \mathcal{I}^\Lambda.
\]

Since the class \( \mathcal{I}^\Lambda \) is stable under translations, the set \( [\langle \xi^*, \cdot \rangle \leq 0] \) also belongs to \( \mathcal{I}^\Lambda \). Now assume that \( \sigma_{X \setminus \Lambda}(-\xi^*) \) is not finite, or equivalently \( \sigma_{X \setminus \Lambda}(-\xi^*) = +\infty \) since \( X \setminus \Lambda \neq \emptyset \) by assumption. Let us determine the set \( ([\langle \xi^*, \cdot \rangle \leq 0]^{-\Lambda})^\Lambda \).

Remark that
\[
[\langle \xi^*, \cdot \rangle \leq 0]^{-\Lambda} \subset [\langle \xi^*, \cdot \rangle < 0]^{-\Lambda}
\]
\[
= [\langle -\xi^*, \cdot \rangle > 0]^{-\Lambda}
\]
\[
= [\langle -\xi^*, \cdot \rangle \leq -\sigma_{\Lambda}(\xi^*)] \quad \text{in view of (i)}.
\]

Since \( \sigma_{\Lambda}(\xi^*) = \sigma_{X \setminus \Lambda}(-\xi^*) = +\infty \), it ensues that \( ([\langle \xi^*, \cdot \rangle \leq 0]^{-\Lambda})^\Lambda = X \neq [\langle \xi^*, \cdot \rangle \leq 0] \).

From Proposition 4.3, we conclude that \( ([\langle \xi^*, \cdot \rangle \leq 0] \notin \mathcal{I}^\Lambda \), which ends the proof of the announced equivalence.

\[\square\]

Let us denote by \( \mathcal{C}(X) \) the class of nonempty closed convex sets of \( X \).

**Theorem 4.1.** Let \( X \) be a locally convex space. Let \( \Lambda \subset X \) be such that \( \Lambda \neq X \).

For every cone \( Q \subset X^* \), the following equivalence holds true
\[
\{C \in \mathcal{C}(X), \text{dom} \sigma_C \subset Q\} \subset \mathcal{I}^\Lambda \iff Q \subset -\text{dom} \sigma_{X \setminus \Lambda}.
\]

**Proof.** Let \( Q \subset X^* \) be a cone and assume that
\[
\{C \in \mathcal{C}(X), \text{dom} \sigma_C \subset Q\} \subset \mathcal{I}^\Lambda.
\]
(30)

Let \( \xi^* \in Q \). Setting \( C = [\langle \xi^*, \cdot \rangle \leq 0] \in \mathcal{C}(X) \), we have \( \sigma_C = \delta_{\mathbb{R}^+ \xi^*} \), and hence \( \text{dom} \sigma_C = \mathbb{R}^+ \xi^* \subset Q \). In view of (30), it ensues that \( C \in \mathcal{I}^\Lambda \). We then deduce from Proposition 4.7 (ii) that \( \xi^* \in -\text{dom} \sigma_{X \setminus \Lambda} \). Since this is true for every \( \xi^* \in Q \), we

\(^4\)If \( \xi^* = 0 \), we have \( \sigma_{X \setminus \Lambda}(-\xi^*) = 0 \) because \( X \setminus \Lambda \neq \emptyset \) by assumption. In this case, the equality \( \langle \xi^*, \xi \rangle = -\sigma_{X \setminus \Lambda}(-\xi^*) \) is satisfied by every \( \xi \in X \).
conclude that \( Q \subset -\text{dom} \sigma_{X \setminus \Lambda} \).
Now assume that \( Q \subset -\text{dom} \sigma_{X \setminus \Lambda} \) and let \( C \in \mathcal{C}(X) \) be such that \( \text{dom} \sigma_C \subset Q \).
Then \( \delta_C \in \Gamma_0(X) \) with \( \text{dom} \delta_C \subset Q \), and since
\[
Q \subset -\text{dom} \sigma_{X \setminus \Lambda} = -\text{dom} \delta_{X \setminus \Lambda} = -\text{dom} (-(\delta_{X \setminus \Lambda}))^* ,
\]
by Theorem 3.1 we have \( \delta_C \in \mathcal{E}^{-\delta_{X \setminus \Lambda}} \) (keep in mind \( \sim \delta_{X \setminus \Lambda} \neq -\omega_X \) since \( \Lambda \neq X \)).
Proposition 4.6 (ii) yields that \( C \in \mathcal{T}_\Lambda \) as desired. Finally, we have shown the inclusion (30), which ends the proof. \( \square \)

Applying Theorem 4.1 with \( Q = X^* \), we immediately obtain the following result.

**Corollary 4.3.** Let \( X \) be a locally convex space. Let \( \Lambda \subset X \) be such that \( \Lambda \neq X \). Then, the following equivalence holds true
\[
\mathcal{C}(X) \subset \mathcal{T}_\Lambda \iff \text{dom} \sigma_{X \setminus \Lambda} = X^* .
\]

5. A preorder relation on \( \mathcal{F}(X, \mathbb{R}) \) based on \( \varphi \)-envelopes

Let \( X \) be a vector space and let \( \mathcal{F}(X, \mathbb{R}) \) be the set of extended real-valued functions on \( X \). We define the relation \( \sim \) on the space \( \mathcal{F}(X, \mathbb{R}) \) as follows: for every \( \varphi, \psi : X \to \mathbb{R} \)
\[
\psi \sim \varphi \iff \text{there exist } \xi \in X \text{ and } \alpha \in \mathbb{R} \text{ such that } \psi = \varphi(\cdot - \xi) + \alpha
\]
\[
\iff \psi \text{ is a } \varphi \text{-elementary function}.
\]
Clearly, the relation \( \sim \) is reflexive, symmetric and transitive, hence defines an equivalence relation. The objective of this section is to determine suitable⁵ subsets \( \mathcal{G} \) of \( \mathcal{F}(X, \mathbb{R}) \) such that the following implication holds true for every \( \varphi, \psi \in \mathcal{G} \)
\[
\psi \in \mathcal{E}^\varphi \text{ and } \varphi \in \mathcal{E}^\psi \implies \psi \sim \varphi .
\]  

5.1. The coercive case. For any function \( \varphi : X \to \mathbb{R} \), the deconvolution function \( \varphi \oplus \varphi \) defined by \((\varphi \oplus \varphi)(x) = \sup_{y \in X}(\varphi(y) - \varphi(z))\) can be expressed as a \( \varphi \)-envelope via the equality \( \varphi \oplus \varphi = (\varphi_-)^\varphi \). The next lemma shows that this function is subadditive. Recall that a function \( f : X \to [\mathbb{R}] \) is said to be subadditive if for any \( x, y \in X \),
\[
f(x + y) \leq f(x) + f(y).
\]

**Lemma 5.1.** Let \( X \) be a vector space and let \( f, \varphi : X \to [\mathbb{R}] \). For any \( x, x' \in X \), we have
\[
f^{\varphi}(x') \leq (\varphi \oplus \varphi)(x' - x) + f^{\varphi}(x).
\]
Moreover, the function \( \varphi \oplus \varphi \) is subadditive.

**Proof.** Fix \( x, x' \in X \). It is immediate to check that for every \( y \in X \),
\[
\varphi(x' - y) - f(y) \leq [\varphi(x' - y) - \varphi(x - y)] + [\varphi(x - y) - f(y)].
\]
Taking the supremum over \( y \in X \) and using [21, Proposition 4.1] we deduce that
\[
f^{\varphi}(x') \leq \sup_{y \in X}[\varphi(x' - y) - \varphi(x - y)] + \sup_{y \in X}[\varphi(x - y) - f(y)],
\]
\[
= (\varphi \oplus \varphi)(x' - x) + f^{\varphi}(x),
\]

⁵The implication (31) is not true for all \( \varphi, \psi \in \mathcal{F}(X, \mathbb{R}) \), see a counterexample in subsection 5.3.
Proof. Let us argue by contradiction and assume that there exists \( \varphi \in \mathcal{E}^\varphi \) such that
\[
(\varphi \ominus \varphi)(x') \leq (\varphi \ominus \varphi)(x' - x) + (\varphi \ominus \varphi)(x),
\]
hence the function \( \varphi \ominus \varphi \) is subadditive. \( \Box \)

If the space \((X, \| \cdot \|)\) is normed and if the function \( \varphi \) satisfies the coercivity property \( \lim_{\|x\| \to +\infty} \varphi(x)/\|x\| = +\infty \), the following lemma shows that \( \varphi \ominus \varphi = +\infty \) on \( X \setminus \{0\} \).

**Lemma 5.2.** Let \((X, \| \cdot \|)\) be a normed space and let \( \varphi : X \to \mathbb{R} \) be an extended real-valued function. Assume that \( \varphi \neq +\omega_X \) and \( \lim_{\|x\| \to +\infty} \varphi(x)/\|x\| = +\infty \). Then we have \( \varphi \ominus \varphi = +\infty \) on \( X \setminus \{0\} \).

**Proof.** Let us argue by contradiction and assume that there exists \( u \neq 0 \) such that \( (\varphi \ominus \varphi)(u) < +\infty \). Let us fix \( x \in \varphi \) and observe that for every \( n \in \mathbb{N}^6 \)
\[
\varphi(x + nu) - \varphi(x) \leq (\varphi \ominus \varphi)(nu)
\]
and taking the upper limit as \( n \to +\infty \), we deduce that
\[
\limsup_{n \to +\infty} \frac{1}{n} \varphi(x + nu) \leq (\varphi \ominus \varphi)(u),
\]
which contradicts the fact that \( \lim_{\|x\| \to +\infty} \varphi(x)/\|x\| = +\infty \). Finally, we obtain that \( \varphi \ominus \varphi = +\infty \) on \( X \setminus \{0\} \). \( \Box \)

**Theorem 5.1.** Let \( X \) be a vector space and let \( \varphi, \psi : X \to \mathbb{R} \) be such that \( \psi \in \mathcal{E}^\varphi \) and \( \varphi \in \mathcal{E}^\psi \).

(i) If \( \varphi \ominus \varphi = +\infty \) on \( X \setminus \{0\} \), then we have \( \psi \sim \varphi \).

(ii) Assume that \((X, \| \cdot \|)\) is a normed space. If \( \lim_{\|x\| \to +\infty} \varphi(x)/\|x\| = +\infty \) (resp. \( \lim_{\|x\| \to +\infty} \varphi(x)/\|x\| = -\infty \)), then we have \( \psi \sim \varphi \).

**Proof.** If \( \varphi = \pm \omega_X \), it is immediate to check that \( \psi = \varphi \). From now on, let us assume that \( \varphi \neq \pm \omega_X \). Since \( \psi \in \mathcal{E}^\varphi \) and \( \varphi \in \mathcal{E}^\psi \), there exist \( f, g : X \to \mathbb{R} \) such that \( -\psi = (-\varphi) \triangledown f \) and \( -\varphi = (-\psi) \triangledown g \). It ensues that
\[
-\varphi = (-\varphi) \triangledown (f \triangledown g).
\]

(32)

Now observe that
\[
(-\varphi) \triangledown (f \triangledown g) \geq -\varphi \iff (-\varphi)(x - y) + (f \triangledown g)(y) \geq -\varphi(x) \quad \text{for all } x, y \in X
\]
\[
\iff (f \triangledown g)(y) \geq \varphi(x - y) - \varphi(x) \quad \text{for all } x, y \in X
\]
\[
\iff (f \triangledown g)(y) \geq \sup_{x \in X} (\varphi(x - y) - \varphi(x)) \quad \text{for all } y \in X
\]
\[
\iff f \triangledown g \geq [\varphi \ominus \varphi]_-. \]

---

\( ^6 \mathbb{N} \) denotes the set of positive integers.
(i) Assume that \( \varphi \odot \varphi = +\infty \) on \( X \setminus \{0\} \). We then deduce from the above inequality that
\[
f \triangledown g = +\infty \quad \text{on} \quad X \setminus \{0\}.
\]
If \( f \triangledown g = \omega_X \), we infer from (32) that \( \varphi = -\omega_X \), thus implying in turn that \( \psi = -\omega_X \). If \( f \triangledown g \neq \omega_X \), equality (33) shows that \( \text{dom}(f \triangledown g) = \{0\} \). Recalling that \( \text{dom}(f \triangledown g) = \text{dom} f + \text{dom} g \), we deduce that \( \text{dom} f + \text{dom} g = \{0\} \). Hence there exists \( \xi \in X \) such that \( \text{dom} f = \{\xi\} \) and \( \text{dom} g = \{-\xi\} \). We infer that
\[
-\psi = (-\varphi) \triangledown f = (-\varphi)(\cdot - \xi) + f(\xi)
\]
and
\[
-\varphi = (-\psi) \triangledown g = (-\psi)(\cdot + \xi) + g(-\xi).
\]
If \( f(\xi) \in \mathbb{R} \), we obtain from (34) that \( \psi = \varphi(\cdot - \xi) - f(\xi) \) and therefore \( \psi \sim \varphi \). If \( g(-\xi) \in \mathbb{R} \), equality (35) shows that \( \varphi = \psi(\cdot + \xi) - g(-\xi) \), and hence \( \varphi \sim \psi \). On the other hand, if \( f(\xi) = g(-\xi) = -\infty \), we deduce from (34)-(35) that
\[
-\psi \leq (-\varphi)(\cdot - \xi) \quad \text{and} \quad -\varphi \leq (-\psi)(\cdot + \xi),
\]
thus implying that \( \psi = \varphi(\cdot - \xi) \) and therefore \( \psi \sim \varphi \).

(ii) First assume that \( \lim_{\|x\| \to +\infty} \varphi(x)/\|x\| = +\infty \). We infer from Lemma 5.2 that \( \varphi \odot \varphi = +\infty \) on \( X \setminus \{0\} \) and we conclude with (i).

Now assume that \( \lim_{\|x\| \to +\infty} \varphi(x)/\|x\| = -\infty \). From Lemma 5.2, we deduce that \( (-\varphi) \odot (-\varphi) = +\infty \) on \( X \setminus \{0\} \). Recalling that
\[
(-\varphi) \odot (-\varphi) = (-\varphi)^- \odot (-\varphi)^- = \varphi^- = [\varphi^-],
\]
we infer that \( \varphi \odot \varphi = +\infty \) on \( X \setminus \{0\} \) and we conclude again with (i).

Let us define the relation \( \preceq \) on \( \mathcal{F}(X, \mathbb{R}) \) by
\[
\psi \preceq \varphi \iff \psi \in \mathcal{E} \varphi.
\]
The relation \( \preceq \) is clearly reflexive, and also transitive in view of Proposition 3.2 (iii).

It is compatible with the equivalence relation \( \sim \), i.e.
\[
\varphi \sim \varphi', \quad \psi \sim \psi' \quad \text{and} \quad \psi \preceq \varphi \implies \psi' \preceq \varphi'.
\]

It ensues that we can properly define the relation \( \preceq \) on the quotient set \( \mathcal{F}(X, \mathbb{R})/\sim \).

The relation \( \preceq \) so defined on \( \mathcal{F}(X, \mathbb{R})/\sim \) is clearly reflexive and transitive, hence it is a preorder. Let us denote by \( \mathcal{G}, \mathcal{G}' \) and \( \mathcal{G}'' \) the following respective sets
\[
\mathcal{G} = \left\{ f : X \to \mathbb{R}, \ f \odot f = +\infty \quad \text{on} \quad X \setminus \{0\} \right\},
\]
\[
\mathcal{G}' = \left\{ f : X \to \mathbb{R}, \ \lim_{\|x\| \to +\infty} f(x)/\|x\| = +\infty \right\},
\]
\[
\mathcal{G}'' = \left\{ f : X \to \mathbb{R}, \ \lim_{\|x\| \to +\infty} f(x)/\|x\| = -\infty \right\}.
\]

Theorem 5.1 expresses that for every \( \varphi, \psi \in \mathcal{G} \) (resp. \( \mathcal{G}', \mathcal{G}'' \)), we have
\[
\psi \preceq \varphi, \quad \varphi \preceq \psi \implies \psi \sim \varphi.
\]

Hence the induced relation \( \preceq \) on the quotient set \( \mathcal{G}/\sim \) (resp. \( \mathcal{G}'/\sim, \mathcal{G}''/\sim \)) is antisymmetric, thus giving rise to an order relation.
Let us now specialize the result of Theorem 5.1 in the case of sets. We define the equivalence relation \( \sim \) on \( P(X) \) by
\[
C \sim D \iff \text{there exists } \xi \in X \text{ such that } D = C + \xi,
\]
along with the preorder relation \( \preceq \) on \( P(X) \) by
\[
C \preceq D \iff C \in \mathcal{I}^D.
\]
Recall that the star-difference \( C^* - C \) is defined by
\[
C^* - C = \bigcap_{x \in C} C - x = (-C)^C.
\]
By applying Theorem 5.1 with indicator functions, we obtain the following corollary.

**Corollary 5.1.** Let \( X \) be a vector space and let \( \Gamma, \Delta \subseteq X \) such that \( \Delta \in \mathcal{I}^\Gamma \) and \( \Gamma \in \mathcal{I}^\Delta \).

(i) If \( \Gamma^* - \Gamma = \{0\} \), then we have \( \Delta \sim \Gamma \).

(ii) Assume that \( (X, \| \cdot \|) \) is a normed space. If the set \( \Gamma \) (resp. \( X \setminus \Gamma \)) is bounded, then we have \( \Delta \sim \Gamma \).

**Proof.** If \( \Gamma \in \{0, X\} \) (resp. \( \Delta \in \{0, X\} \)), it is immediate to check that \( \Delta = \Gamma \).

Let us now assume that \( \Gamma \notin \{0, X\} \) and \( \Delta \notin \{0, X\} \). In view of Corollary 4.1, the assumptions \( \Delta \in \mathcal{I}^\Gamma \) and \( \Gamma \in \mathcal{I}^\Delta \) imply that \( \delta_\Delta \in \mathcal{E}^\delta_\Gamma \) and \( \delta_\Gamma \in \mathcal{E}_{\delta_\Delta} \).

(i) Assume that \( \Gamma^* - \Gamma = \{0\} \). Then, by (5) and Proposition 4.4 (i) we have
\[
\delta_\Gamma \ominus \delta_\Gamma = (\delta_{-\Gamma})^\delta_\Gamma = \delta_{-\Gamma^\delta_\Gamma} = \delta_{\Gamma^* - \Gamma} = \delta_{\{0\}}.
\]
By applying Theorem 5.1 (i) with \( \varphi = \delta_\Gamma \) and \( \psi = \delta_\Delta \), we obtain that \( \delta_\Delta \sim \delta_\Gamma \) and hence \( \Delta \sim \Gamma \).

(ii) First assume that \( \Gamma \) is bounded. Then the indicator function \( \delta_\Gamma \) is coercive and we deduce from Lemma 5.2 that \( \delta_\Gamma \ominus \delta_\Gamma = +\infty \) on \( X \setminus \{0\} \). This implies that \( \Gamma^* - \Gamma = \{0\} \) and we conclude with (i). Now assume that \( X \setminus \Gamma \) is bounded. From what precedes, we have \( (X \setminus \Gamma)^* \subseteq (X \setminus \Gamma) \). Observing that
\[
\Gamma^* - \Gamma = (X \setminus \Gamma)^* \subseteq \Gamma = \Gamma^* - \Gamma = (X \setminus \Gamma)^* (X \setminus \Gamma) \subseteq (X \setminus \Gamma)^{X \setminus \Gamma} = -(X \setminus \Gamma)^{X \setminus \Gamma} = -[(X \setminus \Gamma)^* (X \setminus \Gamma)] = -[(X \setminus \Gamma)^* (X \setminus \Gamma)],
\]
we infer that \( \Gamma^* - \Gamma = \{0\} \) and we conclude again with (i). \( \square \)

Let us denote by \( \mathcal{Q}, \mathcal{Q}' \) and \( \mathcal{Q}'' \) the following respective sets
\[
\mathcal{Q} = \{ C \subseteq X, C^* - C = \{0\} \},
\]
\[
\mathcal{Q}' = \{ C \subseteq X, C \text{ is bounded} \},
\]
\[
\mathcal{Q}'' = \{ C \subseteq X, X \setminus C \text{ is bounded} \}.
\]

The above corollary expresses that for every \( \Gamma, \Delta \in \mathcal{Q} \) (resp. \( \mathcal{Q}', \mathcal{Q}'') \), we have
\[
\Delta \preceq \Gamma, \quad \Gamma \preceq \Delta \implies \Delta \sim \Gamma.
\]
Hence the induced relation \( \preceq \) on the quotient set \( \mathcal{Q}/ \sim \) (resp. \( \mathcal{Q}'/ \sim, \mathcal{Q}''/ \sim \)) is antisymmetric, thus giving rise to an order relation.
5.2. The case \( \varphi, \psi \in -\Gamma_0(X) \). Let us first state a result that will be a key ingredient for the next theorem.

**Lemma 5.3.** Let \( X \) be a vector space, let \( D \subset X \) be a convex set and let us denote by \( \text{Aff}(D) \) the affine space generated by \( D \). Assume that a real-valued function \( h : D \to \mathbb{R} \) is both convex and concave. Then there exists a unique affine function \( \tilde{h} : \text{Aff}(D) \to \mathbb{R} \) such that \( \tilde{h}|_D = h \).

For a proof of this result, the reader is referred to [33]. By extending affinely the function \( \tilde{h} \) to the whole space \( X \), we deduce from the above result that there exists a linear function \( \ell : X \to \mathbb{R} \) with \( \alpha \in \mathbb{R} \) such that \( h = \ell|_D + \alpha \).

In view of stating the next theorem, given a locally convex space \( X \), let \( X^* \) be a convex set and let us denote \( \text{Aff}(X^*,X) \) as the finest locally convex topology on \( X^* \) that is both convex and concave. Then there exists a unique affine function \( h : \text{Aff}(X^*,X) \to \mathbb{R} \) such that \( h|_X = \varphi \) and both convex and concave. By applying the previous lemma with \( h \), the conclusion is satisfied. From now on, let us assume that \( X^* \) is finite-dimensional, or \( \text{dom}(\varphi,\psi) = \text{dom}(\varphi) \).

**Theorem 5.2.** Let \( X \) be a locally convex space. Let \( \varphi, \psi : X \to \mathbb{R} \) be functions such that \( \psi \in \mathcal{E}^c \) and \( \varphi \in \mathcal{E}^\circ \). Assume that either

- the space \( X \) is finite-dimensional, or
- one of the functions \( -\varphi \) and \( -\psi \) is \( \tau(X^*,X) \)-continuous at some point and finite at this point.

Then we have \((\varphi)^* \sim (\psi)^*\). If each of the functions \( -\varphi \) and \( -\psi \) has a continuous affine minorant, then \( \varphi(\varphi) \sim \varphi(\psi) \). In particular, if \( -\varphi \in \Gamma_0(X) \) and \( -\psi \in \Gamma_0(X) \), then we have \( \varphi \sim \psi \).

**Proof.** By assumption, we have \( -\psi = (\varphi) \nabla f \) and \( -\varphi = (\psi) \nabla g \), for some \( f, g : X \to \mathbb{R} \). Taking the conjugates, we obtain that

\[
(\varphi)^* = (\varphi)^* + f^* \quad \text{and} \quad (\psi)^* = (\psi)^* + g^*.
\]

(36)

First observe that if one of the functions \( (\varphi)^* \), \( (\psi)^* \), \( f^* \) or \( g^* \) is equal to \( -\omega_{X^*} \), then equalities (36) imply that \( (\varphi)^* = (\psi)^* = -\omega_{X^*} \). This implies in turn that \( \varphi = \psi = -\omega_X \) and the conclusion is satisfied. From now on, let us assume that the functions \( (\varphi)^* \), \( (\psi)^* \), \( f^* \) and \( g^* \) differ from \( -\omega_{X^*} \). From the first equality of (36), we deduce that \( \text{dom}(\psi)^* \subset \text{dom}(\varphi)^* \), while the second equality of (36) yields \( \text{dom}(\varphi)^* \subset \text{dom}(\psi)^* \). Finally, the domains of \( (\varphi)^* \) and \( (\psi)^* \) coincide and both functions are finite on their common domain \( D \). If the set \( D \) is empty, then \( (\varphi)^* = (\psi)^* = \omega_{X^*} \). This implies that \( (\varphi)^* = (\psi)^* = -\omega_X \), hence the conclusion is trivially satisfied. Without loss of generality, we now assume that \( D \neq \emptyset \). By combining both equalities of (36), we obtain

\[
(\varphi)^* = (\varphi)^* + f^* + g^*.
\]

It ensues that \( f^* + g^* = 0 \) on \( D \). Hence the function \( f^*|_D \) is finite-valued on \( D \) and both convex and concave. By applying the previous lemma with \( h = f^*|_D \), we obtain that there exist a linear function \( \ell : X^* \to \mathbb{R} \) such that \( f^* = \ell + \alpha \) on \( D \). Coming back to the first equality of (36), we deduce that

\[
(\psi)^* = (\varphi)^* + \ell + \alpha.
\]
Observe that the above equality holds true on the whole space $X^*$, since the functions $(-\varphi)^*$ and $(-\psi)^*$ are equal to $+\infty$ outside $D$. Taking the conjugate of each member, we find for every $\xi \in X$

$$(-\psi)^*\star(\xi) = \sup_{x^* \in X^*} [(x^*, \xi) - (-\varphi)^*(x^*) - \ell(x^*) - \alpha]. \quad (37)$$

Let us now show that the linear function $\ell$ is $\tau(X^*, X)$-continuous on $X^*$.

**Lemma 5.1.** Under the assumptions of Theorem 5.2, the function $\ell : X^* \to \mathbb{R}$ is $\tau(X^*, X)$-continuous on $X^*$.

*Proof of Lemma 5.1.* If the space $X$ is finite-dimensional, the assertion is obvious. Now assume that the function $(-\varphi)^*$ is $\tau(X^*, X)$-continuous at some $\pi^* \in X^*$ and finite at this point. There exist a $\tau(X^*, X)$-neighborhood $W$ of $\pi^*$ and $M \in \mathbb{R}$ such that $(-\varphi)^* \leq M$ on $W$. We deduce from (37) that for every $\xi \in X$,

$$(-\psi)^*\star(\xi) \geq \sup_{x^* \in W} [(x^*, \xi) - (-\varphi)^*(x^*) - \ell(x^*) - \alpha] \geq \sup_{x^* \in W} [(x^*, \xi) - \ell(x^*)] - M - \alpha.$$

Let us argue by contradiction and assume that $\ell$ is not $\tau(X^*, X)$-continuous on $X^*$. Since the linear function $\langle \cdot, \xi \rangle - \ell$ is not $\tau(X^*, X)$-continuous on $X^*$, the above supremum equals $+\infty$. It ensues that $(-\psi)^*\star = \omega_X$, and hence $-\psi = \omega_X$. Recalling that $-\varphi = (-\psi) \blacktriangleright g$, we deduce that $-\varphi = \omega_X$. This implies in turn that $(-\varphi)^* = -\omega_{X^*}$, a contradiction with $(-\varphi)^*(\pi^*) \in \mathbb{R}$. We conclude that the linear function $\ell$ is $\tau(X^*, X)$-continuous on $X^*$. Since $\varphi$ and $\psi$ play symmetric roles, the same conclusion holds true if the function $(-\psi)^*$ is assumed to be $\tau(X^*, X)$-continuous at some $\tilde{x}^* \in X^*$ and finite at this point. □

From the previous lemma and the definition of the Mackey topology $\tau(X^*, X)$, there exists $x \in X$ such that $\ell(x^*) = \langle x^*, x \rangle$ for every $x^* \in X^*$. In view of (37), we deduce that

$$(-\psi)^*\star(\xi) = \sup_{x^* \in X^*} [(x^*, \xi - x) - (-\varphi)^*(x^*)] - \alpha = (-\varphi)^*\star(\xi - x) - \alpha.$$

Since this is true for every $\xi \in X$, we conclude that $(-\psi)^*\star \sim (-\varphi)^*\star$. If the function $(-\varphi)$ (resp. $(-\psi)$) admits a continuous affine minorant, we have $(-\varphi)^*\star = \overline{\varphi}(-\varphi)$ (resp. $(-\psi)^*\star = \overline{\varphi}(-\psi)$). We infer that $\overline{\varphi}(-\psi) \sim \overline{\varphi}(-\varphi)$. The last assertion of the statement is a direct consequence of what precedes. □

**Remark 5.1.** If the normed space $(X, \|\cdot\|)$ is reflexive, the $\tau(X^*, X)$-continuity assumption on $(-\varphi)^*$ (resp. $(-\psi)^*$) amounts to the continuity assumption with respect to the dual norm $\|\cdot\|_{X^*}$.

Theorem 5.2 implies that the relation $\preceq$ defines an order on the following set

$$\{ \varphi \in -\Gamma_0(X), \quad (-\varphi)^* \text{ is } \tau(X^*, X) \text{-continuous at some point} \}/ \sim.$$

If the space $X$ is finite-dimensional, the relation $\preceq$ is an order on the set $(-\Gamma_0(X))/ \sim$.

By applying Theorem 5.2 with the opposite of indicator functions, we obtain the following corollary.

**Corollary 5.2.** Let $X$ be a locally convex space. Let $\Gamma, \Delta \subset X$ be such that $\Delta \in \mathcal{I}^\Gamma$ and $\Gamma \in \mathcal{I}^\Delta$. Assume that either

- the space $X$ is finite-dimensional, or
- one of the functions $\sigma_{X,\Gamma}$ and $\sigma_{X,\Delta}$ is $\tau(X^* , X)$-continuous at some point.

Then we have $\overline{\sigma}(X \setminus \Gamma) \sim \overline{\sigma}(X \setminus \Delta)$. In particular, if the sets $X \setminus \Gamma$ and $X \setminus \Delta$ are closed and convex, then $\Gamma \sim \Delta$.

**Proof.** From Proposition 4.6 (i), condition $\Delta \in I^\Gamma$ (resp. $\Gamma \in I^\Delta$) is equivalent to $-\delta_{X,\Delta} \in E^{-X,\Gamma}$ (resp. $-\delta_{X,\Delta} \in E^{-X,\Delta}$). By applying Theorem 5.2 with $\varphi = -\delta_{X,\Gamma}$ and $\psi = -\delta_{X,\Delta}$, we obtain the existence of $\xi \in X$ and $\alpha \in \mathbb{R}$ such that

$$\overline{\sigma}(\delta_{X,\Delta}) = [\overline{\sigma}(\delta_{X,\Gamma})][\cdot \, - \xi] - \alpha.$$

We immediately deduce that $\overline{\sigma}(X \setminus \Delta) = \overline{\sigma}(X \setminus \Gamma) + \xi$. The last assertion of the statement is a direct consequence of what precedes. □

5.3. **A counterexample.** Let us start with a preliminary result.

**Lemma 5.4.** Let $X$ be a topological vector space and let $G$ be a dense additive subgroup of $X$. Assume that $K \subset X$ is an open set such that $K + K \subset K$ and $0 \in \text{cl}(K)$. Then we have

(i) For all $\xi, \xi' \in X$,

$$[G \cap (K + \xi)] + [G \cap (K + \xi')] = G \cap (K + \xi + \xi').$$

(ii) If in addition $\text{cl}(K) \cap -\text{cl}(K) = \{0\}$, then

$$G \cap (K + \xi) = (G \cap K) + \xi' \implies \xi = \xi'.$$

If $G \neq X$ and $\xi \in X \setminus G$, there is no $\xi' \in X$ such that $G \cap (K + \xi) = (G \cap K) + \xi'$.

**Proof.** (i) Let us fix $\xi, \xi' \in X$ and let us prove the inclusion from the left to the right. Observe that

$$[G \cap (K + \xi)] + [G \cap (K + \xi')] \subset G + G$$

and

$$[G \cap (K + \xi)] + [G \cap (K + \xi')] \subset (K + \xi) + (K + \xi').$$

Since $G + G \subset G$ and $K + K \subset K$, we deduce that

$$[G \cap (K + \xi)] + [G \cap (K + \xi')] \subset G \cap (K + \xi + \xi').$$

Now let us establish the reverse inclusion. Let $x \in G \cap (K + \xi + \xi')$. Observe that the open set $K + \xi + \xi' - x$ contains 0. Recalling that $0 \in \text{cl}(K)$, we have $(K + \xi + \xi' - x) \cap -K \neq \emptyset$, hence $(K + \xi - x) \cap (-K - \xi') \neq \emptyset$. Since the set $K$ is open, the set $(K + \xi - x) \cap (-K - \xi')$ is open. By using the density of $G$ in $X$, we deduce that

$$G \cap (K + \xi - x) \cap (-K - \xi') \neq \emptyset.$$

Since $G = -G$, the above property can be rewritten as

$$[G \cap (K + \xi - x)] \cap [-G \cap (-K - \xi')] \neq \emptyset,$$

which is in turn equivalent to

$$0 \in [G \cap (K + \xi - x)] + [G \cap (K + \xi')] .$$

Recalling that $x \in G$, we have $G = G - x$, hence $G \cap (K + \xi - x) = [G \cap (K + \xi)] - x$. In view of the latter inclusion, we conclude that

$$x \in [G \cap (K + \xi)] + [G \cap (K + \xi')].$$
The inclusion
\[ G \cap (K + \xi + \xi') \subset [G \cap (K + \xi)] + [G \cap (K + \xi')] \]
is proved.

(ii) Let us assume that \( G \cap (K + \xi) = (G \cap K) + \xi' \) for some \( \xi, \xi' \in X \). We deduce that \( G \cap (K + \xi) \subset K + \xi' \). By using the openness of the set \( K + \xi \) along with the density of \( G \) in \( X \), we easily infer that \( K + \xi \subset \text{cl}(K) + \xi' \). This implies in turn that \( \text{cl}(K) + \xi \subset \text{cl}(K) + \xi' \) and since \( 0 \in \text{cl}(K) \), we obtain \( \xi - \xi' \in \text{cl}(K) \). By a symmetric argument, we find \( \xi' - \xi \in \text{cl}(K) \), hence \( \xi - \xi' \in \text{cl}(K) \cap \text{cl}(K) \). Since \( \text{cl}(K) \cap \text{cl}(K) = \{0\} \) by assumption, we conclude that \( \xi = \xi' \).

Now let \( \xi \in X \setminus G \) and assume that there exists \( \xi' \in X \) such that \( G \cap (K + \xi) = (G \cap K) + \xi' \). From what precedes, we have \( \xi' = \xi \) and hence \( G \cap (K + \xi) = (G + \xi) \cap (K + \xi) \). On the other hand, the assumption \( \xi \in X \setminus G \) implies that the sets \( G \) and \( G + \xi \) are disjoint. We deduce that \( G \cap (K + \xi) = \emptyset \), a contradiction since the nonempty set \( K \) is open and the set \( G \) is dense in \( X \).

Let us now build an example of sets \( \Gamma, \Delta \subset X \) satisfying \( \Delta \in \mathcal{I}^\Gamma \) and \( \Gamma \in \mathcal{I}^\Delta \), but with \( \Delta \) and \( \Gamma \) not equal up to a translation. We are given an open set \( K \subset X \) such that \( K + K \subset K \) and \( \text{cl}(K) \cap -\text{cl}(K) = \{0\} \), along with a dense additive subgroup \( G \subset X \) such that \( G \neq X \). Define the sets \( C, U, V \subset X \) respectively by

\[ C = G \cap K; \quad U = G \cap (K + \xi); \quad V = G \cap (K - \xi), \]

where \( \xi \in X \setminus G \). In view of Lemma 5.4 (i), the set \( D = C + U \) satisfies

\[ D = G \cap (K + \xi) \quad \text{and} \quad D + V = G \cap K = C. \]

Lemma 5.4 (ii) shows that the set \( D \) is not translated from \( C \). Defining the complementary sets \( \Gamma = X \setminus C \) and \( \Delta = X \setminus D \), we have

\[ \Delta = X \setminus (C + U) = U^X \setminus C = U^\Gamma \in \mathcal{I}^\Gamma \]

and

\[ \Gamma = X \setminus (D + V) = V^X \setminus D = V^\Delta \in \mathcal{I}^\Delta. \]

From what precedes, the set \( \Delta \) is not translated from \( \Gamma \). The above counterexample for sets obviously furnishes a counterexample for functions. Indeed, we deduce from (38)-(39) that the indicator functions \( \delta_\Gamma \) and \( \delta_\Delta \) satisfy \( \delta_\Delta \in \mathcal{E}^{\Gamma} \) and \( \delta_\Gamma \in \mathcal{E}^{\Delta} \), but the functions \( \delta_\Gamma \) and \( \delta_\Delta \) are not equal up to a translation.

By particularizing the above sets \( G, K \subset X \), one obtains various counterexamples. If \( X = \mathbb{R} \), one can take \( G = \mathbb{Q}, K = [0, +\infty) \) and \( \xi \in \mathbb{R} \setminus \mathbb{Q} \). On the other hand, if \( X \) is infinite dimensional, one can assume that \( G \) is a dense subspace of \( X \) and that \( K \) is an open convex cone such that \( \text{cl}(K) \) is pointed. This furnishes a counterexample with convex sets \( C, D \subset X \).

6. CASES OF EITHER SUPERADDITIONITY OR SUBADDITIONITY OF \( \varphi \)

Let us first recall that a function \( \varphi : X \to \mathbb{R} \) is said to be superadditive (resp. subadditive) if for all \( x, y \in X \),

\[ \varphi(x + y) \geq \varphi(x) + \varphi(y) \quad \text{(resp.} \quad \varphi(x + y) \leq \varphi(x) + \varphi(y)). \]

Let us start with a preliminary result.
Lemma 6.1. Let $X$ be a vector space. Let $h, k : X \to \mathbb{R}$ and assume that $k(0) = 0$. Then we have

\[ h = h \triangle k \iff h(x) \geq h(y) + k(x - y) \quad \text{for all } x, y \in X \]
\[ \iff h(y) \leq h(x) + (-k)(y - x) \quad \text{for all } x, y \in X \]
\[ \iff h = h \triangle (-k_\cdot). \]

As a consequence, the function $k$ is superadditive if and only if $k = k \triangle k$, which is in turn equivalent to $k = k \triangledown (-k_\cdot)$.

Proof. If $h = h \triangle k$, then the definition of $h \triangle k$ entails that $h(x) \geq h(y) + k(x - y)$ for all $x, y \in X$. Conversely, if this inequality holds true for every $x, y \in X$, we have

\[ h(x) \geq \sup_{y \in X} h(y) + k(x - y) \geq h(x) + k(0) = h(x), \]

for every $x \in X$. This implies that $h(x) = (h \triangle k)(x)$ for every $x \in X$ and the first equivalence is proved.

For the second equivalence, observe that for all $x, y \in X$

\[ h(x) \geq h(y) + k(x - y) \iff h(y) \leq h(x) + (-k)(y - x) = h(x) + (-k_\cdot)(y - x). \]

The proof of the third equivalence follows the same lines as the first one. For the last assertion, observe that $k$ is superadditive if and only if $k(x) \geq k(y) + k(x - y)$ for all $x, y \in X$. It suffices then to use what precedes with $h = k$. \qed

Through the above lemma, the following theorem provides, in particular, various characterizations of the class $\mathcal{E}^\varphi$ when $\varphi$ is superadditive.

Theorem 6.1. Let $X$ be a vector space. Let $\varphi : X \to \mathbb{R}$ be a superadditive function satisfying $\varphi(0) = 0$.

(a) For a function $g : X \to \mathbb{R}$, the following assertions are equivalent

(i) $g \in \mathcal{E}^\varphi$;
(ii) $g = g \triangle \varphi$;
(iii) $g(x) \geq g(y) + \varphi(x - y)$ for all $x, y \in X$;
(iv) $g(y) \leq g(x) + (-\varphi_\cdot)(y - x)$ for all $x, y \in X$;
(v) $g = g \triangledown (-\varphi_\cdot)$;
(vi) $-g \in \mathcal{E}^{\varphi_-}$.

(b) For every function $f : X \to \overline{\mathbb{R}}$, $f \triangledown (-\varphi_\cdot)$ is the greatest $\varphi$-envelope that is majorized by $f$, while $f \triangle \varphi$ is the lowest $\varphi$-envelope that is minorized by $f$.

(c) The following inclusion holds true $\mathcal{E}^{-\varphi} \subset \mathcal{E}^{\varphi_-}$.

Proof. (a) Let us assume that $g \in \mathcal{E}^\varphi$. Then there exists $f : X \to \overline{\mathbb{R}}$ such that $g = f^\varphi = (-f) \triangle \varphi$. Using the superadditivity of $\varphi$ and the last assertion of Lemma 6.1, we have

\[ g \triangle \varphi = ((-f) \triangle \varphi) \triangle \varphi = (-f) \triangle (\varphi \triangle \varphi) = (-f) \triangle \varphi = g. \]

This shows that (i) $\implies$ (ii). Conversely, if $g = g \triangle \varphi$ then $g = (-g)^\varphi$ and clearly $g \in \mathcal{E}^\varphi$. The equivalences (ii) $\iff$ (iii) $\iff$ (iv) $\iff$ (v) follow directly from
Lemma 6.1. For the equivalence \((v) \iff (vi)\), observe that
\[ g = g \vartriangle (-\varphi_-) \iff -g = (-g) \vartriangle \varphi_- , \]
and invoke the equivalence \((i) \iff (ii)\).

(b) Let \(f : X \to \mathbb{R}\). Observe that
\[
(f \vartriangle (-\varphi_-)) \vartriangle (-\varphi_-) = f \vartriangle ((-\varphi_-) \vartriangle (-\varphi_-)) = f \vartriangle (-\varphi_-) \vartriangle (-\varphi_-)
\]
by Lemma 6.1 since \(\varphi_-\) is superadditive.

In view of the implication \((v) \implies (ii)\) in (a), we deduce that
\[
f \vartriangle (-\varphi_-) = (f \vartriangle (-\varphi_-)) \vartriangle \varphi_- = ((-f) \vartriangle \varphi_-) = (f \vartriangle -\varphi_-)^\circ .
\]
Hence \(f \vartriangle (-\varphi_-)\) coincides with \((f \vartriangle -\varphi_-)^\circ\), which is by property (6) the greatest element of \(\mathcal{E}^\varphi\) that is majorized by \(f\). Replacing \(f\) (resp. \(\varphi\)) with \(-f\) (resp. \(\varphi_-\)) and taking the opposite, we deduce that \(f \vartriangle \varphi_-\) is the lowest element of \(-\mathcal{E}^\varphi_-\) that is minorized by \(f\). It suffices then to recall that \(\mathcal{E}^\varphi_- = -\mathcal{E}^\varphi\), see the equivalence \((i) \iff (vi)\) in (a).

(c) Since \(\varphi \in \mathcal{E}^\varphi\), we have \(-\varphi \in -\mathcal{E}^\varphi = \mathcal{E}^\varphi_-\). In view of Proposition 3.2 (iii), we infer that \(\mathcal{E}^{-\varphi} \subset \mathcal{E}^\varphi_-\).

\[\square\]

Example 6.1. Assume that \((X, \| \cdot \|)\) is a normed space. For \(k \geq 0\) and \(\alpha \in [0, 1]\), take \(\varphi = -k \| \cdot \|^\alpha\). Observe that for all \(x, y \in X\)
\[
\| x + y \|^\alpha \leq (\| x \| + \| y \|)^\alpha \leq \| x \|^\alpha + \| y \|^\alpha. \tag{40}
\]
It ensues that the function \(\| \cdot \|^\alpha\) is subadditive, hence \(\varphi\) is superadditive. From Theorem 6.1 (a), we deduce that
\[
f \in \mathcal{E}^{-k \| \cdot \|^\alpha} \iff f(x) \geq f(y) - k \| x - y \|^\alpha \quad \text{for all} \quad x, y \in X. \tag{41}
\]
By reversing the roles of \(x\) and \(y\), we immediately obtain
\[
f \in \mathcal{E}^{-k \| \cdot \|^\alpha} \iff f(x) \leq f(y) + k \| x - y \|^\alpha \quad \text{for all} \quad x, y \in X. \tag{42}
\]
If \(f(y) = +\infty\) (resp. \(f(y) = -\infty\)) for some \(y \in X\), we deduce from (41) (resp. (42)) that \(f = \omega_X\) (resp. \(f = -\omega_X\)). On the other hand, if the function \(f\) is finite-valued, we infer from (41)-(42) that \(|f(x) - f(y)| \leq k \| x - y \|^\alpha\) for all \(x, y \in X\). This implies that
\[
\mathcal{E}^{-k \| \cdot \|^\alpha} = \{ f : X \to \mathbb{R}, |f(x) - f(y)| \leq k \| x - y \|^\alpha \quad \text{for all} \quad x, y \in X \} \cup \{ \omega_X, -\omega_X \} = \{ f : X \to \mathbb{R}, f \text{ is } \alpha\text{-Hölderian with constant } k \} \cup \{ \omega_X, -\omega_X \}.
\]
From Theorem 6.1 (b), we deduce that \(f \vartriangle k \| \cdot \|^\alpha\) (resp. \(f \vartriangle (-k \| \cdot \|^\alpha)\)) is the greatest (resp. lowest) \(\varphi\)-envelope that is majorized (resp. minorized) by \(f\). Since the map \(\| \cdot \|\alpha\) is even, Theorem 6.1 (c) shows that \(\mathcal{E}^{k \| \cdot \|^\alpha} \subset \mathcal{E}^{-k \| \cdot \|^\alpha}\).

Now assume that \(\alpha = 1\). From what precedes, we obtain that
\[
\mathcal{E}^{-k} = \{ f : X \to \mathbb{R}, f \text{ is } k\text{-Lipschitz continuous} \} \cup \{ \omega_X, -\omega_X \}.
\]
The Pasch-Hausdorff regularization of \(f\), defined by \(l_k(f) = f \vartriangle k \| \cdot \|\), is the greatest function of \(\mathcal{E}^{-k}\) that is majorized by \(f\). On the other hand, \(f \vartriangle (-k \| \cdot \|)\) is the lowest function of \(\mathcal{E}^{-k}\) that is minorized by \(f\). The inclusion \(\mathcal{E}^{k} \subset \mathcal{E}^{-k}\) shows that the \(k\)-envelopes are either \(k\)-Lipschitz continuous or equal to \(\pm \omega_X\).

The convexity of \(\| \cdot \|\) implies that \(k\)-envelopes are also convex, therefore the
inclusion $E^{k\|\cdot\|} \subset E^{-k\|\cdot\|}$ is strict. This ensures that the inclusion in (c) of the above theorem generally fails to be an equality.

As regards the function $\varphi = -k\|\cdot\|^\alpha$ it is also worth mentioning that, for $\eta(x, y) := \|x - y\|^\alpha$ with $\alpha > 0$ (even with more general coupling functions) and taking

$$\Phi := \{r - \sigma \eta(\cdot, y) : r \in \mathbb{R}, \sigma > 0, y \in X\},$$

a lower semicontinuous function on the normed space $X$ is shown in [7, Theorem 4.2] to be $\Phi$-convex (i.e., a pointwise supremum of functions in $\Phi$), whenever it is bounded from below by a function in $\Phi$. The latter property with $\alpha = 2$ was previously proved in [29, Theorem 2]. The function $(x, y) \mapsto -k\|x - y\|^\alpha$ is also used as a particular important example of coupling functions arising in the framework of generalized conjugacy in many papers, see for example [23, p. 204].

Remark 6.1. Given a nonincreasing convex function $\theta : \mathbb{R}_+ \to \mathbb{R}$ such that $\theta(0) = 0$, one can easily check that the function $\theta(\|\cdot\|)$ is superadditive. Hence the previous example can be generalized by taking $\varphi = \theta(\|\cdot\|)$.

Example 6.2. Let $X$ be a vector space. Let $\Lambda \subset X$ be a set containing the origin and such that $\Lambda + \Lambda \subset \Lambda$. The function $\delta_\Lambda$ is clearly subadditive. This implies that the function $\varphi = -\delta_\Lambda$ is superadditive. By Theorem 6.1 (a) it follows that

$$f \in E^{-\delta_\Lambda} \iff f(x) \geq f(y) + (-\delta_\Lambda)(x - y) \quad \text{for all } x, y \in X$$
$$\iff f(x) \geq f(y) \quad \text{if } x - y \in \Lambda$$
$$\iff f \text{ is } \Lambda\text{-nondecreasing.}$$

This and Theorem 6.1 (b) entail that $f \vee \delta_\Lambda$ (resp. $f \wedge (-\delta_\Lambda)$) is the greatest (resp. lowest) $\Lambda$-nondecreasing function that is majorized (resp. minorized) by $f$. Further, Theorem 6.1 (c) says that $E^{\delta_\Lambda} \subset E^{(-\delta_\Lambda)} = E^{-\delta_\Lambda}$, hence the functions of $E^{\delta_\Lambda}$ are $\Lambda$-nonincreasing. In fact, this can be recovered directly by using the characterization of $E^{\delta_\Lambda}$ given by Proposition 4.4 (ii).

7. CASE $\varphi \in \Gamma(X)$

7.1. Expressions of $\varphi$-envelopes as Legendre-Fenchel conjugates. Let us start with the following elementary lemma.

Lemma 7.1. Let $X$ be a locally convex space. For every function $f : X \to \mathbb{R}$, we have $(f^\ast)^\ast = (f_-)^\ast$.

Proof. It suffices to use the definition of the Legendre-Fenchel conjugate. For every $\xi^\ast \in X^\ast$, we have

$$(f^\ast)^\ast(\xi^\ast) = (f^\ast)(-\xi^\ast) = \sup_{x \in X}\{(-\xi^\ast, x) - f(x)\}$$
$$= \sup_{y \in X}\{(\xi^\ast, y) - f(-y)\}$$
$$= \sup_{y \in X}\{(\xi^\ast, y) - f_-(y)\} = (f_-)^\ast(\xi^\ast).$$

□
Theorem 7.1. Let $X$ be a locally convex space. Let us assume that $\varphi \in \Gamma(X)$ and let $\psi : X^* \to \mathbb{R}$ be such that $\psi^* = \varphi$. Then we have for every function $f : X \to \mathbb{R}$

$$f^\varphi = (\psi \ast (f_-)^*)^*.$$  \hfill (43)

Moreover the following equivalences hold

\begin{align*}
g \in \mathcal{E}^\varphi & \iff g = (\psi \ast h)^* \text{ for some } h \in \Gamma(X^*) \\
& \iff g = (\psi \ast (\psi \ast g^*)^{**})^*.
\end{align*}

Proof. For every $x \in X$,

\begin{align*}
f^\varphi(x) & = \sup_{y \in X} \{ \varphi(x - y) - f(y) \} \\
& = \sup_{y \in X} \sup_{\xi^* \in X^*} \{ \langle \xi^*, x - y \rangle - \psi(\xi^*) \} - f(y) \\
& = \sup_{\xi^* \in X^*} \sup_{y \in X} \{ \langle \xi^*, x - y \rangle - \psi(\xi^*) \} - f(y) \\
& = \sup_{\xi^* \in X^*} \sup_{y \in X} \{ \langle \xi^*, y \rangle - f(y) \} - \psi(\xi^*) + \langle \xi^*, x \rangle \\
& = (\psi \ast (f_+)^* - f_+)^*(x) \\
& = (\psi \ast (f_-)^*)^*(x) \text{ in view of Lemma 7.1.}
\end{align*}

For the first equivalence, recall that $g \in \mathcal{E}^\varphi$ if and only if there exists $f : X \to \mathbb{R}$ such that $g = f^\varphi$. Then use the equality $f^\varphi = (\psi \ast (f_-)^*)^*$ and the fact that the range of the Legendre-Fenchel transform is equal to $\Gamma(X^*)$, see, e.g., [20].

For the second equivalence, observe that

\begin{align*}
g \in \mathcal{E}^\varphi & \iff g = (g^\varphi)^* \\
& \iff g = \left[ (\psi \ast (g_-)^*)^* \right]^\varphi \text{ from formula (43)} \\
& \iff g = \left[ ((\psi \ast g^*)^*)^- \right]^\varphi \text{ by Lemma 7.1} \\
& \iff g = \left( \psi \ast (\psi \ast g^*)^{**} \right)^* \text{ from formula (43) again.}
\end{align*}

\[ \square \]

Remark 7.1. Since $\varphi \in \Gamma(X)$ by assumption, we have $\varphi^{**} = \varphi$, hence we can take $\psi = \varphi^*$ in the statement of Theorem 7.1.

Remark 7.2. Formula (43) can be recovered partially by using a formula on the conjugate of the difference of functions. Recall that for $\psi : X \to \mathbb{R} \cup \{+\infty\}$ and
Proof. For the first equality, it suffices to use Theorem 7.1 and the fact that the functions $f$ since
Assume that $\varphi \in \Gamma_0(X)$ and $f \in \Gamma_0(X)$. By reversing the roles of $X$ and $X^*$ and by using
equality (44) with $h = (f_-)^*$ and $\psi : X^* \to \mathbb{R} \cup \{+\infty\}$ such that $\psi^* = \varphi$, we find
\[
(\psi - (f_-)^*)^* = \varphi \ominus (f_-)^*
= \varphi \ominus f_- = f^\varphi.
\]
Hence we recover formula (43) in the case where both functions $\varphi$ and $f$ are in $\Gamma_0(X)$.

The next corollary says in particular that the $\varphi$-envelope of a function coincides with the $\varphi$-envelope of its lower semicontinuous convex hull whenever $\varphi \in \Gamma(X)$.

**Corollary 7.1.** Let $X$ be a locally convex space and $\varphi \in \Gamma(X)$. Then we have for every function $f : X \to \mathbb{R}$ and every function $g : X \to \mathbb{R}$ satisfying $\underline{\varphi} f \leq g \leq f$,
\[
f^\varphi = (\underline{\varphi} f)^\varphi = g^\varphi.
\]

**Proof.** For the first equality, it suffices to use Theorem 7.1 and the fact that the functions $f$ and $\underline{\varphi} f$ have the same Legendre-Fenchel conjugate. On the other hand, since $\underline{\varphi} f \leq g \leq f$, we see that $f^\varphi \leq g^\varphi \leq (\underline{\varphi} f)^\varphi$. Recalling that $f^\varphi = (\underline{\varphi} f)^\varphi$, the second equality immediately follows. $\square$

For every set $D \subset X^*$, we define as in section 3 the classes $\Sigma_D$ and $\Sigma_D^*$ by
\[
\Sigma_D = \{f : X^* \to \mathbb{R}, \text{ dom } f \subset D\} \quad \text{and} \quad \Sigma_D^* = \{f^*, f \in \Sigma_D\}.
\]
In the same vein, let us define the classes $\hat{\Sigma}_D$ and $\hat{\Sigma}_D^*$ by
\[
\hat{\Sigma}_D = \{f : X^* \to \mathbb{R}, \text{ dom } f = D\} \quad \text{and} \quad \hat{\Sigma}_D^* = \{f^*, f \in \hat{\Sigma}_D\}.
\]
The following proposition allows us to characterize the classes $\hat{\Sigma}_D^*$ and $\Sigma_D^*$.

**Proposition 7.1.** Let $X$ be a locally convex space and let $D \subset X^*$ be such that $D = \{a_i^*, i \in I\}$ for some set $I$. Then for every function $f : X \to \mathbb{R}$, we have
\[
f \in \hat{\Sigma}_D^* \quad \text{(resp. } \Sigma_D^*) \quad \text{if and only if there exists a family } (\alpha_i)_{i \in I} \subset \mathbb{R} \cup \{+\infty\} \quad \text{(resp. } \mathbb{R}) \quad \text{such that } f = \sup_{i \in I} (a_i^*, \cdot) + \alpha_i.
\]

**Proof.** Assume that $f \in \hat{\Sigma}_D^*$ (resp. $\Sigma_D^*$). By definition, there exists $g : X^* \to \mathbb{R}$ such that $f = g^*$ and dom $g = D$ (resp. dom $g \subset D$). Hence we have
\[
f = \sup_{x^* \in D} \langle x^*, \cdot \rangle - g(x^*) = \sup_{i \in I} \langle a_i^*, \cdot \rangle - g(a_i^*).
\]
By setting $\alpha_i = -g(a_i^*)$ for every $i \in I$, we obtain $f = \sup_{i \in I} (a_i^*, \cdot) + \alpha_i$ with $\alpha_i \in \mathbb{R} \cup \{+\infty\}$ (resp. $\mathbb{R}$).
Conversely, assume that there exists \((\alpha_i)_{i \in I} \subset \mathbb{R} \cup \{+\infty\}\) (resp. \(\mathbb{R}\)) such that \(f = \sup_{i \in I} \langle a_i^*, \cdot \rangle + \alpha_i\). Then we have
\[
f = \sup_{x^* \in D} \left[ \sup_{\{i \in I, a_i^* = x^*\}} \langle a_i^*, \cdot \rangle + \alpha_i \right] \\
= \sup_{x^* \in D} \left[ \langle x^*, \cdot \rangle + \sup_{\{i \in I, a_i^* = x^*\}} \alpha_i \right].
\]
Defining the function \(h : X^* \to \mathbb{R}\) by
\[
h(x^*) = \begin{cases} \sup_{\{i \in I, a_i^* = x^*\}} \alpha_i & \text{if } x^* \in D \\ -\infty & \text{if } x^* \notin D, \end{cases}
\]
we obtain
\[
f = \sup_{x^* \in D} \langle x^*, \cdot \rangle + h(x^*) \\
= \sup_{x^* \in X^*} \langle x^*, \cdot \rangle + h(x^*).
\]
We conclude that \(f = (-h)^*\) with \(\text{dom}(-h) = D\) (resp. \(\text{dom}(-h) \subset D\)), hence \(f \in \hat{\Sigma}_D^*\) (resp. \(f \in \Sigma_D^*\)). \(\square\)

**Example 7.1.** Take \(D = \{a_1^*, \ldots, a_n^*\} \subset X^*\) for some \(n \geq 1\). The previous proposition shows that, for every function \(f : X \to \mathbb{R}\),
\[
f \in \Sigma_D^* \iff f = \sup_{i=1}^n \langle a_i^*, \cdot \rangle + \alpha_i \quad \text{for some } \alpha_1, \ldots, \alpha_n \in \mathbb{R}.
\]
On the other hand, if \(f \in \Gamma_0(X)\), the following equivalence holds true
\[
\text{dom } f^* \subset D \iff \text{dom } f^* \subset \{a_i^*\} \quad \text{for some } i \in \{1, \ldots, n\}
\]
because the set \(\text{dom } f^*\) is convex. Since \(f^*\) is proper, this is in turn equivalent to \(f^* = \delta_{\{a_i^*\}} - \alpha_i\) for some \(\alpha_i \in \mathbb{R}\). Taking the conjugate, we find \(f = \langle a_i^*, \cdot \rangle + \alpha_i\).
It ensues that the set \(\{f \in \Gamma_0(X), \text{dom } f^* \subset D\}\) coincides with the set of affine continuous functions with slopes in \(D = \{a_1^*, \ldots, a_n^*\}\). This yields an example for which the inclusion (14) is strict. By applying again Proposition 7.1, we obtain that
\[
f \in \Sigma_{\text{co}(D)}^* \iff f = \sup_{x^* \in \text{co}(D)} \langle x^*, \cdot \rangle + \alpha_{x^*},
\]
with \(\alpha_{x^*} \in \mathbb{R}\) for every \(x^* \in \text{co}(D)\). The comparison of (45) and (46) clearly shows that the inclusion \(\Sigma_D^* \subset \Sigma_{\text{co}(D)}^*\) is strict as soon as the set \(D = \{a_1^*, \ldots, a_n^*\}\) is not a singleton. This easily implies that the inclusion (15) is strict for such a set \(D\).

The next result gives several upper bounds (in the sense of inclusion) for the set \(\mathcal{E}^\varphi\), respectively when \(\varphi \in \Gamma(X)\), \(\varphi \in \hat{\Sigma}_D^*\) and \(\varphi \in \Sigma_D^*\).

**Corollary 7.2.** Let \(X\) be a locally convex space and let \(\varphi \in \Gamma(X)\).

(i) The following inclusions hold true
\[
\mathcal{E}^\varphi \subset \bigcap_{\{\psi, \varphi = \psi^*\}} \hat{\Sigma}_{\text{dom } \psi}^* \cup \{-\omega_X\} \subset \bigcap_{\{\psi, \varphi = \psi^*\}} \Sigma_{\text{dom } \psi}^*.
\]
(ii) For every subset $D \subset X^*$, we have
\[
\varphi \in \hat{\Sigma}^*_D \iff \mathcal{E}^\varphi \subset \hat{\Sigma}^*_D \cup \{-\omega_X\} \quad \text{if } \varphi \neq -\omega_X
\]
\[
\varphi \in \Sigma^*_D \iff \mathcal{E}^\varphi \subset \Sigma^*_D.
\]
(iii) Assume that there exist families $(a^*_i)_{i \in I} \subset X^*$ and $(\alpha_i)_{i \in I} \subset \mathbb{R} \cup \{+\infty\}$
(resp. $\mathbb{R}$) such that
\[
\varphi = \sup_{i \in I} (a^*_i, \cdot) + \alpha_i.
\]
Then for every $g \in \mathcal{E}^\varphi \setminus \{-\omega_X\}$ (resp. $g \in \mathcal{E}^\varphi$), there exists $(\beta_i)_{i \in I} \subset \mathbb{R} \cup \{+\infty\}$ (resp. $\mathbb{R}$) such that
\[
g = \sup_{i \in I} (a^*_i, \cdot) + \beta_i.
\]
In particular, if the set $I$ is finite, every $\varphi$-envelope is polyhedral.

Proof. (i) Let $\psi : X^* \to \mathbb{R}$ be such that $\varphi = \psi^*$. Assuming that $g \in \mathcal{E}^\varphi$, Theorem 7.1 shows that $g = (\psi - h)^*$ for some $h \in \Gamma(X^*)$. If $h = -\omega_{X^*}$, we have $\psi - h = \omega_{X^*}$ and therefore $g = -\omega_X$. If $h \neq -\omega_{X^*}$, we see that $\text{dom}(\psi - h) = \text{dom} \psi$, hence $g \in \hat{\Sigma}^*_{\text{dom} \psi}$. We deduce the inclusion $\mathcal{E}^\varphi \subset \hat{\Sigma}^*_\text{dom} \psi \cup \{-\omega_X\}$. Since this is true for every function $\psi : X^* \to \mathbb{R}$ such that $\varphi = \psi^*$, the first inclusion of (47) follows.

For the second inclusion, it suffices to notice that $\hat{\Sigma}^*_\text{dom} \psi \subset \Sigma^*_\text{dom} \psi$. (ii) Let us fix $D \subset X^*$ and assume that $\varphi \in \hat{\Sigma}^*_D$. Then there exists $\psi : X^* \to \mathbb{R}$ such that $\varphi = \psi^*$ and $\text{dom} \psi = D$. We deduce from the first inclusion of (47) that
\[
\mathcal{E}^\varphi \subset \hat{\Sigma}^*_\text{dom} \psi \cup \{-\omega_X\} = \hat{\Sigma}^*_D \cup \{-\omega_X\}.
\]
Conversely, if $\mathcal{E}^\varphi \subset \hat{\Sigma}^*_D \cup \{-\omega_X\}$ and if $\varphi \neq -\omega_X$, then we obtain $\varphi \in \hat{\Sigma}^*_D$ according to the inclusion $\varphi \in \mathcal{E}^\varphi$. The proof of the second equivalence is analogous and left to the reader.

(iii) Let $(a^*_i)_{i \in I} \subset X^*$ and $(\alpha_i)_{i \in I} \subset \mathbb{R} \cup \{+\infty\}$ (resp. $\mathbb{R}$) be such that $\varphi = \sup_{i \in I} (a^*_i, \cdot) + \alpha_i$. Let us set $D = \{a^*_i, i \in I\}$. Proposition 7.1 shows that $\varphi \in \hat{\Sigma}^*_D$ (resp. $\Sigma^*_D$). If $g \in \mathcal{E}^\varphi \setminus \{-\omega_X\}$ (resp. $g \in \mathcal{E}^\varphi$), we deduce from (ii) that $g \in \hat{\Sigma}^*_D$ (resp. $\Sigma^*_D$). By applying Proposition 7.1 again, we derive the existence of $(\beta_i)_{i \in I} \subset \mathbb{R} \cup \{+\infty\}$ (resp. $\mathbb{R}$) such that $g = \sup_{i \in I} (a^*_i, \cdot) + \beta_i$. Finally, if the set $I$ is finite and if $g$ is a $\varphi$-envelope, then either $g = \pm \omega_X$ or the function $g$ is the supremum of a finite collection of continuous affine functions. We then conclude that $g$ is polyhedral.

By applying the second equivalence of Corollary 7.2 (ii) with $D = X^*$, we obtain that $\varphi \in \Gamma(X)$ if and only if $\mathcal{E}^\varphi \subset \Gamma(X)$. Corollary 7.3 below shows that in this case the set $\mathcal{E}^\varphi$ is strictly included in $\Gamma(X)$. Notice that for $\varphi \in \Gamma_0(X)$ satisfying a suitable condition (named generating condition), the functions of the class $\mathcal{E}^\varphi$ have been studied in [24] under the terminology of $\varphi$-strongly convex functions.

Following Theorem 7.1 and Remark 7.1, we have $g \in \mathcal{E}^\varphi$ if and only if $g = (\varphi^* - h)^*$ for some $h \in \Gamma(X^*)$. Let us now have a look at the class of the functions equal to $(\varphi^* - h)^*$ for some $h : X^* \to \mathbb{R} \cup \{+\infty\}$ not necessarily in $\Gamma(X^*)$.

**Proposition 7.2.** Let $X$ be a locally convex space. Assume that $\varphi \in \Gamma_0(X)$ and $g \in \Gamma_0(X)$. 

(i) If \( g = (\varphi^* \div h)^* \) for some \( h : X^* \to \mathbb{R} \cup \{+\infty\} \), then we have \( g^\infty = \varphi^\infty \), which is equivalent to \( \text{cl}w^*(\text{dom} g^*) = \text{cl}w^*(\text{dom} \varphi^*) \).

(ii) If \( \text{dom} g^* = \text{dom} \varphi^* \), then \( g = (\varphi^* \div h)^* \) for \( h : X^* \to \mathbb{R} \cup \{+\infty\} \) given by \( h = \varphi^* \div g^* \).

Proof. (i) Assume that \( g = (\varphi^* \div h)^* \) for some \( h : X^* \to \mathbb{R} \cup \{+\infty\} \). By definition of the Legendre-Fenchel transform, we obtain
\[
g = \sup_{\xi^* \in X^*} \{ (\xi^*, \cdot) + h(\xi^*) - \varphi^*(\xi^*) \}
\]
\[
= \sup_{\xi^* \in \text{dom} \varphi^*} \{ (\xi^*, \cdot) + h(\xi^*) - \varphi^*(\xi^*) \}.
\]

Observe that the function \( h \) cannot take the value \(+\infty\) on \( \text{dom} \varphi^* \) (otherwise we would have \( g = \omega_X \)). Therefore the values \(-\varphi^*(\xi^*)\) and \( h(\xi^*) \) are finite for every \( \xi^* \in \text{dom} \varphi^* \). By taking the recession function of each member of (48), we obtain
\[
g^\infty = \sup_{\xi^* \in \text{dom} \varphi^*} \{ (\xi^*, \cdot) + h(\xi^*) - \varphi^*(\xi^*) \}^\infty.
\]

The recession function of the affine map \( \langle \xi^*, \cdot \rangle + h(\xi^*) - \varphi^*(\xi^*) \) is equal to \( \langle \xi^*, \cdot \rangle \), thus implying that \( g^\infty = \sup_{\xi^* \in \text{dom} \varphi^*} \langle \xi^*, \cdot \rangle = \sigma_{\text{dom} \varphi^*} \). Recalling that \( \sigma_{\text{dom} \varphi^*} = \varphi^\infty \), we deduce that \( g^\infty = \varphi^\infty \), which is in turn equivalent to the equality \( \text{cl}w^*(\text{dom} g^*) = \text{cl}w^*(\text{dom} \varphi^*) \), see [20].

(ii) Assume that \( \text{dom} g^* = \text{dom} \varphi^* \). It is easy to check that for every \( x^* \in X^* \),
\[
(\varphi^* \div (\varphi^* \div g^*))(x^*) = \begin{cases} g^*(x^*) & \text{if } x^* \in \text{dom} g^* \\ +\infty & \text{if } x^* \notin \text{dom} g^*. \end{cases}
\]

It ensues that \( \varphi^* \div (\varphi^* \div g^*) = g^* \). Since \( g \in \Gamma_0(X) \) by assumption, we have \( g = g^\infty \) hence \( g = (\varphi^* \div (\varphi^* \div g^*))^* \). The function \( h = \varphi^* \div g^* \) takes its values in \( \mathbb{R} \cup \{+\infty\} \) because \( \text{dom} g^* = \text{dom} \varphi^* \). \( \square \)

Combining Theorem 7.1 and Proposition 7.2, we derive a necessary (resp. sufficient) condition for a function \( g \in \Gamma_0(X) \) to be a \( \varphi \)-envelope.

**Corollary 7.3.** Let \( X \) be a locally convex space. Assume that \( \varphi \in \Gamma_0(X) \) and \( g \in \Gamma_0(X) \).

(i) If \( g \in \mathcal{E}^\varphi \) then \( g^\infty = \varphi^\infty \).

(ii) If \( \text{dom} g^* = \text{dom} \varphi^* \) and \( \varphi^* \div g^* \in \Gamma_0(X^*) \), then \( g \in \mathcal{E}^\varphi \).

Proof. (i) If \( g \in \mathcal{E}^\varphi \), we deduce from Theorem 7.1 that \( g = (\varphi^* \div h)^* \) for some \( h \in \Gamma(X^*) \). Since \( g \in \Gamma_0(X) \) by assumption, we have \( h \neq -\omega_X \), hence the function \( h \) does not take the value \(-\infty \). Proposition 7.2 (i) then implies that \( g^\infty = \varphi^\infty \).

(ii) If \( \text{dom} g^* = \text{dom} \varphi^* \), Proposition 7.2 (ii) shows that \( g = (\varphi^* \div h)^* \) with \( h = \varphi^* \div g^* \). Since \( h \in \Gamma_0(X^*) \) by assumption, we conclude by Theorem 7.1 that \( g \in \mathcal{E}^\varphi \). \( \square \)
7.2. Klee envelopes. Let \((X, \| \cdot \|)\) be a normed space and let \(f : X \to \mathbb{R}\) be an extended real-valued function. For any reals \(\lambda > 0\) and \(p \geq 1\), we define the Klee envelope of \(f\) with index \(\lambda\) and power \(p\) as

\[
\kappa_{\lambda,p}f(x) = \sup_{y \in X} \left( \frac{1}{p\lambda} \|x - y\|^p - f(y) \right).
\]

In other words, we have \(\kappa_{\lambda,p}f = f^\varphi\) with the function \(\varphi : X \to \mathbb{R}\) defined by \(\varphi(x) = \frac{1}{p\lambda} \|x\|^p\). Applying Theorem 7.1 with \(\varphi = \frac{1}{p\lambda} \| \cdot \|^p\) and denoting by \(\| \cdot \|_X\), the dual norm on \(X^*\) we obtain the following result.

**Corollary 7.4.** Let \((X, \| \cdot \|)\) be a normed space. For any \(\lambda > 0\), \(p > 1\) and for every function \(f : X \to \mathbb{R}\), we have

\[
\kappa_{\lambda,p}f = \left( \frac{\lambda^{q-1}}{q} \| \cdot \|_X^q - (f^\ast)^\ast \right)^\ast,
\]

where \(q > 1\) is the conjugate exponent of \(p\). Moreover the following assertions are equivalent

(i) \(g\) is a Klee envelope with index \(\lambda\) and power \(p\);

(ii) \(g = \left( \frac{\lambda^{q-1}}{q} \| \cdot \|_X^q - h \right)^\ast\) for some \(h \in \Gamma(X^*)\);

(iii) \(g = \left( \frac{\lambda^{q-1}}{q} \| \cdot \|_X^q - \left( \frac{\lambda^{q-1}}{q} \| \cdot \|_X^q - g^\ast \right)^{\ast\ast} \right)^\ast\).

These assertions are satisfied whenever the following stronger condition is fulfilled

(iv) \(g \in \Gamma(X)\) and \(\frac{\lambda^{q-1}}{q} \| \cdot \|_X^q - g^\ast \in \Gamma(X^*)\).

**Proof.** It suffices to apply Theorem 7.1 with \(\varphi = \frac{1}{p\lambda} \| \cdot \|^p\) and \(\psi = \varphi^\ast = \frac{\lambda^{q-1}}{q} \| \cdot \|_X^q\). Let us now establish the implication \((iv) \implies (ii)\). Assume that \(g \in \Gamma(X)\) and that \(\frac{\lambda^{q-1}}{q} \| \cdot \|_X^q - g^\ast \in \Gamma(X^*)\). The function \(q^\ast\) can be written as \(g^\ast = \frac{\lambda^{q-1}}{q} \| \cdot \|_X^q - h\) for some \(h \in \Gamma(X^*)\). Since \(g \in \Gamma(X)\) by assumption, we have \(g = g^{\ast\ast}\). Hence we deduce that \(g = \left( \frac{\lambda^{q-1}}{q} \| \cdot \|_X^q - h \right)^\ast\) and assertion \((ii)\) is proved. \(\square\)

**Corollary 7.5.** Let \((X, \| \cdot \|)\) be a normed space. For every \(p > 1\) and every \(C \subset X\), the farthest distance function \(\Delta_C = \sup_{y \in C} \| \cdot - y \|^p\) satisfies

\[
\frac{1}{p} \Delta_C^p = \left( \frac{1}{q} \| \cdot \|_X^q - \sigma_C \right)^\ast.
\]

**Proof.** Observe that

\[
\kappa_{1,p} \delta_C = \sup_{y \in X} \left\{ \frac{1}{p} \| \cdot - y \|^p - \delta_C(y) \right\} = \sup_{y \in C} \frac{1}{p} \| \cdot - y \|^p = \frac{1}{p} \Delta_C^p.
\]

It suffices then to apply formula (49) of Corollary 7.4 with \(f = \delta_C\) and \(\lambda = 1\). \(\square\)

Additional properties of the Klee envelopes can be obtained in the case when \((X, \| \cdot \|)\) is a Hilbert space and \(p = 2\).

**Theorem 7.2.** Assume that \(X\) is a Hilbert space endowed with the scalar product \(\langle \cdot, \cdot \rangle\) and the corresponding norm \(\| \cdot \|\).
(a) For every \( \lambda > 0 \) and every function \( f : X \to \mathbb{R} \), we have
\[
\kappa_{\lambda, 2} f = \left( \frac{\lambda}{2} \Vert \cdot \Vert^2 - (f_-)^* \right)^* \\
= \left( f - \frac{1}{2\lambda} \Vert \cdot \Vert^2 \right)^* \left( -\frac{\cdot}{\lambda} \right) + \frac{1}{2\lambda} \Vert \cdot \Vert^2 ;
\]
(50)
\[
\kappa_{\lambda, 2} (\kappa_{\lambda, 2} f) = \left( f - \frac{1}{2\lambda} \Vert \cdot \Vert^2 \right)^{**} + \frac{1}{2\lambda} \Vert \cdot \Vert^2.
\]
(52)

(b) For \( \lambda > 0 \) and \( f : X \to \mathbb{R} \) the following assertions are equivalent
(i) \( f \) is a Klee envelope with index \( \lambda \) and power \( 2 \);
(ii) \( f = \left( \frac{1}{2} \Vert \cdot \Vert^2 - h \right)^* \) for some \( h \in \Gamma(X) \);
(iii) \( f = \left( \frac{1}{2} \Vert \cdot \Vert^2 - \left( \frac{1}{2} \Vert \cdot \Vert^2 - f^* \right)^{**} \right)^* \);
(iv) \( f - \frac{1}{2\lambda} \Vert \cdot \Vert^2 \in \Gamma(X) \);
(v) \( f \in \Gamma(X) \) and \( \frac{1}{2} \Vert \cdot \Vert^2 - f^* \in \Gamma(X) \).

Proof. (a) For the equality (50), it suffices to apply Corollary 7.4 with \( p = 2 \). For the equality (51), observe that for every \( x \in X \),
\[
\kappa_{\lambda, 2} f(x) = \sup_{y \in X} \left\{ \frac{1}{2\lambda} \Vert x - y \Vert^2 - f(y) \right\} \\
= \sup_{y \in X} \left\{ \frac{1}{2\lambda} \Vert x \Vert^2 + \frac{1}{2\lambda} \Vert y \Vert^2 - \frac{1}{\lambda} (x, y) - f(y) \right\} \\
= \left( f - \frac{1}{2\lambda} \Vert \cdot \Vert^2 \right)^* \left( -\frac{x}{\lambda} \right) + \frac{1}{2\lambda} \Vert \cdot \Vert^2.
\]
By iterating we deduce that
\[
\kappa_{\lambda, 2} (\kappa_{\lambda, 2} f) = \left( \kappa_{\lambda, 2} f - \frac{1}{2\lambda} \Vert \cdot \Vert^2 \right)^* \left( -\frac{\cdot}{\lambda} \right) + \frac{1}{2\lambda} \Vert \cdot \Vert^2 \\
= \left[ \left( f - \frac{1}{2\lambda} \Vert \cdot \Vert^2 \right)^* \left( -\frac{\cdot}{\lambda} \right) \right]^* \left( -\frac{\cdot}{\lambda} \right) + \frac{1}{2\lambda} \Vert \cdot \Vert^2 \\
= \left( f - \frac{1}{2\lambda} \Vert \cdot \Vert^2 \right)^{**} + \frac{1}{2\lambda} \Vert \cdot \Vert^2,
\]
which proves the equality (52).

(b) We now show that assertions (i) to (v) are equivalent. The equivalences (i) \( \iff \) (ii) \( \iff \) (iii) are consequences of Corollary 7.4 applied with \( p = 2 \). Let us show the equivalence (i) \( \iff \) (iv). Observe that \( f \) is a Klee envelope with index \( \lambda \) and power \( 2 \) if and only if \( f \in \mathcal{E}^\varphi \) with \( \varphi = \frac{1}{2\lambda} \Vert \cdot \Vert^2 \). From the equivalence (7) \( \iff \) (8) and the fact that \( \varphi_- = \varphi \), this is in turn equivalent to \( f = (f^\varphi)^\varphi \). Since \( (f^\varphi)^\varphi = \kappa_{\lambda, 2} (\kappa_{\lambda, 2} f) \) and using the equality (52), we infer that
\[
f \text{ is a Klee envelope with index } \lambda \text{ and power } 2
\]
\[
f - \frac{1}{2\lambda} \Vert \cdot \Vert^2 = \left( f - \frac{1}{2\lambda} \Vert \cdot \Vert^2 \right)^{**}
\]
Hence the equivalence $(i) \iff (iv)$ is proved. Let us now show that $(iv) \implies (v)$. If 
\[ f - \frac{1}{2\lambda} \| \cdot \|^2 \in \Gamma(X), \]
then assertion $(v)$ is trivially satisfied. Hence we can assume that 
\[ f = \frac{1}{2\lambda} \| \cdot \|^2 + h \text{ with } h \in \Gamma_0(X). \]
This clearly implies that $f \in \Gamma_0(X)$. Taking the conjugate, we obtain that 
\[ f^* = \frac{\lambda}{2} \| \cdot \|^2 \vee h^* \] since the classical qualification condition is satisfied. It ensues that for every $x \in X$, 
\[
f^*(x) = \inf_{y \in X} \left\{ \frac{\lambda}{2} \| x - y \|^2 + h^*(y) \right\} = \frac{\lambda}{2} \| x \|^2 + \inf_{y \in X} \left\{ -\lambda \langle x, y \rangle + \frac{\lambda}{2} \| y \|^2 + h^*(y) \right\},\]
Therefore, 
\[
\frac{\lambda}{2} \| x \|^2 - f^*(x) = \sup_{y \in X} \left\{ \lambda \langle x, y \rangle - \frac{\lambda}{2} \| y \|^2 - h^*(y) \right\} = \left( h^* + \frac{\lambda}{2} \| \cdot \|^2 \right)^* (\lambda x).
\]
This clearly implies that $\frac{\lambda}{2} \| \cdot \|^2 - f^* \in \Gamma_0(X)$ and $(v)$ is proved. Let us finally observe that the implication $(v) \implies (ii)$ has been established in Corollary 7.4. As a conclusion, we have shown the equivalences $(i) \iff (ii) \iff (iii) \iff (iv)$ along with the implications $(iv) \implies (v) \implies (ii)$, which clearly establishes that all assertions $(i)$ to $(v)$ are equivalent.

The equalities (51) and (52) have been previously established by Wang [37] respectively in Proposition 4.5 and at the end of the proof of Proposition 4.13. As noticed in [37, Proposition 4.13] those equalities directly yield, for $f$ proper and lower semicontinuous, that $\kappa_{\lambda,2}(\kappa_{\lambda,2}f) = f$ if and only if $f - \frac{1}{2\lambda} \| \cdot \|^2$ is convex.

Taking $f$ as the indicator function of a set $C$ gives the following corollary.

**Corollary 7.6.** Assume that $X$ is a Hilbert space. For every $C \subset X$, the farthest distance function $\Delta_C$ satisfies
\[
\frac{1}{2} \Delta_C^2 = \left( \frac{1}{2} \| \cdot \|^2 - \sigma_{-C} \right)^* = \left( \delta_{-C} - \frac{1}{2} \| \cdot \|^2 \right)^* + \frac{1}{2} \| \cdot \|^2.
\]

**Proof.** It suffices to apply formulas (50)-(51) of Theorem 7.2 with $f = \delta_C$ and $\lambda = 1$. \qed

### 7.3. Case of a positively homogeneous function $\varphi$.

In this subsection, we assume that $X$ is a locally convex space and that the function $\varphi \in \Gamma_0(X)$ is positively homogeneous, i.e., $\varphi = \sigma_D$ for a nonempty set $D \subset X^*$. By applying Theorem 7.1 with $\psi = \delta_D$, we immediately obtain the following result.

**Corollary 7.7.** Let $X$ be a locally convex space. Take $\varphi = \sigma_D$ for a nonempty set $D \subset X^*$. Then we have for every function $f : X \to \mathbb{R}$,
\[
f^\varphi = (\delta_D - (f_-)^*)^* = \sup_{\xi^* \in D} \{ \langle \xi^*, \cdot \rangle + f^*(\xi^*) \}.
\]
Moreover,
\[ g \in \mathcal{E}^\varphi \iff g = (\delta_D \hat{-} h)^* = \sup_{\xi^* \in D} \{\langle \xi^*, \cdot \rangle + h(\xi^*)\} \text{ for some } h \in \Gamma(X^*) \]
\[ \iff g = (\delta_D \hat{-} (\delta_D \hat{-} g)^{**})^*. \]

Let us now particularize to the case of a normed space \((X, \|\cdot\|)\) and take \(\varphi = \|\cdot\|\).

**Corollary 7.8.** Let \((X, \|\cdot\|)\) be a normed space. For every function \(f : X \to \mathbb{R}\), we have
\[
\kappa_{1,1} f = (\delta_{B_X^*} \hat{-} (f_-)^*)^* = \sup_{\xi^* \in B_{X^*}} \{\langle \xi^*, \cdot \rangle + f^*(-\xi^*)\}
\]
\[
= (\delta_{S_X^*} \hat{-} (f_-)^*)^* = \sup_{\xi^* \in S_{X^*}} \{\langle \xi^*, \cdot \rangle + f^*(-\xi^*)\}.
\]
Moreover,
\[
g \text{ is a Klee envelope with index 1 and power 1}
\]
\[
g = (\delta_{B_X^*} \hat{-} h)^* = \sup_{\xi^* \in B_{X^*}} \{\langle \xi^*, \cdot \rangle + h(\xi^*)\} \text{ for some } h \in \Gamma(X^*)
\]
\[
\iff g = (\delta_{B_X^*} \hat{-} (\delta_{B_X^*} \hat{-} g)^{**})^*.
\]

**Proof.** For the equalities \(\kappa_{1,1} f = (\delta_{B_X^*} \hat{-} (f_-)^*)^*\) and \(\kappa_{1,1} f = (\delta_{S_X^*} \hat{-} (f_-)^*)^*\), use Corollary 7.7 respectively with \(D = B_X^*\) and \(D = S_{X^*}\). The characterizations of Klee envelopes with index 1 and power 1 follow immediately. \(\square\)

Assuming that \(f = \delta_C\), we have
\[
\kappa_{1,1} \delta_C = \sup_{x \in X} \{\|\cdot - x\| - \delta_C(x)\} = \sup_{x \in C} \|\cdot - x\| = \Delta_C,
\]
where \(\Delta_C\) is the farthest distance function. Taking into account the previous corollary, we then obtain
\[
\Delta_C = (\delta_{B_X^*} \hat{-} \sigma_C)^* = (\delta_{S_X^*} \hat{-} \sigma_C)^*.
\]
It is interesting to compare this expression with the one of the signed distance \(\operatorname{sgd}(\cdot, C) := d(\cdot, C) - d(\cdot, X \setminus C)\), for which it is known that \(\operatorname{sgd}(\cdot, C) = (\delta_{S_X^*} + \sigma_C)^*\), see [22].

Consider now the case of a finite set \(D = \{a_1^*, \ldots, a_n^*\} \subset X^*\) for \(n \geq 1\). By applying Corollary 7.7, we obtain the following result.
Corollary 7.9. Let $X$ be a locally convex space. Take $\varphi = \sigma_{\{a_1^*, \ldots, a_n^*\}}$ with $a_1^*, \ldots, a_n^* \in X^*$ and $n \geq 1$. Then we have for every function $f : X \to \mathbb{R}$

$$f^\varphi = \sup_{i=1}^n (a_i^*, \cdot) + f^*(-a_i^*).$$

Moreover,

$$g \in \mathcal{E}^\varphi \iff g = \sup_{i=1}^n (a_i^*, \cdot) + h(a_i^*) \quad \text{for some } h \in \Gamma(X^*).$$

8. Case $\varphi \in -\Gamma(X)$

8.1. Links between $\varphi$-envelopes and Legendre-Fenchel conjugates.

Proposition 8.1. Let $X$ be a locally convex space and let $\varphi, g : X \to \mathbb{R}$ be extended real-valued functions.

(i) If $g \in \mathcal{E}^\varphi$, then there exists $h \in \Gamma(X^*)$ such that $(-g)^* = (-\varphi)^* + h$. If in addition $g \in -\Gamma(X)$, then $-g = ((-\varphi)^* + h)^*$. Hence the expected equality holds with $h = f^* \in \Gamma(X^*)$. If in addition $g \in -\Gamma(X)$, we have $-g = (-g)^*$, hence we deduce from what precedes that $-g = ((-\varphi)^* + h)^*$.

(ii) Assume that $X$ is normed. If $\varphi \in -\Gamma(X)$ and if there exists $h \in \Gamma(X^*)$ satisfying the equality $-g = ((-\varphi)^* + h)^*$ along with the condition $0 \in \text{int} (\text{dom} h - \text{dom} (-\varphi)^*)$, then $g \in \mathcal{E}^\varphi$.

Proof. (i) Since $g \in \mathcal{E}^\varphi$, there exists $f : X \to \mathbb{R}$ such that $g = f^\varphi$, hence $-g = (-\varphi)^* + f$. Taking the conjugate of each member, we find $(-g)^* = (-\varphi)^* + f^*$. Hence the expected equality holds with $h = f^* \in \Gamma(X^*)$. If in addition $g \in -\Gamma(X)$, we have $-g = (-g)^*$, hence we deduce from what precedes that $-g = ((-\varphi)^* + h)^*$.

(ii) Assume that $-g = ((-\varphi)^* + h)^*$ for some $h \in \Gamma(X^*)$. If $h = -\omega_X$ or if $(-\varphi)^* = -\omega_X$, then $-g = ((-\omega_X)^* + h)^* = \omega_X$ and the inclusion $g \in \mathcal{E}^\varphi$ trivially holds. Now assume that $h \not= -\omega_X$ and $(-\varphi)^* \not= -\omega_X$. Since $0 \in \text{int} (\text{dom} h - \text{dom} (-\varphi)^*)$, the functions $(-\varphi)^*$ and $h$ are proper and according to the fact that $X^*$ is a Banach space, we have

$$-g = (-\varphi)^* \nabla h^*$$

$$= (-\varphi) \nabla h^* \quad \text{because } \varphi \in -\Gamma(X).$$

We conclude that $g = \varphi \triangle (-h^*) = (h^*)^\varphi \in \mathcal{E}^\varphi$. \qed

Corollary 8.1. Let $X$ be a normed space and let $\varphi \in -\Gamma_0(X)$ be such that $\text{dom} (-\varphi)^* = X^*$. For every $g \in -\Gamma(X)$, the following equivalences hold true

$$g \in \mathcal{E}^\varphi \iff (-g)^* - (-\varphi)^* \in \Gamma(X^*)$$

$$\iff -g = ((-\varphi)^* + h)^* \quad \text{for some } h \in \Gamma(X^*).$$

Proof. Fix $g \in -\Gamma(X)$. Since $\text{dom} (-\varphi)^* = X^*$ and $-\varphi \in \Gamma_0(X)$, the function $(-\varphi)^*$ is finite-valued on $X^*$, so the implication

$$g \in \mathcal{E}^\varphi \implies h := (-g)^* - (-\varphi)^* \in \Gamma(X^*)$$

follows from Proposition 8.1 (i). Recalling that $g \in -\Gamma(X)$, the right-hand inclusion implies in turn that $-g = ((-\varphi)^* + h)^*$.

Now assume that $-g = ((-\varphi)^* + h)^*$ for some $h \in \Gamma(X^*)$. If $\text{dom} h \not= 0$, the qualification assumption $0 \in \text{int} (\text{dom} h - \text{dom} (-\varphi)^*)$ is automatically satisfied. We then deduce from Proposition 8.1 (ii) that $g \in \mathcal{E}^\varphi$. On the other hand, if
dom \( h = \emptyset \), then we have \( h = \omega_X \), and hence \(-g = (\omega_X)^* = -\omega_X\). Then the inclusion \( g \in \mathcal{E}^\varnothing \) trivially holds.

8.2. Moreau envelopes. Let \((X, \| \cdot \|)\) be a normed space and let \( f : X \to \mathbb{R} \) be an extended real-valued function. For \( \lambda > 0 \) and \( p \geq 1 \), we define the Moreau envelope of \( f \) with index \( \lambda \) and power \( p \) as

\[
e_{\lambda,p} f = \inf_{y \in X} \left( \frac{1}{p\lambda} \| -y \|^p + f(y) \right) = \frac{1}{p\lambda} \| \cdot \|^p \nabla f.
\]

Observe that \(-e_{\lambda,p} f = \left( -\frac{1}{p\lambda} \| \cdot \|^p \right) \nabla (-f) = f^\varnothing\), with the function \( \varphi : X \to \mathbb{R} \) defined by \( \varphi = -\frac{1}{p\lambda} \| \cdot \|^p \). It ensues that \( g \) is a Moreau envelope with index \( \lambda \) and power \( p \) if and only if \(-g \in \mathcal{E}^\varnothing \), for \( \varphi = -\frac{1}{p\lambda} \| \cdot \|^p \). By applying the results of the previous subsection with \( \varphi = -\frac{1}{p\lambda} \| \cdot \|^p \), we obtain the following statement.

**Corollary 8.2.** Assume that \((X, \| \cdot \|)\) is a normed space. Let \( \lambda > 0 \), \( p > 1 \) and let \( q \) be the conjugate exponent of \( p \).

(i) If \( g \) is a Moreau envelope with index \( \lambda \) and power \( p \), then the function \( g^* = \frac{\lambda^{q-1}}{q} \| \cdot \|^q_X \in \Gamma(X^*) \).

(ii) If moreover \( g \in \Gamma(X) \), the following equivalences hold true

\[
g^* = \frac{\lambda^{q-1}}{q} \| \cdot \|^q_X \in \Gamma(X^*) \quad \text{and only if} \quad g = \left( \frac{\lambda^{q-1}}{q} \| \cdot \|^q_X + h \right)^* \quad \text{for some} \ h \in \Gamma(X^*). \]

**Proof.** (i) It suffices to apply Proposition 8.1 (i) with \( \varphi = -\frac{1}{p\lambda} \| \cdot \|^p \) and to recall that \( \left( \frac{1}{p\lambda} \| \cdot \|^p \right)^* = \frac{\lambda^{q-1}}{q} \| \cdot \|^q_X \).

(ii) The equivalences follow from Corollary 8.1 applied with \( \varphi = -\frac{1}{p\lambda} \| \cdot \|^p \). □

When \( X \) is a Hilbert space, we obtain a more precise characterization of Moreau envelopes with power 2, as shown by the following proposition.

**Proposition 8.2.** Assume that \( X \) is a Hilbert space endowed with the scalar product \( \langle \cdot, \cdot \rangle \) and the corresponding norm \( \| \cdot \| \).

(a) For every \( \lambda > 0 \) and every function \( f : X \to \mathbb{R} \), we have

\[
e_{\lambda,2} f = -\left( f + \frac{1}{2\lambda} \| \cdot \|^2 \right)^* \left( \frac{1}{\lambda} \right) + \frac{1}{2\lambda} \| \cdot \|^2.
\]

The \( \lambda \)-proximal hull of \( f \) defined by \( h_\lambda = -e_{\lambda,2} f \) is given by

\[
-e_{\lambda,2}(-e_{\lambda,2} f) = \left( f + \frac{1}{2\lambda} \| \cdot \|^2 \right)^{**} - \frac{1}{2\lambda} \| \cdot \|^2.
\]

(b) A function \( f : X \to \mathbb{R} \) is a Moreau envelope with index \( \lambda \) and power 2 if and only if \( f - \frac{1}{2\lambda} \| \cdot \|^2 \in -\Gamma(X) \).
Proof. (a) For every $x \in X$, we have

\[ e_{\lambda,2}f(x) = \inf_{y \in X} \left\{ \frac{1}{2\lambda} \|x - y\|^2 + f(y) \right\} \]

\[ = \inf_{y \in X} \left\{ \frac{1}{2\lambda} \|x\|^2 + \frac{1}{2\lambda} \|y\|^2 - \frac{1}{\lambda}(x, y) + f(y) \right\} \]

\[ = - \left( f + \frac{1}{2\lambda} \|\cdot\|^2 \right)^* \left( x/\lambda \right) + \frac{1}{2\lambda} \|x\|^2, \]

which proves the equality (53). By iterating we deduce that

\[ -e_{\lambda,2}(-e_{\lambda,2}f) = \left( -e_{\lambda,2} f + \frac{1}{2\lambda} \|\cdot\|^2 \right)^* \left( \frac{1}{\lambda} \right) - \frac{1}{2\lambda} \|\cdot\|^2 \]

\[ = \left[ \left( f + \frac{1}{2\lambda} \|\cdot\|^2 \right)^* \left( \frac{1}{\lambda} \right) \right]^* \left( \frac{1}{\lambda} \right) - \frac{1}{2\lambda} \|\cdot\|^2 \]

\[ = \left( f + \frac{1}{2\lambda} \|\cdot\|^2 \right)^{**} - \frac{1}{2\lambda} \|\cdot\|^2, \]

which proves the equality (54).

(b) Observe that $f$ is a Moreau envelope with index $\lambda$ and power 2 if and only if $-f \in \mathcal{E}^\varphi$ with $\varphi = -\frac{1}{2\lambda} \|\cdot\|^2$. From the equivalence (7) $\leftrightarrow$ (8) and the fact that $\varphi_\ast = \varphi$, this is in turn equivalent to $-f = ((-f)^\varphi)^\varphi$. Since $((-f)^\varphi)^\varphi = -e_{\lambda,2}(-e_{\lambda,2}(-f))$ and using the equality (54), we infer that

$f$ is a Moreau envelope with index $\lambda$ and power 2

\[ f \in \Gamma(X). \]

The coupling functional $(x, y) \mapsto -\frac{1}{2\lambda} \|x - y\|^2$ was considered in [7, Section 5] in the framework of generalized conjugacy. Equalities (53)-(54) were established by Penot and Volle [23, p. 206] and Martinez-Legaz [17, p. 182-184]. These equalities were also observed in [30, Example 11.26(c)] and [37, Lemma 3.3]. The characterization (b) above has been noticed in the aforementioned references, and it amounts to the previous characterization in [7, p. 288] of $Q^c$-convex functions with $c := 1/(2\lambda)$. \[ \square \]
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