ENVELOPES FOR SETS AND FUNCTIONS: REGULARIZATION AND GENERALIZED CONJUGACY

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ABSTRACT. Let X be a vector space and let $\varphi : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be an extended real-valued function. For every function $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$, let us define the φ-envelope of f by

$$f^{\varphi}(x) = \sup_{y \in X} \varphi(x - y) \dot{-} f(y),$$

where $\dot{-}$ denotes the lower subtraction in $\mathbb{R} \cup \{-\infty, +\infty\}$. The main purpose of this paper is to study in great details the properties of the important generalized conjugation map $f \mapsto f^{\varphi}$. When the function φ is closed and convex, φ-envelopes can be expressed as Legendre-Fenchel conjugates. By particularizing with $\varphi = \frac{1}{p\lambda} \| \cdot \|^p$, for $\lambda > 0$ and $p \geq 1$, this allows us to derive new expressions of the Klee envelopes with index λ and power p. Links between φ-envelopes and Legendre-Fenchel conjugates are also explored when $-\varphi$ is closed and convex. The case of Moreau envelopes is examined as a particular case.

Besides the φ-envelopes of functions, a parallel notion of envelope is introduced for subsets of X. Given subsets Λ, $C \subset X$, we define the Λ-envelope of C as $C^{\Lambda} = \bigcap_{x \in C} (x + \Lambda)$. Connections between the transform $C \mapsto C^{\Lambda}$ and the aforestated φ-conjugation are investigated.

1. Introduction

Given two topological vector spaces X, Y and a function $c : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$, extending the Legendre-Fenchel conjugacy, Moreau [20, Chapter 14, Section 3] defined, for any function $g : Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ its c-conjugacy as the function $g^c : X \to \mathbb{R} \cup \{-\infty, +\infty\}$

$$g^c(x) := \sup_{y \in Y} \big(c(x, y) \dot{-} g(y) \big) \quad \text{for all } x \in X;$$

see Section 2 for the (extended) lower subtraction $\dot{-}$. We refer to [4, 6, 7, 9, 17, 20, 27, 35] and the references therein, for various duality results in such a context and for several applications. Given a function $\varphi : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ we will focus on the case $c(x, y) := \varphi(x - y)$ and $Y = X$. Otherwise stated, for a function $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ we will be interested in the function f^{φ}, that we call the φ-envelope of f, defined by

$$f^{\varphi}(x) := \sup_{y \in X} \big(\varphi(x - y) \dot{-} f(y) \big) \quad \text{for all } x \in X.$$

Our first aim in this paper is to study in great details the structure of the transform $f \mapsto f^{\varphi}$ and provide various properties of φ-envelopes.

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On the other hand, considering the class $\mathcal{B}_X$ of closed balls of a Banach space X, Mazur [19] studied some Banach spaces X for which every closed bounded convex subset is the intersection of some subclass of $\mathcal{B}_X$; we refer to [10] for a rich survey on the subject. Any such Banach space is actually called in the literature a Banach space with the Mazur intersection property. In his 1983 paper [34] Vial defined strongly convex sets of a normed space as convex sets which are intersections of closed balls with a common radius; sets which are intersections, for a fixed real $r > 0$, of closed balls with radius equal to r are called r-strongly convex sets in [34]. This class of convex sets is thoroughly studied by Polovinkin [24] (see also [25] and the references therein). Denoting by $\mathbb{B}_X$ the closed unit ball of X centered at zero, any r-strongly convex set can be represented in the form

$$\bigcap_{x \in S} (x + r\mathbb{B}_X) \quad \text{with some subset } S \subset X.$$

So, given a subset Λ of the space X, our second aim in the paper is to analyze properties of the transform which assigns to each subset C of X the set

$$C^\Lambda := \bigcap_{x \in C} (x + \Lambda).$$

We will also provide the connections between the latter transform and the aforestated transform related to φ-envelopes.

In Section 2 we recall the lower and upper additions (resp. subtractions), and we also recall various concepts and results in Convex Analysis which will be needed in the paper. Section 3 offers a large list of general properties of φ-envelopes. Section 4 establishes the connections between φ-envelopes and the aforementioned transform $C \mapsto C^\Lambda$; many properties of sets which can be represented in this form are also provided. In Section 5 we examine the question whether $\psi = \varphi(\cdot - a) - \alpha$ (for some $a \in X$ and $\alpha \in \mathbb{R}$) whenever ψ is a φ-envelope and φ is a ψ-envelope. A counter-example is constructed and various sufficient conditions are given. The analogous question is also investigated with sets instead of functions. Section 6 considers additional properties in the case when the function φ is either superadditive or subadditive. In Section 7, assuming that φ is convex and lower semicontinuous, we provide several links between φ-envelope of a function and Legendre-Fenchel conjugates of other functions related to f. Taking φ as a power of the norm, we also provide various results concerning the Klee envelope $\kappa_{\lambda,p} f$ (with index λ and power p) of a function f, where

$$\kappa_{\lambda,p} f(x) := \sup_{y \in X} \big(\frac{1}{p\lambda} \|x - y\|^p - f(y) \big) \quad \text{for all } x \in X.$$

Finally in Section 8, assuming that $-\varphi$ is convex and lower semicontinuous, we continue to explore the links between φ-envelopes and Legendre-Fenchel conjugates. By particularizing with $\varphi = -\frac{1}{p\lambda}\|\cdot\|^p$, for $\lambda > 0$ and $p \geq 1$, we obtain several properties of Moreau envelopes with index λ and power p.

2. Preliminaries

Following Moreau [20], we extend the usual addition on $\mathbb{R}$ to $\overline{\mathbb{R}} = [-\infty, +\infty]$. We define the upper addition $\dot{+}$ and the lower addition $\underset{\cdot}{+}$ as the laws extending the usual addition via the following conventions

$$\begin{array}{rcccl}(-\infty) \dot{+} (+\infty) & = & (+\infty) \dot{+} (-\infty) & = & +\infty \\ (-\infty) \underset{\cdot}{+} (+\infty) & = & (+\infty) \underset{\cdot}{+} (-\infty) & = & -\infty.\end{array}$$

This leads to introduce the upper subtraction $\dot{-}$ and the lower subtraction $\underset{\cdot}{-}$, respectively defined by

$$s \dot{-} t = s \dot{+} (-t) \quad \text{and} \quad s \underset{\cdot}{-} t = s \underset{\cdot}{+} (-t) \quad \text{for all } s, t \in \overline{\mathbb{R}}.$$

Let X be a vector space; all vector spaces will be real vector spaces. Given two extended real-valued functions $f,\ g : X \to \overline{\mathbb{R}}$, the (Moreau) *inf-convolution* (also called *infimal convolution*) of f and g is defined as follows: for every $x \in X$,

$$\begin{aligned}(f \bigtriangledown g)(x) &= \inf_{y+z=x} \Big[f(y) \dot{+} g(z)\Big] \\ &= \inf_{y \in X} \Big[f(y) \dot{+} g(x-y)\Big] \\ &= \inf_{z \in X} \Big[f(x-z) \dot{+} g(z)\Big].\end{aligned}$$

In a symmetric way, the (Moreau) *sup-convolution* (or *supremal convolution*) of f and g is defined by

$$\begin{aligned}(f \bigtriangleup g)(x) &= \sup_{y+z=x} \Big[f(y) \underset{\cdot}{+} g(z)\Big] \\ &= \sup_{y \in X} \Big[f(y) \underset{\cdot}{+} g(x-y)\Big] \\ &= \sup_{z \in X} \Big[f(x-z) \underset{\cdot}{+} g(z)\Big].\end{aligned}$$

For the function f as above, the set $\operatorname{dom} f = \{x \in X,\ f(x) < +\infty\}$ is called the *effective domain* of f. We call f a proper function if $f(x) < +\infty$ for at least one $x \in X$, and $f(x) > -\infty$ for all $x \in X$, or in other words, if $\operatorname{dom} f$ is a nonempty set on which f is finite. The function which is constantly equal to $+\infty$ (resp. $-\infty$) on X is denoted by ω_X (resp. $-\omega_X$).

Now assume that X is a locally convex space; all such spaces in the paper will be Hausdorff. We will denote by X^* the topological dual of X. Then, following again [20] we set

$$\begin{aligned}\Gamma(X) \;:=\; &\{f : X \to \overline{\mathbb{R}},\ f \text{ is a pointwise supremum of a family of continuous} \\ &\text{affine functions with slopes in } X^*\}\end{aligned}$$

and

$$\begin{aligned}\Gamma(X^*) \;:=\; &\{g : X^* \to \overline{\mathbb{R}},\ g \text{ is a pointwise supremum of a family of continuous} \\ &\text{affine functions with slopes in } X\}.\end{aligned}$$

We denote by $\Gamma_0(X)$ the set of $f \in \Gamma(X)$ which differ from ω_X and $-\omega_X$. In the same way, $\Gamma_0(X^*)$ is the set $\Gamma_0(X^*) = \Gamma(X^*) \setminus \{\omega_{X^*}, -\omega_{X^*}\}$. The classes $\Gamma_0(X)$ and $\Gamma_0(X^*)$ are respectively characterized by

$$\begin{aligned}\Gamma_0(X) &= \{f : X \to \overline{\mathbb{R}},\ f \text{ is closed, convex and proper}\} \\ &= \{f : X \to \overline{\mathbb{R}},\ f \text{ is } w(X, X^*) \text{ closed, convex and proper}\},\end{aligned}$$

and

$$\Gamma_0(X^*) = \{g : X^* \to \overline{\mathbb{R}},\ g \text{ is } w(X^*, X) \text{ closed, convex and proper}\},$$

see for example [1, 8, 20]. Above and in all the paper, $w(X, X^*)$ and $w(X^*, X)$ stand for the weak topology on X and the weak star topology on X^* respectively.

With the function $f : X \to \overline{\mathbb{R}}$ is associated, in the duality pairing from X to X^*, its *Legendre-Fenchel conjugate* $f^* : X^* \to \overline{\mathbb{R}}$ defined by

$$\forall x^* \in X^*, \quad f^*(x^*) = \sup_{\xi \in X} \{\langle x^*, \xi\rangle - f(\xi)\}.$$

In the same way, throughout the paper (unless ontherwise stated) the Legendre-Fenchel conjugate of a function $g : X^* \to \overline{\mathbb{R}}$ defined on the dual space X^* will be taken in the duality pairing from X^* to X, that is, $g^* : X \to \overline{\mathbb{R}}$ is defined on X by

$$\forall x \in X, \quad g^*(x) = \sup_{\xi^* \in X^*} \{\langle \xi^*, x\rangle - g(\xi^*)\}.$$

The Legendre-Fenchel transform $f \mapsto f^*$ (see, for example, [20]) is known to be a one-to-one mapping from $\Gamma_0(X)$ onto $\Gamma_0(X^*)$. For any $f \in \Gamma_0(X)$ one has $f = f^{**}$ and for any $g \in \Gamma_0(X^*)$ one has $g = g^{**}$, see for example [1, 8, 20].

Given a set $C \subset X$, we denote as usual by δ_C the *indicator function* of C, *i.e.* , $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ if $x \notin C$. The *support function* $\sigma_C : X^* \to \overline{\mathbb{R}}$ of C is defined by

$$\forall x^* \in X^*, \quad \sigma_C(x^*) = \sup_{\xi \in C} \langle x^*, \xi\rangle\,,$$

so σ_C coincides with the Legendre-Fenchel conjugate of δ_C. For a nonempty cone $K \subset X$, the support function σ_K is equal to the indicator function of the polar cone K° of K defined by

$$K^\circ = \{x^* \in X^*,\ \langle x^*, x\rangle \le 0 \ \text{ for all } x \in K\}.$$

For a set $C \subset X$, we denote by $\operatorname{co}(C)$ (resp. $\overline{\operatorname{co}}(C)$) the convex hull (resp. closed convex hull) of C. The $w(X^*, X)$-closed convex hull of a set $D \subset X^*$ is denoted by $\overline{\operatorname{co}}^{w*}(D)$. For a function $f : X \to \overline{\mathbb{R}}$, its convex hull $\operatorname{co}(f)$ (resp. lower semicontinuous convex hull $\overline{\operatorname{co}}(f)$) is the greatest convex (resp. lower semicontinuous convex) function less or equal to f. The $w(X^*, X)$-lower semicontinuous convex hull of a function $g : X^* \to \overline{\mathbb{R}}$ is denoted by $\overline{\operatorname{co}}^{w*}(g)$.

If $f \in \Gamma_0(X)$ and if $\overline{x} \in \operatorname{dom} f$, the *recession function* f^∞ is defined by

$$\forall u \in X, \quad f^\infty(u) = \lim_{t \to +\infty} \frac{f(\overline{x} + tu) - f(\overline{x})}{t} = \sup_{t>0} \frac{f(\overline{x} + tu) - f(\overline{x})}{t}.$$

The function $f^\infty : X \to \mathbb{R} \cup \{+\infty\}$ does not depend on the point $\overline{x} \in \operatorname{dom} f$ since it is also given by

$$\forall u \in X, \quad f^\infty(u) = \sup_{x \in \operatorname{dom} f} (f(x + u) - f(x)).$$

The function f^∞ satisfies $f^\infty \in \Gamma_0(X)$, it is positively homogeneous and we have $f^\infty = \sigma_{\operatorname{dom} f^*}$. Given a closed convex set $C \subset X$ and $\overline{x} \in C$, the recession cone C^∞ is defined by

$$C^\infty = \{u \in X, \quad \overline{x} + tu \in C \ \text{ for all } t \ge 0\}.$$

The set C^∞ does not depend on $\overline{x} \in C$ and is also given by

$$C^\infty = \{u \in X,\ u + C \subset C\}.$$

It follows from the definition that C^∞ is a closed convex cone and we have $\delta_{C^\infty} = (\delta_C)^\infty$. For more details on recession analysis, see, for example, [1, 2, 13, 28].

Let us end these preliminaries with the subdifferential of convex analysis. We recall that the *subdifferential* $\partial f(x)$ of a convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ at $x \in \operatorname{dom} f$ is the set

$$\partial f(x) = \{\xi^* \in X^*,\ f(y) \geq f(x) + \langle \xi^*, y - x\rangle \ \text{ for every } y \in X\}. \tag{1}$$

When $x \notin \operatorname{dom} f$, then $\partial f(x) = \emptyset$ by convention. The *domain* and the *range* of the operator $\partial f : X \rightrightarrows X^*$ are respectively given by

$$\operatorname{dom}(\partial f) = \{x \in X,\ \partial f(x) \neq \emptyset\} \text{ and } \operatorname{Rge}(\partial f) = \{x^* \in X^*,\ \exists x \in X,\ x^* \in \partial f(x)\}.$$

If $f \in \Gamma_0(X)$, the subdifferentials of f and f^* are connected through the following relation

$$x^* \in \partial f(x) \iff x \in \partial f^*(x^*), \tag{2}$$

for all $x \in X$ and $x^* \in X^*$. For further details, the reader is referred to the classical textbooks on convex analysis, see for example [13, 28].

3. Definitions. General properties

Let X be a vector space. For functions $\varphi : X \to \overline{\mathbb{R}}$ and $f : X \to \overline{\mathbb{R}}$, the φ-*envelope* of f is defined as follows:

$$\forall x \in X, \quad f^\varphi(x) = \sup_{y \in X}\{\varphi(x-y) \dot{-} f(y)\} = \sup_{z \in X}\{\varphi(z) \dot{-} f(x-z)\}.$$

A function $g : X \to \overline{\mathbb{R}}$ is said to be a φ-envelope if there exists $f : X \to \overline{\mathbb{R}}$ such that $g = f^\varphi$. It is immediate to check that for every function $f : X \to \overline{\mathbb{R}}$,

$$f^{-\omega_X} = -\omega_X, \ \text{ while } f^{\omega_X} = \begin{cases} \omega_X & \text{if } \ f \neq \omega_X \\ -\omega_X & \text{if } \ f = \omega_X. \end{cases}$$

It ensues that the unique $(-\omega_X)$-envelope is the function $-\omega_X$, while the ω_X-envelopes are $\pm\omega_X$. The function f^φ can be expressed via the inf-convolution and sup-convolution operators

$$f^\varphi = \varphi \bigtriangleup (-f) = -\left((-\varphi) \bigtriangledown f\right). \tag{3}$$

The roles played by f and φ in the definition of f^φ are opposite in the sense that

$$(-\varphi)^{(-f)} = (-f) \bigtriangleup (-(-\varphi)) = (-f) \bigtriangleup \varphi = f^\varphi. \tag{4}$$

The definition of f^φ is closely connected to the deconvolution operation. For any $g,\, h : X \to \overline{\mathbb{R}}$, the *deconvolution* of g and h is the function $g \ominus h$ defined by

$$(g \ominus h)(x) = \sup_{y - z = x} (g(y) \dot{-} h(z)),$$

for every $x \in X$. Denoting by h_- the function defined by $h_-(x) = h(-x)$ for every $x \in X$, we deduce immediately from the above definition that

$$g \ominus h = g \bigtriangleup (-h_-) = (h_-)^g. \tag{5}$$

It ensues that for any $f,\, \varphi : X \to \overline{\mathbb{R}}$,

$$f^\varphi = \varphi \ominus f_-.$$

The deconvolution operation has been studied in details by many authors, see for example [3, 12, 14, 36].

Following the terminology of Moreau [21], we call φ-*elementary function* a function of the form $\varphi(\cdot - y) + \lambda$ with $y \in X$ and $\lambda \in \mathbb{R}$. By using a generalized conjugacy argument, one can show that for any φ, $f : X \to \overline{\mathbb{R}}$

$$(f^{\varphi_-})^{\varphi} \text{ is the upper envelope of the } \varphi\text{-elementary functions that minorize } f, \tag{6}$$

see for example [21, Section 4] and [30, Section 11.L]. It can easily be deduced the following characterization of φ-envelopes: for any $g : X \to \overline{\mathbb{R}}$,

$$g \text{ is the upper envelope of a family of } \varphi\text{-elementary functions} \tag{7}$$
$$\Updownarrow$$
$$g = (g^{\varphi_-})^{\varphi} \tag{8}$$
$$\Updownarrow$$
$$g \text{ is a } \varphi\text{-envelope.} \tag{9}$$

The expression of the double envelope $(g^{\varphi_-})^{\varphi}$ can be developed as follows

$$\begin{aligned} (g^{\varphi_-})^{\varphi} &= \varphi \bigtriangleup (-g^{\varphi_-}) \\ &= \varphi \bigtriangleup \big(-(\varphi_- \bigtriangleup (-g))\big) \\ &= \varphi \bigtriangleup ((-\varphi_-) \bigtriangledown g). \end{aligned}$$

By using the deconvolution operation, we obtain

$$\begin{aligned} (g^{\varphi_-})^{\varphi} &= \varphi \ominus (\varphi \bigtriangleup (-g_-)) \\ &= \varphi \ominus (\varphi \ominus g). \end{aligned}$$

From the equivalence (8) $\Leftrightarrow$ (9), we deduce that

$$g \text{ is a } \varphi\text{-envelope}$$
$$\Updownarrow$$
$$g = \varphi \bigtriangleup ((-\varphi_-) \bigtriangledown g) \tag{10}$$
$$\Updownarrow$$
$$g = \varphi \ominus (\varphi \ominus g).$$

Now let $f, \psi : X \to \overline{\mathbb{R}}$. Following the terminology of Martinez-Legaz & Penot [18], the function f is said to be (exactly) ψ-regular if $f = (f \ominus \psi) \bigtriangledown \psi$. By taking the opposite in each member of the equality (10), we find

$$\begin{aligned} -g &= (-\varphi) \bigtriangledown (\varphi_- \bigtriangleup (-g)) \\ &= (-\varphi) \bigtriangledown ((-g) \ominus (-\varphi)). \end{aligned}$$

In view of the above equivalences, this implies that

$$g \text{ is a } \varphi\text{-envelope} \iff -g \text{ is } (-\varphi)\text{-regular in the sense of [18].}$$

We denote by $\mathcal{E}^{\varphi}(X)$, or $\mathcal{E}^{\varphi}$ if there is no risk of confusion, the set of φ-envelopes and by $F_{\varphi} : \mathcal{E}^{\varphi_-} \to \mathcal{E}^{\varphi}$ the map defined by $F_{\varphi}(f) = f^{\varphi}$ for every $f \in \mathcal{E}^{\varphi_-}$. The equivalence (8) $\Leftrightarrow$ (9) says that $F_{\varphi} \circ F_{\varphi_-} = Id_{\mathcal{E}^{\varphi}}$ and $F_{\varphi_-} \circ F_{\varphi} = Id_{\mathcal{E}^{\varphi_-}}$, otherwise stated we have:

Proposition 3.1. *The map $F_{\varphi} : \mathcal{E}^{\varphi_-} \to \mathcal{E}^{\varphi}$ is bijective and $(F_{\varphi})^{-1} = F_{\varphi_-}$.*

As a consequence of the previous proposition, if φ is even the map $F_{\varphi} : \mathcal{E}^{\varphi} \to \mathcal{E}^{\varphi}$ is bijective and $(F_{\varphi})^{-1} = F_{\varphi}$.

Let us now state several general properties of φ-envelopes.

Proposition 3.2. *Let X be a vector space and let $\varphi : X \to \overline{\mathbb{R}}$.*

(*i*) *For every function $f : X \to \overline{\mathbb{R}}$ and every $a \in X$ and $\beta \in \mathbb{R}$, we have $(f(\cdot - a) - \beta)^\varphi = f^\varphi(\cdot - a) + \beta$. If $g \in \mathcal{E}^\varphi$, then $g(\cdot - a) + \beta \in \mathcal{E}^\varphi$ for every $a \in X$ and $\beta \in \mathbb{R}$.*

(*ii*) *Given a family $(f_i)_{i\in I}$ of functions $f_i : X \to \overline{\mathbb{R}}$, we have $(\inf_{i\in I} f_i)^\varphi = \sup_{i\in I} f_i^\varphi$. If $g = \sup_{i\in I} g_i$ with $g_i \in \mathcal{E}^\varphi$ for every $i \in I$, then $g \in \mathcal{E}^\varphi$.*

(*iii*) *For f_1, $f_2 : X \to \overline{\mathbb{R}}$, we have $(f_1 \bigtriangledown f_2)^\varphi = f_1^{(f_2^\varphi)}$. Let g, $h : X \to \overline{\mathbb{R}}$. If $h \in \mathcal{E}^g$ and $g \in \mathcal{E}^\varphi$, then $h \in \mathcal{E}^\varphi$. Otherwise stated, if $g \in \mathcal{E}^\varphi$, then $\mathcal{E}^g \subset \mathcal{E}^\varphi$.*

(*iv*) *For $f : X \to \overline{\mathbb{R}}$, we have $(f^\varphi)_- = {f_-}^{\varphi_-}$. As a consequence, $g \in \mathcal{E}^\varphi$ if and only if $g_- \in \mathcal{E}^{\varphi_-}$.*

Proof. (*i*) Let $a \in X$ and $\beta \in \mathbb{R}$. For every $x \in X$, we have

$$\begin{aligned}(f(\cdot - a) - \beta)^\varphi(x) &= \sup_{y\in X}\{\varphi(x-y) \dot{-} f(y-a) + \beta\}\\ &= \sup_{y'\in X}\{\varphi(x-a-y') \dot{-} f(y') + \beta\} = f^\varphi(x-a) + \beta.\end{aligned}$$

For the second assertion of (*i*), it suffices to apply the first part with $g = f^\varphi$.
(*ii*) By definition, we have

$$\begin{aligned}(\inf_{i\in I} f_i)^\varphi &= \varphi \bigtriangleup \big(-\inf_{i\in I} f_i\big)\\ &= \varphi \bigtriangleup \sup_{i\in I}(-f_i)\\ &= \sup_{i\in I}(\varphi \bigtriangleup (-f_i)) = \sup_{i\in I} f_i^\varphi, \quad \text{see for example [20].}\end{aligned}$$

Now assume that $g = \sup_{i\in I} g_i$ with $g_i \in \mathcal{E}^\varphi$ for every $i \in I$. Then, for each $i \in I$, we have $g_i = f_i^\varphi$ for some f_i. It ensues that $g = \sup_{i\in I} f_i^\varphi = (\inf_{i\in I} f_i)^\varphi$, hence $g \in \mathcal{E}^\varphi$.
(*iii*) By definition, we have

$$\begin{aligned}f_1^{(f_2^\varphi)} &= f_2^\varphi \bigtriangleup (-f_1)\\ &= (\varphi \bigtriangleup (-f_2)) \bigtriangleup (-f_1)\\ &= \varphi \bigtriangleup ((-f_2) \bigtriangleup (-f_1))\\ &= \varphi \bigtriangleup (-(f_2 \bigtriangledown f_1))\\ &= (f_2 \bigtriangledown f_1)^\varphi = (f_1 \bigtriangledown f_2)^\varphi.\end{aligned}$$

Now assume that $h \in \mathcal{E}^g$ and $g \in \mathcal{E}^\varphi$. Then there exist f_1, $f_2 : X \to \overline{\mathbb{R}}$ such that $h = f_1^g$ and $g = f_2^\varphi$. It ensues that $h = f_1^{(f_2^\varphi)} = (f_1 \bigtriangledown f_2)^\varphi$, hence $h \in \mathcal{E}^\varphi$.
(*iv*) For every $x \in X$, we have

$$\begin{aligned}(f^\varphi)_-(x) &= \sup_{y\in X}\{\varphi(-x-y) \dot{-} f(y)\}\\ &= \sup_{\xi\in X}\{\varphi(-x+\xi) \dot{-} f(-\xi)\}\\ &= \sup_{\xi\in X}\{\varphi_-(x-\xi) \dot{-} f_-(\xi)\} = {f_-}^{\varphi_-}(x).\end{aligned}$$

If $g \in \mathcal{E}^\varphi$, there exists $f : X \to \overline{\mathbb{R}}$ such that $g = f^\varphi$. It ensues that $g_- = (f^\varphi)_- = (f_-)^{\varphi_-}$, hence $g_- \in \mathcal{E}^{\varphi_-}$. The proof of the reverse assertion is identical. $\square$

In the next proposition, we show that the φ-envelope of a continuous linear functional is affine and we characterize the elements of $\mathcal{E}^\varphi$ that are linear.

Proposition 3.3. *Let X be a locally convex space. Let $\varphi : X \to \overline{\mathbb{R}}$ and $\xi^* \in X^*$. Then we have*

(i) $\langle \xi^*, \cdot\rangle^\varphi = -\langle \xi^*, \cdot\rangle + (-\varphi)^*(\xi^*)$.

(ii) If $\varphi \neq -\omega_X$, the following equivalence holds

$$\langle \xi^*, \cdot\rangle \in \mathcal{E}^\varphi \quad \Longleftrightarrow \quad \xi^* \in -\mathrm{dom}\,(-\varphi)^*.$$

Proof. (i) For every $x \in X$, we have

$$\begin{aligned}\langle \xi^*, \cdot\rangle^\varphi(x) &= \sup_{y\in X}\{\varphi(y) - \langle \xi^*, x-y\rangle\} \\ &= -\langle \xi^*, x\rangle + (-\varphi)^*(\xi^*).\end{aligned}$$

(ii) Let $g = \langle \xi^*, \cdot\rangle$. We deduce from (i) that

$$g^{\varphi_-} = -\langle \xi^*, \cdot\rangle + (-\varphi_-)^*(\xi^*) = -\langle \xi^*, \cdot\rangle + (-\varphi)^*(-\xi^*). \tag{11}$$

First assume that $(-\varphi)^*(-\xi^*) = +\infty$. Then we have $g^{\varphi_-} = \omega_X$, thus implying that $(g^{\varphi_-})^\varphi = -\omega_X$. It ensues that $(g^{\varphi_-})^\varphi \neq g$, which shows that $g \notin \mathcal{E}^\varphi$ according to the equivalence (7) $\Longleftrightarrow$ (8). Now assume that $(-\varphi)^*(-\xi^*) < +\infty$. Observe that $(-\varphi)^*(-\xi^*) \in \mathbb{R}$ since

$$(-\varphi)^*(-\xi^*) = -\infty \quad \Longrightarrow \quad \sup_{x\in X}\langle -\xi^*, x\rangle + \varphi(x) = -\infty \quad \Longrightarrow \quad \varphi = -\omega_X,$$

which is impossible by assumption. Since $(-\varphi)^*(-\xi^*) \in \mathbb{R}$, we deduce from (11), (i) above and Proposition 3.2 (i) that

$$(g^{\varphi_-})^\varphi = \langle \xi^*, \cdot\rangle + (-\varphi)^*(-\xi^*) - (-\varphi)^*(-\xi^*) = \langle \xi^*, \cdot\rangle = g,$$

and therefore $g \in \mathcal{E}^\varphi$. □

For every set $C \subset X$, let us set

$$\Sigma_C = \{f : X \to \overline{\mathbb{R}},\ \mathrm{dom}\, f \subset C\} \quad \text{and} \quad \Sigma_C^* = \{f^*,\ f \in \Sigma_C\}.$$

We adopt the same notations Σ_D and Σ_D^* for a subset $D \subset X^*$.

Theorem 3.1. *Let X be a locally convex space and let $\varphi : X \to \overline{\mathbb{R}}$ be such that $\varphi \neq -\omega_X$. For every subset D of X^*, the following assertions are equivalent*

(i) $\Sigma_D^* \subset \mathcal{E}^\varphi$*;*

(ii) $\{f \in \Gamma_0(X),\ \mathrm{dom}\, f^* \subset D\} \subset \mathcal{E}^\varphi$*;*

(iii) $D \subset -\mathrm{dom}\,(-\varphi)^*$*.*

Proof. $(i) \Rightarrow (ii)$ Let $D \subset X^*$. Observe that

$$\begin{aligned}\{f \in \Gamma_0(X),\ \mathrm{dom}\, f^* \subset D\} &= \{g^*,\ \mathrm{dom}\, g \subset D \text{ and } g \in \Gamma_0(X)\} \\ &\subset \{g^*,\ \mathrm{dom}\, g \subset D\} = \Sigma_D^*.\end{aligned}$$

The implication $(i) \Rightarrow (ii)$ follows immediately.
$(ii) \Rightarrow (iii)$ Assume that

$$\{f \in \Gamma_0(X),\ \mathrm{dom}\, f^* \subset D\} \subset \mathcal{E}^\varphi. \tag{12}$$

Let $\xi^* \in D$. Observe that $\langle \xi^*, .\rangle \in \Gamma_0(X)$ and that

$$\mathrm{dom}\,(\langle \xi^*, .\rangle)^* = \mathrm{dom}\,\delta_{\{\xi^*\}} = \{\xi^*\} \subset D,$$

hence $\langle \xi^*, .\rangle \in \mathcal{E}^\varphi$ in view of (12). We then deduce from Proposition 3.3 (ii) that $\xi^* \in -\mathrm{dom}\,(-\varphi)^*$. Since this is true for every $\xi^* \in D$, we conclude that $D \subset -\mathrm{dom}\,(-\varphi)^*$.
$(iii) \Rightarrow (i)$ Now assume that $D \subset -\mathrm{dom}\,(-\varphi)^*$ and let $f \in \Sigma^*_D$. There exists $g : X^* \to \overline{\mathbb{R}}$ such that $f = g^*$ and $\mathrm{dom}\, g \subset D$. The definition of the Legendre-Fenchel conjugate yields

$$\begin{aligned} f &= \sup_{\xi^* \in X^*} \{\langle \xi^*, .\rangle - g(\xi^*)\} \\ &= \sup_{\xi^* \in \mathrm{dom}\, g} \{\langle \xi^*, .\rangle - g(\xi^*)\}. \end{aligned} \tag{13}$$

Recalling that $\mathrm{dom}\, g \subset D \subset -\mathrm{dom}\,(-\varphi)^*$, we deduce from Proposition 3.3 (ii) that the linear function $\langle \xi^*, .\rangle$ is a φ-envelope for every $\xi^* \in \mathrm{dom}\, g$. In view of Proposition 3.2 (i), the affine function $\langle \xi^*, .\rangle - g(\xi^*)$ is also a φ-envelope for every $\xi^* \in \mathrm{dom}\, g$. Coming back to formula (13), we infer from Proposition 3.2 (ii) that f is a φ-envelope as a supremum of φ-envelopes. Finally, we have shown that $f \in \mathcal{E}^\varphi$, which proves the inclusion $\Sigma^*_D \subset \mathcal{E}^\varphi$. □

Given a set $D \subset X^*$, the following result explores the links between the class Σ^*_D and the class of functions $f \in \Gamma_0(X)$ satisfying $\mathrm{dom}\, f^* \subset D$. When the set D is $w(X^*, X)$-closed and convex, these classes can be characterized via the support function of D.

Proposition 3.4. *Let X be a locally convex space and let D be a nonempty subset of X^*.*

(i) *We have*

$$\{f \in \Gamma_0(X),\, \mathrm{dom}\, f^* \subset D\} \cup \{\omega_X, -\omega_X\} \subset \Sigma^*_D, \tag{14}$$

$$\Sigma^*_D \subset \{f \in \Gamma_0(X),\, \mathrm{dom}\, f^* \subset \overline{\mathrm{co}}^{w*}(D)\} \cup \{\omega_X, -\omega_X\}. \tag{15}$$

As a consequence, if the set $D \subset X^*$ is $w(X^*, X)$-closed and convex, the following equality holds true

$$\Sigma^*_D = \{f \in \Gamma_0(X),\, \mathrm{dom}\, f^* \subset D\} \cup \{\omega_X, -\omega_X\}. \tag{16}$$

(ii) *If the set $D \subset X^*$ is $w(X^*, X)$-closed and convex, then*

$$\{f \in \Gamma_0(X),\, \mathrm{dom}\, f^* \subset D\} = \{f \in \Gamma_0(X),\, f^\infty \leq \sigma_D\} \tag{17}$$

$$= \{f \in \Gamma_0(X),\, f(y) \leq f(x) + \sigma_D(y - x),\, \forall x, y \in X\}. \tag{18}$$

Proof. (i) We have already shown the inclusion $\{f \in \Gamma_0(X),\, \mathrm{dom}\, f^* \subset D\} \subset \Sigma^*_D$, see the proof of Theorem 3.1. On the other hand, we always have $-\omega_X \in \Sigma^*_D$. Since $D \neq \emptyset$, we also have $\omega_X \in \Sigma^*_D$. This proves the inclusion (14). Let us now establish (15). Assume that $f \in \Sigma^*_D$. There exists $g : X^* \to \overline{\mathbb{R}}$ such that $\mathrm{dom}\, g \subset D$ and $f = g^*$. We distinguish the cases $\overline{\mathrm{co}}^{w*}(g)$ proper and $\overline{\mathrm{co}}^{w*}(g)$ improper. If $\overline{\mathrm{co}}^{w*}(g) = \omega_{X^*}$, we have $g = \omega_{X^*}$, hence $f = -\omega_X$. If $\overline{\mathrm{co}}^{w*}(g)$ takes the value $-\infty$, we infer that $g^* = (\overline{\mathrm{co}}^{w*}(g))^* = \omega_X$, whence $f = \omega_X$. Let us now assume that $\overline{\mathrm{co}}^{w*}(g) \in \Gamma_0(X^*)$. It ensues that $f = g^* = (\overline{\mathrm{co}}^{w*}(g))^* \in \Gamma_0(X)$. This implies in turn that $f^* = \overline{\mathrm{co}}^{w*}(g)$, thus

$$\mathrm{dom}\, f^* = \mathrm{dom}\,(\overline{\mathrm{co}}^{w*}(g)) \subset \overline{\mathrm{co}}^{w*}(\mathrm{dom}\, g) \subset \overline{\mathrm{co}}^{w*}(D),$$

which ends the proof of (15). When the set D is $w(X^*, X)$-closed and convex, equality (16) is an immediate consequence of the inclusions (14)-(15).

(ii) Assuming that the set D is $w(X^*,X)$-closed and convex, we have $\operatorname{dom} f^* \subset D$ if and only if $\sigma_{\operatorname{dom} f^*} \leq \sigma_D$. Recalling that $\sigma_{\operatorname{dom} f^*} = f^\infty$ (see section 2), we derive equality (17). Since $f^\infty = \sup_{x \in \operatorname{dom} f} \big(f(\cdot + x) - f(x)\big)$, we deduce in turn equality (18). □

Remark 3.1. In general, the inclusions (14) and (15) are strict, as will be shown in Example 7.1.

If X is a Banach space and if the set $D \subset X^*$ is closed, the class of functions $f \in \Gamma_0(X)$ satisfying $\operatorname{dom} f^* \subset D$ can be expressed via the subdifferential of f.

Proposition 3.5. *Let X be a Banach space and let D be a closed subset of X^*. Then we have*

$$\{f \in \Gamma_0(X),\, \operatorname{dom} f^* \subset D\} = \{f \in \Gamma_0(X),\, \partial f(x) \subset D \textit{ for all } x \in X\}.$$

Proof. Let us first state as a lemma the following direct consequence of the Brønsted-Rockafellar theorem (see [5, Theorem 2]) concerning the conjugate of a function in $\Gamma_0(X)$.

Lemma 3.1 (See Theorem 2 in [5])**.** *If X is a Banach space and if $f \in \Gamma_0(X)$, then* $\operatorname{cl}(\operatorname{dom} f^*) = \operatorname{cl}(\operatorname{Rge}(\partial f))$.

Assume that the set $D \subset X^*$ is closed. From Lemma 3.1, we have for every $f \in \Gamma_0(X)$

$$\begin{aligned} \operatorname{dom} f^* \subset D \quad &\Longleftrightarrow \quad \operatorname{Rge}(\partial f) \subset D \\ &\Longleftrightarrow \quad \partial f(x) \subset D \quad \text{ for all } x \in X. \end{aligned}$$

The announced equality follows immediately. □

Applying Theorem 3.1 with particular sets D, we obtain the following corollaries.

Corollary 3.1. *Let X be a locally convex space and let $\varphi : X \to \overline{\mathbb{R}}$ be such that $\varphi \neq -\omega_X$. Then the following equivalence holds*

$$\Gamma(X) \subset \mathcal{E}^\varphi \quad \Longleftrightarrow \quad \operatorname{dom}(-\varphi)^* = X^*.$$

Proof. It suffices to take $D = X^*$ in the equivalence $(i) \Leftrightarrow (iii)$ of Theorem 3.1. □

Remark 3.2. Under the assumption $\operatorname{dom}(-\varphi)^* = X^*$, the function φ cannot be convex (see hereafter). Therefore the set $\mathcal{E}^\varphi$ is strictly larger than $\Gamma(X)$, since it contains the nonconvex function φ.
If $\operatorname{dom}(-\varphi)^* = X^*$, we have $(-\varphi)^*(0) < +\infty$. Recalling that $(-\varphi)^*(0) = \sup \varphi$, we deduce that the function φ is bounded from above on the whole space X. If moreover the function φ is convex, we infer from a classical result that it is constant, say $\varphi \equiv \beta$ for some $\beta \in \mathbb{R}$. It ensues that $(-\varphi)^* = \beta + \delta_{\{0\}}$, hence $\operatorname{dom}(-\varphi)^* = \{0\}$, a contradiction. This confirms that functions φ with $\operatorname{dom}(-\varphi)^* = X^*$ cannot be convex.

Given a set $K \subset X$, recall that a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be K-nonincreasing (resp. K-nondecreasing) if $f(y) \leq f(x)$ (resp. $f(y) \geq f(x)$) for all $x, y \in X$ such that $y - x \in K$.

Corollary 3.2. *Let X be a locally convex space. Let $K \subset X$ be a closed convex cone and let $\varphi : X \to \overline{\mathbb{R}}$ be such that $\varphi \neq -\omega_X$. Then the set $\mathcal{E}^\varphi$ contains all the functions of $\Gamma_0(X)$ which are K-nonincreasing if and only if $-K^\circ \subset \operatorname{dom}(-\varphi)^*$.*

Proof. Take $D = K^\circ$ in the equivalence $(ii) \Leftrightarrow (iii)$ of Theorem 3.1 to obtain that

$$\begin{aligned} \{f \in \Gamma_0(X),\, \mathrm{dom}\, f^* \subset K^\circ\} \subset \mathcal{E}^\varphi &\iff K^\circ \subset -\mathrm{dom}\,(-\varphi)^* \\ &\iff -K^\circ \subset \mathrm{dom}\,(-\varphi)^*. \end{aligned} \tag{19}$$

On the other hand, observe by (18) that for $f \in \Gamma_0(X)$,

$$\begin{aligned} \mathrm{dom}\, f^* \subset K^\circ &\iff f(y) \le f(x) + \sigma_{K^\circ}(y - x) \quad \text{for all } x, y \in X \\ &\iff f(y) \le f(x) + \delta_K(y - x) \quad \text{for all } x, y \in X \\ &\iff f \text{ is } K\text{-nonincreasing}. \end{aligned} \tag{20}$$

The announced equivalence then follows immediately from (19) and (20). □

In the sequel, when X is a normed space we will denote by $\mathbb{B}_X$ (resp. $\mathbb{B}_{X^*}$) the closed unit ball of X (resp. X^*).

Corollary 3.3. *Let $(X, \|\cdot\|)$ be a normed space. Let a real $k \ge 0$ and let $\varphi : X \to \overline{\mathbb{R}}$ be such that $\varphi \ne -\omega_X$. Then the set $\mathcal{E}^\varphi$ contains all the functions of $\Gamma_0(X)$ which are k-Lipschitz continuous on X if and only if $k\mathbb{B}_{X^*} \subset \mathrm{dom}\,(-\varphi)^*$.*

Proof. Take $D = k\mathbb{B}_{X^*}$ in the equivalence $(ii) \Leftrightarrow (iii)$ of Theorem 3.1 to obtain that

$$\begin{aligned} \{f \in \Gamma_0(X),\, \mathrm{dom}\, f^* \subset k\mathbb{B}_{X^*}\} \subset \mathcal{E}^\varphi &\iff k\mathbb{B}_{X^*} \subset -\mathrm{dom}\,(-\varphi)^* \\ &\iff k\mathbb{B}_{X^*} \subset \mathrm{dom}\,(-\varphi)^*. \end{aligned} \tag{21}$$

Then observe by (18) that for $f \in \Gamma_0(X)$,

$$\begin{aligned} \mathrm{dom}\, f^* \subset k\mathbb{B}_{X^*} &\iff f(y) \le f(x) + k\,\|y - x\| \quad \text{for all } x, y \in X \\ &\iff f \text{ is } k\text{-Lipschitz on } X, \end{aligned} \tag{22}$$

where the last equivalence is obtained by reversing the roles of x and y. The announced equivalence then follows immediately from (21) and (22). □

4. Equivalence between functions and sets

Recall that for $f : X \to \overline{\mathbb{R}}$, the epigraph (resp. hypograph) of f is defined by

$$\mathrm{epi}\, f = \{(x, \lambda) \in X \times \mathbb{R},\, f(x) \le \lambda\} \quad (\text{resp. } \mathrm{hypo}\, f = \{(x, \lambda) \in X \times \mathbb{R},\, f(x) \ge \lambda\}).$$

The strict epigraph and strict hypograph of f are obtained by replacing the above inequalities with strict inequalities

$$\mathrm{epi}_s f = \{(x, \lambda) \in X \times \mathbb{R},\, f(x) < \lambda\} \quad (\text{resp. } \mathrm{hypo}_s f = \{(x, \lambda) \in X \times \mathbb{R},\, f(x) > \lambda\}).$$

The following lemma gives a geometrical interpretation for the inf-convolution and sup-convolution operations. Assertion (i) is well known. For completeness and convenience of the reader we provide a proof of (ii).

Lemma 4.1. *Let X be a vector space and let $f,\, g : X \to \overline{\mathbb{R}}$. Then we have*
(i) $\mathrm{epi}_s(f \bigtriangledown g) = \mathrm{epi}_s f + \mathrm{epi}_s g$.
(ii) $\mathrm{hypo}_s(f \bigtriangleup g) = \mathrm{hypo}_s f + \mathrm{hypo}_s g$.

Proof. Point (i) is classical, see for example [20, 30]. Point (ii) is deduced easily from (i) by observing that

$$\begin{aligned}
(x,\lambda) \in \operatorname{hypo}_s f \bigtriangleup g &\iff (x,-\lambda) \in \operatorname{epi}_s[-(f \bigtriangleup g)] \\
&\iff (x,-\lambda) \in \operatorname{epi}_s[(-f) \bigtriangledown (-g)] \\
&\iff (x,-\lambda) \in \operatorname{epi}_s(-f) + \operatorname{epi}_s(-g) \\
&\iff (x,\lambda) \in \operatorname{hypo}_s(f) + \operatorname{hypo}_s(g).
\end{aligned}$$

□

Since f^φ is defined via a sup-convolution operation, we derive the following consequence of Lemma 4.1.

Proposition 4.1. *Let X be a vector space and let $\varphi : X \to \overline{\mathbb{R}}$.*

(i) *For every $f : X \to \overline{\mathbb{R}}$, we have*

$$\begin{aligned}
\operatorname{hypo}_s f^\varphi &= \operatorname{hypo}_s(-f) + \operatorname{hypo}_s \varphi \\
\operatorname{epi} f^\varphi &= \bigcap_{u \in \operatorname{hypo}_s(-f)} u + \operatorname{epi}\varphi.
\end{aligned}$$

(ii) *For every $g : X \to \overline{\mathbb{R}}$, the following equivalences hold*

$$\begin{aligned}
g \in \mathcal{E}^\varphi &\iff \operatorname{hypo}_s g = U + \operatorname{hypo}_s \varphi \quad \textit{for some } U \subset X \times \mathbb{R} \\
&\iff \operatorname{epi} g = \bigcap_{u \in U} u + \operatorname{epi}\varphi \quad \textit{for some } U \subset X \times \mathbb{R}.
\end{aligned}$$

Proof. (i) Let f, $\varphi : X \to \overline{\mathbb{R}}$. Recalling that $f^\varphi = \varphi \bigtriangleup (-f)$, we deduce from Lemma 4.1 (ii) that

$$\begin{aligned}
\operatorname{hypo}_s f^\varphi &= \operatorname{hypo}_s(-f) + \operatorname{hypo}_s \varphi \\
&= \bigcup_{u \in \operatorname{hypo}_s(-f)} u + \operatorname{hypo}_s \varphi.
\end{aligned}$$

Taking the complement of each member of the above equality, we infer that

$$\operatorname{epi} f^\varphi = \bigcap_{u \in \operatorname{hypo}_s(-f)} u + \operatorname{epi}\varphi.$$

(ii) Let $g : X \to \overline{\mathbb{R}}$. If $g \in \mathcal{E}^\varphi$, there exists $f : X \to \overline{\mathbb{R}}$ such that $g = f^\varphi$. In view of (i), we obtain that $\operatorname{hypo}_s g = U + \operatorname{hypo}_s \varphi$ with $U = \operatorname{hypo}_s(-f)$. Conversely, assume that $\operatorname{hypo}_s g = U + \operatorname{hypo}_s \varphi$ for some $U \subset X \times \mathbb{R}$. Then we have

$$\begin{aligned}
\operatorname{hypo}_s g &= \bigcup_{(x,\lambda) \in U} (x,\lambda) + \operatorname{hypo}_s \varphi \\
&= \bigcup_{(x,\lambda) \in U} \operatorname{hypo}_s[\varphi(\cdot - x) + \lambda] \\
&= \operatorname{hypo}_s\Big[\sup_{(x,\lambda) \in U} \varphi(\cdot - x) + \lambda\Big].
\end{aligned}$$

Hence we deduce that $g = \sup_{(x,\lambda)\in U}(\varphi(\cdot - x) + \lambda)$, which shows that $g \in \mathcal{E}^\varphi$. This proves the first equivalence of (ii). For the other equivalence, it suffices to take the complement of the sets arising in each member of the equality concerning $\operatorname{hypo}_s g$. □

Given a set $\Lambda \subset X$, the previous result suggests to consider the class $\mathcal{I}^\Lambda$ of subsets of X defined as follows[1]

$$\mathcal{I}^\Lambda = \{C^\Lambda,\, C \subset X\}, \quad \text{where } C^\Lambda = \bigcap_{x\in C} x + \Lambda.$$

By convention[2], we take $\emptyset^\Lambda = \bigcap_{x\in\emptyset} x + \Lambda = X$ for every set $\Lambda \subset X$. This implies that $X \in \mathcal{I}^\Lambda$ for every $\Lambda \subset X$. It is immediate to check that $\mathcal{I}^X = \{X\}$, while $\mathcal{I}^\emptyset = \{\emptyset, X\}$. A set $D \subset X$ belongs to the class $\mathcal{I}^\Lambda$ if it is equal to some intersection of translated sets from Λ. It ensues immediately that the class $\mathcal{I}^\Lambda$ is stable under translation and intersection.

Example 4.1. Take $r > 0$ and $\Lambda = r\,\mathbb{B}_X$. The class $\mathcal{I}^{r\,\mathbb{B}_X}$ corresponds to the class studied by Vial [34] under the terminology of r-strongly convex sets. More generally, for a closed convex set $\Lambda \subset X$, the sets of the form C^Λ are called Λ-strongly convex. The Λ-strongly convex sets are thoroughly studied by Polovinkin [24], under an additional condition on the set Λ (which is assumed to be generating, see [24] for more details).

The definition of C^Λ is directly linked to the star-difference of sets. For every $C_1, C_2 \subset X$, the star-difference of C_1 with C_2 is the set $C_1 \overset{*}{-} C_2$ given by

$$C_1 \overset{*}{-} C_2 = \bigcap_{x\in C_2} C_1 - x.$$

We deduce immediately from the above definition that $C^\Lambda = \Lambda \overset{*}{-} (-C)$ for every $C, \Lambda \subset X$. The star-difference of sets was used in [26] in the context of differential games. See also [12] for the links between the star-difference of sets and the deconvolution operation, also called epigraphical star-difference.

Given $C \subset X$ and $\Lambda \subset X$, the next proposition gives several expressions for the set C^Λ.

Proposition 4.2. *Let X be a vector space. For any sets $C \subset X$ and $\Lambda \subset X$, we have*

(i) $C^\Lambda = \{x \in X,\, x - C \subset \Lambda\} = \{x \in X,\, C \subset x - \Lambda\}$;
(ii) $X \setminus C^\Lambda = C + (X \setminus \Lambda)$ *or equivalently* $C^{X\setminus\Lambda} = X \setminus (C + \Lambda)$.
(iii) $(X \setminus \Lambda)^{X\setminus C} = C^\Lambda$.

Proof. (i) It suffices to observe that

$$\begin{aligned} x \in C^\Lambda &\iff \forall u \in C,\, x \in u + \Lambda \\ &\iff \forall u \in C,\, x - u \in \Lambda \\ &\iff x - C \subset \Lambda \\ &\iff C \subset x - \Lambda. \end{aligned}$$

(ii) From the definition of C^Λ, we deduce immediately that

$$X \setminus C^\Lambda = \bigcup_{u\in C} u + (X \setminus \Lambda) = C + (X \setminus \Lambda),$$

[1] We draw the attention of the reader to the fact that the notation C^Λ must not be confused with that of the set of maps from Λ into C.

[2] In particular, we obtain $\emptyset^\emptyset = X$.

which is the first equality in (ii). From this equality with $X \setminus \Lambda$ in place of Λ, we obtain that $X \setminus C^{X\setminus\Lambda} = C + \Lambda$, or equivalently $C^{X\setminus\Lambda} = X \setminus (C + \Lambda)$.
(iii) We infer from the previous assertion that

$$X \setminus \left[(X \setminus \Lambda)^{X\setminus C}\right] = (X \setminus \Lambda) + C = X \setminus C^{\Lambda},$$

whence the equality $(X \setminus \Lambda)^{X\setminus C} = C^{\Lambda}$. □

The elements D of $\mathcal{I}^{\Lambda}$ can be characterized by the equality $(D^{-\Lambda})^{\Lambda} = D$. This is the subject of the next proposition.

Proposition 4.3. *Let X be a vector space and let $\Lambda \subset X$. For any set $D \subset X$, the set $(D^{-\Lambda})^{\Lambda}$ is the smallest element of $\mathcal{I}^{\Lambda}$ containing the set D. As a consequence, the following equivalence holds true*

$$D \in \mathcal{I}^{\Lambda} \quad \Longleftrightarrow \quad (D^{-\Lambda})^{\Lambda} = D.$$

Proof. Let S be the subset of X defined by

$$S = \bigcap_{x\in X,\, x+\Lambda\supset D} x + \Lambda.$$

We clearly have $S \in \mathcal{I}^{\Lambda}$ and $S \supset D$. Now let any $S' \in \mathcal{I}^{\Lambda}$ with $S' \supset D$. By definition, there exists some $C \subset X$ such that $S' = \bigcap_{x\in C} x + \Lambda$. The inclusion $S' \supset D$ implies that $x + \Lambda \supset D$ for every $x \in C$ and therefore

$$S' = \bigcap_{x\in C} x + \Lambda \supset \bigcap_{x\in X,\, x+\Lambda\supset D} x + \Lambda = S.$$

This proves that the set S is the smallest element of $\mathcal{I}^{\Lambda}$ containing D. Recall now from Proposition 4.2 (i) that condition $x + \Lambda \supset D$ is equivalent to $x \in D^{-\Lambda}$. We deduce that

$$S = \bigcap_{x\in D^{-\Lambda}} x + \Lambda = (D^{-\Lambda})^{\Lambda}.$$

This finishes the proof of the first assertion. The second assertion is an immediate consequence of the first one. □

Let us write the expression of the double envelope $(D^{-\Lambda})^{\Lambda}$ by using the star-difference operation

$$\begin{aligned}(D^{-\Lambda})^{\Lambda} &= \Lambda \overset{*}{-} (-(D^{-\Lambda}))\\ &= \Lambda \overset{*}{-} ((-D)^{\Lambda})\\ &= \Lambda \overset{*}{-} (\Lambda \overset{*}{-} D).\end{aligned} \tag{23}$$

In view of Proposition 4.2, the complement of the set $(D^{-\Lambda})^{\Lambda}$ can be expressed as

$$\begin{aligned}X \setminus (D^{-\Lambda})^{\Lambda} &= D^{-\Lambda} + X \setminus \Lambda\\ &= (-(X \setminus \Lambda))^{X\setminus D} + X \setminus \Lambda\\ &= \left((X \setminus D) \overset{*}{-} (X \setminus \Lambda)\right) + X \setminus \Lambda.\end{aligned} \tag{24}$$

From equalities (23)-(24) and Proposition 4.3, we deduce that

$$D \in \mathcal{I}^{\Lambda}$$
$$\Updownarrow$$
$$D = \Lambda \overset{*}{-} (\Lambda \overset{*}{-} D)$$

$$\Updownarrow$$
$$X \setminus D = \big((X \setminus D) \overset{*}{-} (X \setminus \Lambda)\big) + X \setminus \Lambda.$$

The last equality amounts to saying that the set $X \setminus D$ is exactly $(X \setminus \Lambda)$-regular in the sense of [18].

With the notations introduced above, for f, $g : X \to \overline{\mathbb{R}}$, the results of Proposition 4.1 can be restated as

$$\operatorname{epi} f^{\varphi} = (\operatorname{hypo}_s(-f))^{\operatorname{epi}\varphi}$$

and

$$g \in \mathcal{E}^{\varphi} \iff \operatorname{epi} g \in \mathcal{I}^{\operatorname{epi}\varphi}.$$

This shows that the study of φ-envelopes amounts to that of the class $\mathcal{I}^{\operatorname{epi}\varphi}$. Conversely, given a set $\Lambda \subset X$, the class $\mathcal{I}^{\Lambda}$ can be fully described via the δ_Λ-envelopes.

Proposition 4.4. *Let X be a vector space and let $\Lambda \subset X$.*

(i) *For every function $f : X \to \overline{\mathbb{R}}$, we have*[3]

$$f^{\delta_\Lambda} = -\inf_X f + \delta_{(\operatorname{dom} f)^\Lambda}. \tag{25}$$

As a consequence, the equality $(\delta_C)^{\delta_\Lambda} = \delta_{C^\Lambda}$ holds for any nonempty set $C \subset X$.

(ii) *For every function $g : X \to \overline{\mathbb{R}}$ such that $g \neq \pm\omega_X$, we have*

$$g \in \mathcal{E}^{\delta_\Lambda} \quad \iff \quad g = \beta + \delta_{C^\Lambda} \quad \textit{for some } \beta \in \mathbb{R} \textit{ and some } C \neq \emptyset.$$

Proof. (i) For every function $f : X \to \overline{\mathbb{R}}$ and every $x \in X$, the definition of f^{δ_Λ} gives

$$f^{\delta_\Lambda}(x) = \sup_{y \in X}\{\delta_\Lambda(x-y) \dot{-} f(y)\} = \sup_{y \in \operatorname{dom} f}\{\delta_\Lambda(x-y) - f(y)\}.$$

First assume that $x - \operatorname{dom} f \subset \Lambda$. For every $y \in \operatorname{dom} f$, we then have $x - y \in \Lambda$, whence $\delta_\Lambda(x-y) = 0$. It ensues that

$$f^{\delta_\Lambda}(x) = \sup_{y \in \operatorname{dom} f} -f(y) = \sup_X(-f) = -\inf_X f.$$

Now assume that $x - \operatorname{dom} f \not\subset \Lambda$. In this case, there exists $y \in \operatorname{dom} f$ such that $x - y \notin \Lambda$. We then have $\delta_\Lambda(x-y) = +\infty$, whence $f^{\delta_\Lambda}(x) = +\infty$. Finally, we obtain for every $x \in X$

$$f^{\delta_\Lambda}(x) = \begin{cases} -\inf_X f & \text{if } x - \operatorname{dom} f \subset \Lambda \\ +\infty & \text{otherwise.} \end{cases}$$

Condition $x - \operatorname{dom} f \subset \Lambda$ is equivalent to $x \in (\operatorname{dom} f)^\Lambda$ in view of Proposition 4.2 (i). Formula (25) follows immediately. For the last assertion, it suffices to take $f = \delta_C$.
(ii) Let $g \in \mathcal{E}^{\delta_\Lambda}$ be such that $g \neq \pm\omega_X$. There exists $f : X \to \overline{\mathbb{R}}$ such that $g = f^{\delta_\Lambda}$, hence we deduce from (i) that $g = -\inf_X f + \delta_{(\operatorname{dom} f)^\Lambda}$. Since $g \neq \pm\omega_X$, we have $\inf_X f \in \mathbb{R}$ and $\operatorname{dom} f \neq \emptyset$. It suffices then to take $\beta = -\inf_X f$ and $C = \operatorname{dom} f$. Conversely, assume that $g = \beta + \delta_{C^\Lambda}$ for some $\beta \in \mathbb{R}$ and some $C \neq \emptyset$. Assertion (i) then shows that $g = f^{\delta_\Lambda}$ for the function f defined by $f = -\beta + \delta_C$, hence $g \in \mathcal{E}^{\delta_\Lambda}$. $\square$

[3]If $\inf_X f = +\infty$ we have $\operatorname{dom} f = \emptyset$, hence $(\operatorname{dom} f)^\Lambda = X$ and $\delta_{(\operatorname{dom} f)^\Lambda} \equiv 0$. Therefore the addition in the right-hand side of (25) is well-defined.

Remark 4.1. The previous proposition shows that for every C, $\Lambda \subset X$ with $C \neq \emptyset$

$$(\delta_C)^{\delta_\Lambda} = (-\delta_C) \bigtriangleup \delta_\Lambda = \delta_{C^\Lambda}. \tag{26}$$

It is interesting to compare this formula with the following one

$$(\delta_C)^{-\delta_\Lambda} = (-\delta_C) \bigtriangleup (-\delta_\Lambda) = -\delta_{C+\Lambda}, \tag{27}$$

that is obtained as a consequence of the equality $\delta_{C+\Lambda} = \delta_C \bigtriangledown \delta_\Lambda$.

Corollary 4.1. *Let X be a vector space. For every set $\Lambda \subset X$ and every set $D \subset X$ such that $D \neq \emptyset$ and $D \neq X$, the following equivalence holds*

$$\delta_D \in \mathcal{E}^{\delta_\Lambda} \quad \Longleftrightarrow \quad D \in \mathcal{I}^\Lambda.$$

In fact, the implication from the left to the right is true as soon as $D \neq \emptyset$, while the reverse one is true if $D \neq X$.

Proof. First assume that $\delta_D \in \mathcal{E}^{\delta_\Lambda}$ and that $D \neq \emptyset$. There exists $f : X \to \overline{\mathbb{R}}$ such that $\delta_D = f^{\delta_\Lambda}$, hence we deduce from Proposition 4.4 (i) that $\delta_D = -\inf_X f + \delta_{(\operatorname{dom} f)^\Lambda}$. Since $D \neq \emptyset$, we have $\inf_X f = 0$ and $D = (\operatorname{dom} f)^\Lambda \in \mathcal{I}^\Lambda$. Conversely, assume that $D \in \mathcal{I}^\Lambda$ and that $D \neq X$. This implies that $D = C^\Lambda$ for some $C \neq \emptyset$, and hence by Proposition 4.4 (i) again $\delta_D = \delta_{C^\Lambda} = (\delta_C)^{\delta_\Lambda} \in \mathcal{E}^{\delta_\Lambda}$. □

Let us now study the class $\mathcal{E}^{-\delta_\Lambda}$. From the generalized conjugation point of view, the case $\varphi = -\delta_\Lambda$ is a special instance of a coupling functional $c : X \times Y \to \overline{\mathbb{R}}$ of the type $c = -\delta_G$, where G is a subset of $X \times Y$. The corresponding conjugation operator, which arises in quasiconvex analysis, has been considered in many papers, see for example [17, 31, 35].

Proposition 4.5. *Let X be a vector space. Let Λ be a nonempty subset of X and let $f : X \to \overline{\mathbb{R}}$. Then we have*

(i)

$$f \in \mathcal{E}^{-\delta_\Lambda} \quad \Longleftrightarrow \quad f = \sup_{y\in\Lambda} \inf_{z\in\Lambda} f(\cdot - y + z). \tag{28}$$

This means equivalently that for every $x \in X$ and every $\lambda < f(x)$, there exists $y \in \Lambda$ such that $f(x - y + z) \geq \lambda$ for every $z \in \Lambda$.

(ii) *If $f \in \mathcal{E}^{-\delta_\Lambda}$ and if $\Lambda + \Lambda \subset \Lambda$, then f is Λ-nondecreasing. Conversely, if f is Λ-nondecreasing and if $0 \in \Lambda$, then $f \in \mathcal{E}^{-\delta_\Lambda}$.*

Proof. (i) The equivalence (7) $\Longleftrightarrow$ (8) yields

$$f \in \mathcal{E}^{-\delta_\Lambda} \quad \Longleftrightarrow \quad f = \left(f^{(-\delta_\Lambda)_-}\right)^{-\delta_\Lambda}.$$

On the other hand, we have

$$f^{(-\delta_\Lambda)_-} = \sup_{\xi\in X} -\delta_\Lambda(-\xi) \dot{-} f(\cdot - \xi) = \sup_{-\xi\in\Lambda} -f(\cdot - \xi) = -\inf_{z\in\Lambda} f(\cdot + z)$$

and hence

$$\left(f^{(-\delta_\Lambda)_-}\right)^{-\delta_\Lambda} = \sup_{y\in\Lambda} -f^{(-\delta_\Lambda)_-}(\cdot - y) = \sup_{y\in\Lambda} \inf_{z\in\Lambda} f(\cdot - y + z).$$

We deduce immediately the equivalence (28).

Since the inequality $\left(f^{(-\delta_\Lambda)_-}\right)^{-\delta_\Lambda} \leq f$ is always satisfied, we infer that $f \in \mathcal{E}^{-\delta_\Lambda}$ if and only if for every $x \in X$,

$$\sup_{y\in\Lambda} \inf_{z\in\Lambda} f(x - y + z) \geq f(x).$$

The last assertion of (i) follows immediately.
(ii) Assume that $f \in \mathcal{E}^{-\delta_\Lambda}$ and that $\Lambda + \Lambda \subset \Lambda$. Let $\xi \in \Lambda$. In view of (28), we have for every $x \in X$

$$\begin{aligned} f(x+\xi) &= \sup_{y\in\Lambda} \inf_{z\in\Lambda} f(x+\xi-y+z) \\ &= \sup_{y\in\Lambda} \inf_{z'\in\xi+\Lambda} f(x-y+z') \\ &\geq \sup_{y\in\Lambda} \inf_{z'\in\Lambda} f(x-y+z') \quad \text{ since } \xi+\Lambda\subset\Lambda \\ &= f(x). \end{aligned}$$

Since this is true for every $\xi \in \Lambda$, we infer that f is Λ-nondecreasing.
Conversely, assume that f is Λ-nondecreasing and that $0 \in \Lambda$. For every y, $z \in \Lambda$, we have

$$f(\cdot - y) \leq f(\cdot - y + z) \leq f(\cdot + z).$$

It ensues immediately that

$$\sup_{y\in\Lambda} f(\cdot - y) \leq \sup_{y\in\Lambda} \inf_{z\in\Lambda} f(\cdot - y + z) \leq \inf_{z\in\Lambda} f(\cdot + z).$$

Since $0 \in \Lambda$, we obtain $\sup_{y\in\Lambda} f(\cdot - y) = \inf_{z\in\Lambda} f(\cdot + z) = f$, and hence $f = \sup_{y\in\Lambda} \inf_{z\in\Lambda} f(\cdot - y + z)$. In view of (28), we conclude that $f \in \mathcal{E}^{-\delta_\Lambda}$. $\square$

Remark 4.2. When $\Lambda + \Lambda \subset \Lambda$ and $0 \in \Lambda$, the equivalence

$$f \in \mathcal{E}^{-\delta_\Lambda} \quad \Longleftrightarrow \quad f \text{ is } \Lambda\text{-nondecreasing}$$

can be recovered by using the subadditivity of the function δ_Λ, see section 6.

Proposition 4.6. *Let X be a vector space and let Λ, $D \subset X$.*

(i) *The following equivalence holds*

$$-\delta_D \in \mathcal{E}^{-\delta_\Lambda} \quad \Longleftrightarrow \quad X \setminus D \in \mathcal{I}^{X\setminus\Lambda}.$$

(ii) *If moreover $\Lambda \neq \emptyset$, we have*

$$\delta_D \in \mathcal{E}^{-\delta_\Lambda} \quad \Longleftrightarrow \quad D \in \mathcal{I}^{X\setminus\Lambda}.$$

Proof. (i) First observe that the equivalence trivially holds if $\Lambda = \emptyset$. Now assume that $\Lambda \neq \emptyset$. Recall that

$$-\delta_D \in \mathcal{E}^{-\delta_\Lambda} \quad \Longleftrightarrow \quad -\delta_D = \left((-\delta_D)^{(-\delta_\Lambda)_-}\right)^{-\delta_\Lambda}. \tag{29}$$

Further, note by (4) that

$$\begin{aligned} (-\delta_D)^{(-\delta_\Lambda)_-} &= (-\delta_D)^{(-\delta_{-\Lambda})} = (\delta_{-\Lambda})^{\delta_D} \\ &= \delta_{(-\Lambda)^D} \quad \text{ from formula (26) and the nonvacuity of } \Lambda. \end{aligned}$$

In view of formula (27), we deduce that

$$\left((-\delta_D)^{(-\delta_\Lambda)_-}\right)^{-\delta_\Lambda} = \left(\delta_{(-\Lambda)^D}\right)^{-\delta_\Lambda} = -\delta_{(-\Lambda)^D+\Lambda}.$$

Coming back to (29), we infer that

$$\begin{aligned}
-\delta_D \in \mathcal{E}^{-\delta_\Lambda} &\iff D = (-\Lambda)^D + \Lambda \\
&\iff D = (X \setminus D)^{-(X\setminus\Lambda)} + \Lambda, \quad \text{see Proposition 4.2 } (iii) \\
&\iff X \setminus D = \left((X \setminus D)^{-(X\setminus\Lambda)}\right)^{X\setminus\Lambda}, \quad \text{in view of Proposition 4.2 } (ii) \\
&\iff X \setminus D \in \mathcal{I}^{X\setminus\Lambda}, \quad \textit{cf.} \text{ Proposition 4.3.}
\end{aligned}$$

(ii) Assume that $\Lambda \neq \emptyset$. By arguing as in (i), we find

$$(\delta_D)^{(-\delta_\Lambda)_-} = (\delta_D)^{-\delta_{-\Lambda}} = -\delta_{D-\Lambda},$$

and hence by (4) and (26)

$$\left((\delta_D)^{(-\delta_\Lambda)_-}\right)^{-\delta_\Lambda} = (-\delta_{D-\Lambda})^{-\delta_\Lambda} = (\delta_\Lambda)^{\delta_{D-\Lambda}} = \delta_{\Lambda^{D-\Lambda}}.$$

Recalling that

$$\delta_D \in \mathcal{E}^{-\delta_\Lambda} \quad \iff \quad \delta_D = \left((\delta_D)^{(-\delta_\Lambda)_-}\right)^{-\delta_\Lambda},$$

we deduce that

$$\begin{aligned}
\delta_D \in \mathcal{E}^{-\delta_\Lambda} &\iff D = \Lambda^{D-\Lambda} \\
&\iff D = (X \setminus (D - \Lambda))^{X\setminus\Lambda} \quad \text{by Proposition 4.2 } (iii) \\
&\iff D = \left(D^{-(X\setminus\Lambda)}\right)^{X\setminus\Lambda} \quad \text{by Proposition 4.2 } (ii) \\
&\iff D \in \mathcal{I}^{X\setminus\Lambda}, \quad \text{see Proposition 4.3.}
\end{aligned}$$

□

Combining Corollary 4.1 and Proposition 4.6, we derive the following corollary giving various characterizations of $\mathcal{I}^\Lambda$ via the classes $\mathcal{E}^{\delta_\Lambda}$ and $\mathcal{E}^{-\delta_{X\setminus\Lambda}}$.

Corollary 4.2. *For every set $\Lambda \subset X$ and every set $D \subset X$ such that $D \neq \emptyset$ and $D \neq X$, the following equivalences hold*

$$D \in \mathcal{I}^\Lambda \iff \delta_D \in \mathcal{E}^{\delta_\Lambda} \iff -\delta_{X\setminus D} \in \mathcal{E}^{-\delta_{X\setminus\Lambda}} \iff \delta_D \in \mathcal{E}^{-\delta_{X\setminus\Lambda}}.$$

Proof. The first equivalence is a consequence of Corollary 4.1, under the assumptions $D \neq \emptyset$ and $D \neq X$. The equivalence $D \in \mathcal{I}^\Lambda \iff -\delta_{X\setminus D} \in \mathcal{E}^{-\delta_{X\setminus\Lambda}}$ follows from Proposition 4.6 (i) applied with $X \setminus D$ (resp. $X \setminus \Lambda$) in place of D (resp. Λ). If $\Lambda \neq X$, the equivalence $D \in \mathcal{I}^\Lambda \iff \delta_D \in \mathcal{E}^{-\delta_{X\setminus\Lambda}}$ is a consequence of Proposition 4.6 (ii) applied with $X \setminus \Lambda$ in place of Λ. If $\Lambda = X$, the equivalence becomes $D \in \mathcal{I}^X \iff \delta_D \in \mathcal{E}^{-\omega_X}$. Since $\mathcal{I}^X = \{X\}$ and $\mathcal{E}^{-\omega_X} = \{-\omega_X\}$, the equivalence amounts to $D = X \iff \delta_D = -\omega_X$. The condition $D = X$ is not realized by assumption, while the condition $\delta_D = -\omega_X$ is never realized. It ensues that the equivalence trivially holds true if $\Lambda = X$. □

For a function $f : X \to \overline{\mathbb{R}}$ and $r \in \mathbb{R}$, the notation $[f \geq r]$ (resp. $[f > r]$) denotes the set $\{x \in X,\ f(x) \geq r\}$ (resp. $\{x \in X,\ f(x) > r\}$). We adopt the corresponding notations for the sublevel sets. Adapting Proposition 3.3 to the framework of sets, we obtain the following statement.

Proposition 4.7. *Let X be a locally convex space. Let $\Lambda \subset X$ and $\xi^* \in X^*$. Then we have*

(i) $\big[\langle \xi^*, \cdot\rangle > 0\big]^\Lambda = \big[\langle \xi^*, \cdot\rangle \leq -\sigma_{X\setminus\Lambda}(-\xi^*)\big].$

(*ii*) *If* $\Lambda \neq X$*, the following equivalence holds*

$$\big[\langle \xi^*, \cdot\rangle \leq 0\big] \in \mathcal{I}^\Lambda \quad\iff\quad \xi^* \in -\mathrm{dom}\,\sigma_{X\setminus\Lambda}.$$

Proof. (*i*) Set $C = \big[\langle \xi^*, \cdot\rangle > 0\big]$ and observe that

$$\begin{aligned} x \in C^\Lambda &\iff x - C \subset \Lambda \\ &\iff X\setminus\Lambda \subset x - X \setminus C \\ &\iff X\setminus\Lambda \subset \{y \in X,\, \langle \xi^*, y\rangle \geq \langle \xi^*, x\rangle\} \\ &\iff \forall y \in X\setminus\Lambda,\, \langle \xi^*, y\rangle \geq \langle \xi^*, x\rangle \\ &\iff \inf_{X\setminus\Lambda} \langle \xi^*, \cdot\rangle \geq \langle \xi^*, x\rangle \\ &\iff -\sigma_{X\setminus\Lambda}(-\xi^*) \geq \langle \xi^*, x\rangle. \end{aligned}$$

Item (*i*) follows immediately.

(*ii*) First assume that $\sigma_{X\setminus\Lambda}(-\xi^*) \in \mathbb{R}$. Recall from (*i*) that the set $\big[\langle \xi^*, \cdot\rangle \leq -\sigma_{X\setminus\Lambda}(-\xi^*)\big]$ belongs to $\mathcal{I}^\Lambda$. Let $\xi \in X$ satisfying[4] the equality $\langle \xi^*, \xi\rangle = -\sigma_{X\setminus\Lambda}(-\xi^*)$. We then have

$$\big[\langle \xi^*, \cdot - \xi\rangle \leq 0\big] = \big[\langle \xi^*, \cdot\rangle \leq -\sigma_{X\setminus\Lambda}(-\xi^*)\big] \in \mathcal{I}^\Lambda.$$

Since the class $\mathcal{I}^\Lambda$ is stable under translations, the set $\big[\langle \xi^*, \cdot\rangle \leq 0\big]$ also belongs to $\mathcal{I}^\Lambda$. Now assume that $\sigma_{X\setminus\Lambda}(-\xi^*)$ is not finite, or equivalently $\sigma_{X\setminus\Lambda}(-\xi^*) = +\infty$ since $X \setminus \Lambda \neq \emptyset$ by assumption. Let us determine the set $\big(\big[\langle \xi^*, \cdot\rangle \leq 0\big]^{-\Lambda}\big)^\Lambda$. Remark that

$$\begin{aligned} \big[\langle \xi^*, \cdot\rangle \leq 0\big]^{-\Lambda} &\subset \big[\langle \xi^*, \cdot\rangle < 0\big]^{-\Lambda} \\ &= \big[\langle -\xi^*, \cdot\rangle > 0\big]^{-\Lambda} \\ &= \big[\langle -\xi^*, \cdot\rangle \leq -\sigma_{-(X\setminus\Lambda)}(\xi^*)\big] \quad \text{in view of } (i). \end{aligned}$$

Since $\sigma_{-(X\setminus\Lambda)}(\xi^*) = \sigma_{X\setminus\Lambda}(-\xi^*) = +\infty$, it ensues that $\big[\langle \xi^*, \cdot\rangle \leq 0\big]^{-\Lambda} = \emptyset$, thus implying that

$$\big(\big[\langle \xi^*, \cdot\rangle \leq 0\big]^{-\Lambda}\big)^\Lambda = X \neq \big[\langle \xi^*, \cdot\rangle \leq 0\big].$$

From Proposition 4.3, we conclude that $\big[\langle \xi^*, \cdot\rangle \leq 0\big] \notin \mathcal{I}^\Lambda$, which ends the proof of the announced equivalence. $\square$

Let us denote by $\mathcal{C}(X)$ the class of nonempty closed convex sets of X.

Theorem 4.1. *Let* X *be a locally convex space. Let* $\Lambda \subset X$ *be such that* $\Lambda \neq X$*. For every cone* $Q \subset X^*$*, the following equivalence holds true*

$$\{C \in \mathcal{C}(X),\, \mathrm{dom}\,\sigma_C \subset Q\} \subset \mathcal{I}^\Lambda \quad\iff\quad Q \subset -\mathrm{dom}\,\sigma_{X\setminus\Lambda}.$$

Proof. Let $Q \subset X^*$ be a cone and assume that

$$\{C \in \mathcal{C}(X),\, \mathrm{dom}\,\sigma_C \subset Q\} \subset \mathcal{I}^\Lambda. \tag{30}$$

Let $\xi^* \in Q$. Setting $C = [\langle \xi^*, \cdot\rangle \leq 0] \in \mathcal{C}(X)$, we have $\sigma_C = \delta_{\mathbb{R}_+\xi^*}$, and hence $\mathrm{dom}\,\sigma_C = \mathbb{R}_+\xi^* \subset Q$. In view of (30), it ensues that $C \in \mathcal{I}^\Lambda$. We then deduce from Proposition 4.7 (*ii*) that $\xi^* \in -\mathrm{dom}\,\sigma_{X\setminus\Lambda}$. Since this is true for every $\xi^* \in Q$, we

[4]If $\xi^* = 0$, we have $\sigma_{X\setminus\Lambda}(-\xi^*) = 0$ because $X \setminus \Lambda \neq \emptyset$ by assumption. In this case, the equality $\langle \xi^*, \xi\rangle = -\sigma_{X\setminus\Lambda}(-\xi^*)$ is satisfied by every $\xi \in X$.

conclude that $Q \subset -\mathrm{dom}\, \sigma_{X\setminus\Lambda}$.
Now assume that $Q \subset -\mathrm{dom}\, \sigma_{X\setminus\Lambda}$ and let $C \in \mathcal{C}(X)$ be such that $\mathrm{dom}\, \sigma_C \subset Q$. Then $\delta_C \in \Gamma_0(X)$ with $\mathrm{dom}\, \delta_C^* \subset Q$, and since

$$Q \subset -\mathrm{dom}\, \sigma_{X\setminus\Lambda} = -\mathrm{dom}\, \delta^*_{X\setminus\Lambda} = -\mathrm{dom}\, (-(-\delta_{X\setminus\Lambda}))^*,$$

by Theorem 3.1 we have $\delta_C \in \mathcal{E}^{-\delta_{X\setminus\Lambda}}$ (keep in mind $-\delta_{X\setminus\Lambda} \neq -\omega_X$ since $\Lambda \neq X$). Proposition 4.6 (ii) yields that $C \in \mathcal{I}^\Lambda$ as desired. Finally, we have shown the inclusion (30), which ends the proof. $\square$

Applying Theorem 4.1 with $Q = X^*$, we immediately obtain the following result.

Corollary 4.3. *Let X be a locally convex space. Let $\Lambda \subset X$ be such that $\Lambda \neq X$. Then, the following equivalence holds true*

$$\mathcal{C}(X) \subset \mathcal{I}^\Lambda \quad \Longleftrightarrow \quad \mathrm{dom}\, \sigma_{X\setminus\Lambda} = X^*.$$

5. A preorder relation on $\mathcal{F}(X, \overline{\mathbb{R}})$ based on φ-envelopes

Let X be a vector space and let $\mathcal{F}(X, \overline{\mathbb{R}})$ be the set of extended real-valued functions on X. We define the relation $\sim$ on the space $\mathcal{F}(X, \overline{\mathbb{R}})$ as follows: for every φ, $\psi : X \to \overline{\mathbb{R}}$

$$\begin{aligned} \psi \sim \varphi \quad &\Longleftrightarrow \quad \text{there exist } \xi \in X \text{ and } \alpha \in \mathbb{R} \text{ such that } \psi = \varphi(\cdot - \xi) + \alpha \\ &\Longleftrightarrow \quad \psi \text{ is a } \varphi\text{-elementary function.} \end{aligned}$$

Clearly, the relation $\sim$ is reflexive, symmetric and transitive, hence defines an equivalence relation. The objective of this section is to determine suitable[5] subsets $\mathcal{G}$ of $\mathcal{F}(X, \overline{\mathbb{R}})$ such that the following implication holds true for every φ, $\psi \in \mathcal{G}$

$$\psi \in \mathcal{E}^\varphi \text{ and } \varphi \in \mathcal{E}^\psi \quad \Longrightarrow \quad \psi \sim \varphi. \tag{31}$$

5.1. **The coercive case.** For any function $\varphi : X \to \overline{\mathbb{R}}$, the deconvolution function $\varphi \ominus \varphi$ defined by $(\varphi \ominus \varphi)(x) = \sup_{y-z=x}(\varphi(y) \dot{-} \varphi(z))$ can be expressed as a φ-envelope via the equality $\varphi \ominus \varphi = (\varphi_-)^\varphi$. The next lemma shows that this function is subadditive. Recall that a function $f : X \to \overline{\mathbb{R}}$ is said to be subadditive if for any x, $y \in X$,

$$f(x+y) \leq f(x) \dot{+} f(y).$$

Lemma 5.1. *Let X be a vector space and let f, $\varphi : X \to \overline{\mathbb{R}}$. For any x, $x' \in X$, we have*

$$f^\varphi(x') \leq (\varphi \ominus \varphi)(x' - x) \dot{+} f^\varphi(x).$$

Moreover, the function $\varphi \ominus \varphi$ is subadditive.

Proof. Fix x, $x' \in X$. It is immediate to check that for every $y \in X$,

$$\varphi(x'-y) \dot{-} f(y) \leq [\varphi(x'-y) \dot{-} \varphi(x-y)] \dot{+} [\varphi(x-y) \dot{-} f(y)].$$

Taking the supremum over $y \in X$ and using [21, Proposition 4.a] we deduce that

$$\begin{aligned} f^\varphi(x') \quad &\leq \quad \sup_{y\in X}[\varphi(x'-y) \dot{-} \varphi(x-y)] \dot{+} \sup_{y\in X}[\varphi(x-y) \dot{-} f(y)], \\ &= \quad (\varphi \ominus \varphi)(x'-x) \dot{+} f^\varphi(x), \end{aligned}$$

[5] The implication (31) is not true for all φ, $\psi \in \mathcal{F}(X, \overline{\mathbb{R}})$, see a counterexample in subsection 5.3.

which yields the desired inequality. Further taking $f = \varphi_-$ in the above inequality and using the identity $(\varphi_-)^\varphi = \varphi \ominus \varphi$, we obtain

$$(\varphi \ominus \varphi)(x') \leq (\varphi \ominus \varphi)(x' - x) \dot{+} (\varphi \ominus \varphi)(x),$$

hence the function $\varphi \ominus \varphi$ is subadditive. $\square$

If the space $(X, \|\cdot\|)$ is normed and if the function φ satisfies the coercivity property $\lim_{\|x\| \to +\infty} \varphi(x)/\|x\| = +\infty$, the following lemma shows that $\varphi \ominus \varphi = +\infty$ on $X \setminus \{0\}$.

Lemma 5.2. *Let $(X, \|\cdot\|)$ be a normed space and let $\varphi : X \to \overline{\mathbb{R}}$ be an extended real-valued function. Assume that $\varphi \neq +\omega_X$ and $\lim_{\|x\| \to +\infty} \varphi(x)/\|x\| = +\infty$. Then we have $\varphi \ominus \varphi = +\infty$ on $X \setminus \{0\}$.*

Proof. Let us argue by contradiction and assume that there exists $u \neq 0$ such that $(\varphi \ominus \varphi)(u) < +\infty$. Let us fix $\overline{x} \in \operatorname{dom} \varphi$ and observe that for every $n \in \mathbb{N}$,[6]

$$\begin{aligned} \varphi(\overline{x} + nu) \dot{-} \varphi(\overline{x}) &\leq (\varphi \ominus \varphi)(nu) \\ &\leq n\,(\varphi \ominus \varphi)(u) \quad \text{since } \varphi \ominus \varphi \text{ is subadditive.} \end{aligned}$$

It ensues that

$$\frac{1}{n}\varphi(\overline{x} + nu) \leq \frac{1}{n}\varphi(\overline{x}) + (\varphi \ominus \varphi)(u),$$

and taking the upper limit as $n \to +\infty$, we deduce that

$$\limsup_{n \to +\infty} \frac{1}{n}\varphi(\overline{x} + nu) \leq (\varphi \ominus \varphi)(u),$$

which contradicts the fact that $\lim_{\|x\| \to +\infty} \varphi(x)/\|x\| = +\infty$. Finally, we obtain that $\varphi \ominus \varphi = +\infty$ on $X \setminus \{0\}$. $\square$

Theorem 5.1. *Let X be a vector space and let φ, $\psi : X \to \overline{\mathbb{R}}$ be such that $\psi \in \mathcal{E}^\varphi$ and $\varphi \in \mathcal{E}^\psi$.*

(*i*) *If $\varphi \ominus \varphi = +\infty$ on $X \setminus \{0\}$, then we have $\psi \sim \varphi$.*
(*ii*) *Assume that $(X, \|\cdot\|)$ is a normed space. If $\lim_{\|x\| \to +\infty} \varphi(x)/\|x\| = +\infty$ (resp. $\lim_{\|x\| \to +\infty} \varphi(x)/\|x\| = -\infty$), then we have $\psi \sim \varphi$.*

Proof. If $\varphi = \pm\omega_X$, it is immediate to check that $\psi = \varphi$. From now on, let us assume that $\varphi \neq \pm\omega_X$. Since $\psi \in \mathcal{E}^\varphi$ and $\varphi \in \mathcal{E}^\psi$, there exist f, $g : X \to \overline{\mathbb{R}}$ such that $-\psi = (-\varphi) \bigtriangledown f$ and $-\varphi = (-\psi) \bigtriangledown g$. It ensues that

$$-\varphi = (-\varphi) \bigtriangledown (f \bigtriangledown g). \tag{32}$$

Now observe that

$$\begin{aligned} (-\varphi) \bigtriangledown (f \bigtriangledown g) \geq -\varphi \quad &\Longleftrightarrow \quad (-\varphi)(x - y) \dot{+} (f \bigtriangledown g)(y) \geq -\varphi(x) \quad \text{for all } x, y \in X \\ &\Longleftrightarrow \quad (f \bigtriangledown g)(y) \geq \varphi(x - y) \dot{-} \varphi(x) \quad \text{for all } x, y \in X \\ &\Longleftrightarrow \quad (f \bigtriangledown g)(y) \geq \sup_{x \in X} (\varphi(x - y) \dot{-} \varphi(x)) \quad \text{for all } y \in X \\ &\Longleftrightarrow \quad f \bigtriangledown g \geq [\varphi \ominus \varphi]_-. \end{aligned}$$

[6]$\mathbb{N}$ denotes the set of positive integers.

(i) Assume that $\varphi \ominus \varphi = +\infty$ on $X \setminus \{0\}$. We then deduce from the above inequality that

$$f \bigtriangledown g = +\infty \quad \text{on } X \setminus \{0\}. \tag{33}$$

If $f \bigtriangledown g = \omega_X$, we infer from (32) that $\varphi = -\omega_X$, thus implying in turn that $\psi = -\omega_X$. If $f \bigtriangledown g \neq \omega_X$, equality (33) shows that $\mathrm{dom}\,(f \bigtriangledown g) = \{0\}$. Recalling that $\mathrm{dom}\,(f \bigtriangledown g) = \mathrm{dom}\, f + \mathrm{dom}\, g$, we deduce that $\mathrm{dom}\, f + \mathrm{dom}\, g = \{0\}$. Hence there exists $\xi \in X$ such that $\mathrm{dom}\, f = \{\xi\}$ and $\mathrm{dom}\, g = \{-\xi\}$. We infer that

$$-\psi = (-\varphi) \bigtriangledown f = (-\varphi)(\cdot - \xi) \dot{+} f(\xi) \tag{34}$$

and

$$-\varphi = (-\psi) \bigtriangledown g = (-\psi)(\cdot + \xi) \dot{+} g(-\xi). \tag{35}$$

If $f(\xi) \in \mathbb{R}$, we obtain from (34) that $\psi = \varphi(\cdot - \xi) - f(\xi)$ and therefore $\psi \sim \varphi$. If $g(-\xi) \in \mathbb{R}$, equality (35) shows that $\varphi = \psi(\cdot + \xi) - g(-\xi)$, and hence $\varphi \sim \psi$. On the other hand, if $f(\xi) = g(-\xi) = -\infty$, we deduce from (34)-(35) that

$$-\psi \leq (-\varphi)(\cdot - \xi) \quad \text{and} \quad -\varphi \leq (-\psi)(\cdot + \xi),$$

thus implying that $\psi = \varphi(\cdot - \xi)$ and therefore $\psi \sim \varphi$.

(ii) First assume that $\lim_{\|x\| \to +\infty} \varphi(x)/\|x\| = +\infty$. We infer from Lemma 5.2 that $\varphi \ominus \varphi = +\infty$ on $X \setminus \{0\}$ and we conclude with (i).
Now assume that $\lim_{\|x\| \to +\infty} \varphi(x)/\|x\| = -\infty$. From Lemma 5.2, we deduce that $(-\varphi) \ominus (-\varphi) = +\infty$ on $X \setminus \{0\}$. Recalling that

$$(-\varphi) \ominus (-\varphi) = (-\varphi_-)^{-\varphi} = \varphi^{\varphi_-} = [(\varphi_-)^{\varphi}]_- = [\varphi \ominus \varphi]_-\,,$$

we infer that $\varphi \ominus \varphi = +\infty$ on $X \setminus \{0\}$ and we conclude again with (i). $\square$

Let us define the relation $\preceq$ on $\mathcal{F}(X, \overline{\mathbb{R}})$ by

$$\psi \preceq \varphi \quad \Longleftrightarrow \quad \psi \in \mathcal{E}^{\varphi}.$$

The relation $\preceq$ is clearly reflexive, and also transitive in view of Proposition 3.2 (iii). It is compatible with the equivalence relation $\sim$, *i.e.*

$$\varphi \sim \varphi', \quad \psi \sim \psi' \quad \text{and} \quad \psi \preceq \varphi \quad \Longrightarrow \quad \psi' \preceq \varphi'.$$

It ensues that we can properly define the relation $\preceq$ on the quotient set $\mathcal{F}(X, \overline{\mathbb{R}})/\sim$. The relation $\preceq$ so defined on $\mathcal{F}(X, \overline{\mathbb{R}})/\sim$ is clearly reflexive and transitive, hence it is a preorder. Let us denote by $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{G}''$ the following respective sets

$$\mathcal{G} = \left\{ f : X \to \overline{\mathbb{R}},\, f \ominus f = +\infty \quad \text{on } X \setminus \{0\} \right\},$$

$$\mathcal{G}' = \left\{ f : X \to \overline{\mathbb{R}},\, \lim_{\|x\| \to +\infty} f(x)/\|x\| = +\infty \right\},$$

$$\mathcal{G}'' = \left\{ f : X \to \overline{\mathbb{R}},\, \lim_{\|x\| \to +\infty} f(x)/\|x\| = -\infty \right\}.$$

Theorem 5.1 expresses that for every φ, $\psi \in \mathcal{G}$ (resp. $\mathcal{G}'$, $\mathcal{G}''$), we have

$$\psi \preceq \varphi, \quad \varphi \preceq \psi \quad \Longrightarrow \quad \psi \sim \varphi.$$

Hence the induced relation $\preceq$ on the quotient set $\mathcal{G}/\sim$ (resp. $\mathcal{G}'/\sim$, $\mathcal{G}''/\sim$) is antisymmetric, thus giving rise to an order relation.

Let us now specialize the result of Theorem 5.1 in the case of sets. We define the equivalence relation $\sim$ on $\mathcal{P}(X)$ by

$$C \sim D \quad \Longleftrightarrow \quad \text{there exists } \xi \in X \text{ such that } D = C + \xi,$$

along with the preorder relation $\preceq$ on $\mathcal{P}(X)$ by

$$C \preceq D \quad \Longleftrightarrow \quad C \in \mathcal{I}^D.$$

Recall that the star-difference $C \overset{*}{-} C$ is defined by

$$C \overset{*}{-} C = \bigcap_{x \in C} C - x = (-C)^C.$$

By applying Theorem 5.1 with indicator functions, we obtain the following corollary.

Corollary 5.1. *Let X be a vector space and let Γ, $\Delta \subset X$ be such that $\Delta \in \mathcal{I}^\Gamma$ and $\Gamma \in \mathcal{I}^\Delta$.*

(*i*) *If $\Gamma \overset{*}{-} \Gamma = \{0\}$, then we have $\Delta \sim \Gamma$.*
(*ii*) *Assume that $(X, \|\cdot\|)$ is a normed space. If the set Γ (resp. $X \setminus \Gamma$) is bounded, then we have $\Delta \sim \Gamma$.*

Proof. If $\Gamma \in \{\emptyset, X\}$ (resp. $\Delta \in \{\emptyset, X\}$), it is immediate to check that $\Delta = \Gamma$. Let us now assume that $\Gamma \notin \{\emptyset, X\}$ and $\Delta \notin \{\emptyset, X\}$. In view of Corollary 4.1, the assumptions $\Delta \in \mathcal{I}^\Gamma$ and $\Gamma \in \mathcal{I}^\Delta$ imply that $\delta_\Delta \in \mathcal{E}^{\delta_\Gamma}$ and $\delta_\Gamma \in \mathcal{E}^{\delta_\Delta}$.
(*i*) Assume that $\Gamma \overset{*}{-} \Gamma = \{0\}$. Then, by (5) and Proposition 4.4 (*i*) we have

$$\delta_\Gamma \ominus \delta_\Gamma = (\delta_{-\Gamma})^{\delta_\Gamma} = \delta_{(-\Gamma)^\Gamma} = \delta_{\Gamma \overset{*}{-} \Gamma} = \delta_{\{0\}}.$$

By applying Theorem 5.1 (*i*) with $\varphi = \delta_\Gamma$ and $\psi = \delta_\Delta$, we obtain that $\delta_\Delta \sim \delta_\Gamma$ and hence $\Delta \sim \Gamma$.
(*ii*) First assume that Γ is bounded. Then the indicator function δ_Γ is coercive and we deduce from Lemma 5.2 that $\delta_\Gamma \ominus \delta_\Gamma = +\infty$ on $X \setminus \{0\}$. This implies that $\Gamma \overset{*}{-} \Gamma = \{0\}$ and we conclude with (*i*). Now assume that $X \setminus \Gamma$ is bounded. From what precedes, we have $(X \setminus \Gamma) \overset{*}{-} (X \setminus \Gamma) = \{0\}$. Observing that

$$\Gamma \overset{*}{-} \Gamma = (-\Gamma)^\Gamma = (X \setminus \Gamma)^{-X \setminus \Gamma} = -[(-X \setminus \Gamma)^{X \setminus \Gamma}] = -[(X \setminus \Gamma) \overset{*}{-} (X \setminus \Gamma)],$$

we infer that $\Gamma \overset{*}{-} \Gamma = \{0\}$ and we conclude again with (*i*). $\square$

Let us denote by $\mathcal{Q}$, $\mathcal{Q}'$ and $\mathcal{Q}''$ the following respective sets

$$\mathcal{Q} = \{C \subset X,\, C \overset{*}{-} C = \{0\}\},$$

$$\mathcal{Q}' = \{C \subset X,\, C \text{ is bounded}\},$$

$$\mathcal{Q}'' = \{C \subset X,\, X \setminus C \text{ is bounded}\}.$$

The above corollary expresses that for every Γ, $\Delta \in \mathcal{Q}$ (resp. $\mathcal{Q}'$, $\mathcal{Q}''$), we have

$$\Delta \preceq \Gamma, \quad \Gamma \preceq \Delta \quad \Longrightarrow \quad \Delta \sim \Gamma.$$

Hence the induced relation $\preceq$ on the quotient set $\mathcal{Q}/\sim$ (resp. $\mathcal{Q}'/\sim$, $\mathcal{Q}''/\sim$) is antisymmetric, thus giving rise to an order relation.

5.2. **The case φ, $\psi \in -\Gamma_0(X)$.** Let us first state a result that will be a key ingredient for the next theorem.

Lemma 5.3. *Let X be a vector space, let $D \subset X$ be a convex set and let us denote by $\mathrm{Aff}\,(D)$ the affine space generated by D. Assume that a real-valued function $h : D \to \mathbb{R}$ is both convex and concave. Then there exists a unique affine function $\tilde{h} : \mathrm{Aff}\,(D) \to \mathbb{R}$ such that $\tilde{h}_{|D} = h$.*

For a proof of this result, the reader is referred to [33]. By extending affinely the function $\tilde{h}$ to the whole space X, we deduce from the above result that there exists a linear function $\ell : X \to \mathbb{R}$ along with $\alpha \in \mathbb{R}$ such that $h = \ell_{|D} + \alpha$.

In view of stating the next theorem, given a locally convex space X recall that the Mackey topology $\tau(X^*, X)$ on X^* is defined as the finest locally convex topology $\mathcal{T}$ on X^* such that the topological dual of $(X^*, \mathcal{T})$ coincides with X. If $(X, \|\cdot\|)$ is normed, this topology is exactly the one associated with the dual norm $\|\cdot\|_{X^*}$ provided that $(X, \|\cdot\|)$ is a reflexive Banach space.

Theorem 5.2. *Let X be a locally convex space. Let φ, $\psi : X \to \overline{\mathbb{R}}$ be functions such that $\psi \in \mathcal{E}^\varphi$ and $\varphi \in \mathcal{E}^\psi$. Assume that either*

-the space X is finite-dimensional, or

-one of the functions $(-\varphi)^*$ and $(-\psi)^*$ is $\tau(X^*, X)$-continuous at some point and finite at this point.

Then we have $(-\varphi)^{**} \sim (-\psi)^{**}$. If each of the functions $-\varphi$ and $-\psi$ has a continuous affine minorant, then $\overline{\mathrm{co}}(-\varphi) \sim \overline{\mathrm{co}}(-\psi)$. In particular, if $-\varphi \in \Gamma_0(X)$ and $-\psi \in \Gamma_0(X)$, then we have $\varphi \sim \psi$.

Proof. By assumption, we have $-\psi = (-\varphi) \bigtriangledown f$ and $-\varphi = (-\psi) \bigtriangledown g$, for some f, $g : X \to \overline{\mathbb{R}}$. Taking the conjugates, we obtain that

$$(-\psi)^* = (-\varphi)^* \dot{+} f^* \quad \text{and} \quad (-\varphi)^* = (-\psi)^* \dot{+} g^*. \tag{36}$$

First observe that if one of the functions $(-\varphi)^*$, $(-\psi)^*$, f^* or g^* is equal to $-\omega_{X^*}$, then equalities (36) imply that $(-\varphi)^* = (-\psi)^* = -\omega_{X^*}$. This implies in turn that $\varphi = \psi = -\omega_X$ and the conclusion is satisfied. From now on, let us assume that the functions $(-\varphi)^*$, $(-\psi)^*$, f^* and g^* differ from $-\omega_{X^*}$. From the first equality of (36), we deduce that $\mathrm{dom}\,(-\psi)^* \subset \mathrm{dom}\,(-\varphi)^*$, while the second equality of (36) yields $\mathrm{dom}\,(-\varphi)^* \subset \mathrm{dom}\,(-\psi)^*$. Finally, the domains of $(-\varphi)^*$ and $(-\psi)^*$ coincide and both functions are finite on their common domain D. If the set D is empty, then $(-\varphi)^* = (-\psi)^* = \omega_{X^*}$. This implies that $(-\varphi)^{**} = (-\psi)^{**} = -\omega_X$, hence the conclusion is trivially satisfied. Without loss of generality, we now assume that $D \neq \emptyset$. By combining both equalities of (36), we obtain

$$(-\varphi)^* = (-\varphi)^* + f^* + g^*.$$

It ensues that $f^* + g^* = 0$ on D. Hence the function ${f^*}_{|D}$ is finite-valued on D and both convex and concave. By applying the previous lemma with $h = {f^*}_{|D}$, we obtain that there exist a linear function $\ell : X^* \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that $f^* = \ell + \alpha$ on D. Coming back to the first equality of (36), we deduce that

$$(-\psi)^* = (-\varphi)^* + \ell + \alpha.$$

Observe that the above equality holds true on the whole space X^*, since the functions $(-\varphi)^*$ and $(-\psi)^*$ are equal to $+\infty$ outside D. Taking the conjugate of each member, we find for every $\xi \in X$

$$(-\psi)^{**}(\xi) = \sup_{x^* \in X^*} [\langle x^*, \xi\rangle - (-\varphi)^*(x^*) - \ell(x^*) - \alpha]. \tag{37}$$

Let us now show that the linear function ℓ is $\tau(X^*, X)$-continuous on X^*.

Lemma 5.1. *Under the assumptions of Theorem 5.2, the function $\ell : X^* \to \mathbb{R}$ is $\tau(X^*, X)$-continuous on X^*.*

Proof of Lemma 5.1. If the space X is finite-dimensional, the assertion is obvious. Now assume that the function $(-\varphi)^*$ is $\tau(X^*, X)$-continuous at some $\overline{x}^* \in X^*$ and finite at this point. There exist a $\tau(X^*, X)$-neighborhood W of $\overline{x}^*$ and $M \in \mathbb{R}$ such that $(-\varphi)^* \leq M$ on W. We deduce from (37) that for every $\xi \in X$,

$$\begin{aligned}(-\psi)^{**}(\xi) \quad &\geq \quad \sup_{x^* \in W} [\langle x^*, \xi\rangle - (-\varphi)^*(x^*) - \ell(x^*) - \alpha] \\ &\geq \quad \sup_{x^* \in W} [\langle x^*, \xi\rangle - \ell(x^*)] - M - \alpha.\end{aligned}$$

Let us argue by contradiction and assume that ℓ is not $\tau(X^*, X)$-continuous on X^*. Since the linear function $\langle \cdot, \xi\rangle - \ell$ is not $\tau(X^*, X)$-continuous on X^*, the above supremum equals $+\infty$. It ensues that $(-\psi)^{**} = \omega_X$, and hence $-\psi = \omega_X$. Recalling that $-\varphi = (-\psi) \bigtriangledown g$, we deduce that $-\varphi = \omega_X$. This implies in turn that $(-\varphi)^* = -\omega_{X^*}$, a contradiction with $(-\varphi)^*(\overline{x}^*) \in \mathbb{R}$. We conclude that the linear function ℓ is $\tau(X^*, X)$-continuous on X^*. Since φ and ψ play symmetric roles, the same conclusion holds true if the function $(-\psi)^*$ is assumed to be $\tau(X^*, X)$-continuous at some $\tilde{x}^* \in X^*$ and finite at this point. $\square$

From the previous lemma and the definition of the Mackey topology $\tau(X^*, X)$, there exists $x \in X$ such that $\ell(x^*) = \langle x^*, x\rangle$ for every $x^* \in X^*$. In view of (37), we deduce that

$$(-\psi)^{**}(\xi) = \sup_{x^* \in X^*} [\langle x^*, \xi - x\rangle - (-\varphi)^*(x^*)] - \alpha = (-\varphi)^{**}(\xi - x) - \alpha.$$

Since this is true for every $\xi \in X$, we conclude that $(-\psi)^{**} \sim (-\varphi)^{**}$. If the function $(-\varphi)$ (resp. $(-\psi)$) admits a continuous affine minorant, we have $(-\varphi)^{**} = \overline{\text{co}}(-\varphi)$ (resp. $(-\psi)^{**} = \overline{\text{co}}(-\psi)$). We infer that $\overline{\text{co}}(-\psi) \sim \overline{\text{co}}(-\varphi)$. The last assertion of the statement is a direct consequence of what precedes. $\square$

Remark 5.1. If the normed space $(X, \|\cdot\|)$ is reflexive, the $\tau(X^*, X)$-continuity assumption on $(-\varphi)^*$ (resp. $(-\psi)^*$) amounts to the continuity assumption with respect to the dual norm $\|\cdot\|_{X^*}$.

Theorem 5.2 implies that the relation $\preceq$ defines an order on the following set

$$\{\varphi \in -\Gamma_0(X), \quad (-\varphi)^* \text{ is } \tau(X^*, X)\text{-continuous at some point}\}/\sim.$$

If the space X is finite-dimensional, the relation $\preceq$ is an order on the set $(-\Gamma_0(X))/\sim$.

By applying Theorem 5.2 with the opposite of indicator functions, we obtain the following corollary.

Corollary 5.2. *Let X be a locally convex space. Let Γ, $\Delta \subset X$ be such that $\Delta \in \mathcal{I}^\Gamma$ and $\Gamma \in \mathcal{I}^\Delta$. Assume that either*

-the space X is finite-dimensional, or

-one of the functions $\sigma_{X\setminus\Gamma}$ and $\sigma_{X\setminus\Delta}$ is $\tau(X^*, X)$-continuous at some point.

Then we have $\overline{\mathrm{co}}(X \setminus \Gamma) \sim \overline{\mathrm{co}}(X \setminus \Delta)$. In particular, if the sets $X \setminus \Gamma$ and $X \setminus \Delta$ are closed and convex, then $\Gamma \sim \Delta$.

Proof. From Proposition 4.6 (i), condition $\Delta \in \mathcal{I}^{\Gamma}$ (resp. $\Gamma \in \mathcal{I}^{\Delta}$) is equivalent to $-\delta_{X\setminus\Delta} \in \mathcal{E}^{-\delta_{X\setminus\Gamma}}$ (resp. $-\delta_{X\setminus\Gamma} \in \mathcal{E}^{-\delta_{X\setminus\Delta}}$). By applying Theorem 5.2 with $\varphi = -\delta_{X\setminus\Gamma}$ and $\psi = -\delta_{X\setminus\Delta}$, we obtain the existence of $\xi \in X$ and $\alpha \in \mathbb{R}$ such that

$$\overline{\mathrm{co}}(\delta_{X\setminus\Delta}) = [\overline{\mathrm{co}}(\delta_{X\setminus\Gamma})](\cdot - \xi) - \alpha.$$

We immediately deduce that $\overline{\mathrm{co}}(X \setminus \Delta) = \overline{\mathrm{co}}(X \setminus \Gamma) + \xi$. The last assertion of the statement is a direct consequence of what precedes. □

5.3. **A counterexample.** Let us start with a preliminary result.

Lemma 5.4. *Let X be a topological vector space and let G be a dense additive subgroup of X. Assume that $K \subset X$ is an open set such that $K + K \subset K$ and $0 \in \mathrm{cl}\,(K)$. Then we have*

(i) *For all ξ, $\xi' \in X$,*

$$[G \cap (K + \xi)] + [G \cap (K + \xi')] = G \cap (K + \xi + \xi').$$

(ii) *If in addition $\mathrm{cl}\,(K) \cap -\mathrm{cl}\,(K) = \{0\}$, then*

$$G \cap (K + \xi) = (G \cap K) + \xi' \quad \Longrightarrow \quad \xi = \xi'.$$

If $G \neq X$ and $\xi \in X \setminus G$, there is no $\xi' \in X$ such that $G \cap (K + \xi) = (G \cap K) + \xi'$.

Proof. (i) Let us fix ξ, $\xi' \in X$ and let us prove the inclusion from the left to the right. Observe that

$$[G \cap (K + \xi)] + [G \cap (K + \xi')] \subset G + G$$

and

$$[G \cap (K + \xi)] + [G \cap (K + \xi')] \subset (K + \xi) + (K + \xi').$$

Since $G + G \subset G$ and $K + K \subset K$, we deduce that

$$[G \cap (K + \xi)] + [G \cap (K + \xi')] \subset G \cap (K + \xi + \xi').$$

Now let us establish the reverse inclusion. Let $x \in G \cap (K + \xi + \xi')$. Observe that the open set $K + \xi + \xi' - x$ contains 0. Recalling that $0 \in \mathrm{cl}\,(K)$, we have $(K + \xi + \xi' - x) \cap -K \neq \emptyset$, hence $(K + \xi - x) \cap (-K - \xi') \neq \emptyset$. Since the set K is open, the set $(K + \xi - x) \cap (-K - \xi')$ is open. By using the density of G in X, we deduce that

$$G \cap (K + \xi - x) \cap (-K - \xi') \neq \emptyset.$$

Since $G = -G$, the above property can be rewritten as

$$[G \cap (K + \xi - x)] \cap [-G \cap (-K - \xi')] \neq \emptyset,$$

which is in turn equivalent to

$$0 \in [G \cap (K + \xi - x)] + [G \cap (K + \xi')].$$

Recalling that $x \in G$, we have $G = G - x$, hence $G \cap (K + \xi - x) = [G \cap (K + \xi)] - x$. In view of the latter inclusion, we conclude that

$$x \in [G \cap (K + \xi)] + [G \cap (K + \xi')].$$

The inclusion

$$G \cap (K + \xi + \xi') \subset [G \cap (K + \xi)] + [G \cap (K + \xi')]$$

is proved.
(ii) Let us assume that $G \cap (K+\xi) = (G \cap K) + \xi'$ for some ξ, $\xi' \in X$. We deduce that $G \cap (K+\xi) \subset K + \xi'$. By using the openness of the set $K+\xi$ along with the density of G in X, we easily infer that $K + \xi \subset \mathrm{cl}\,(K) + \xi'$. This implies in turn that $\mathrm{cl}\,(K) + \xi \subset \mathrm{cl}\,(K) + \xi'$ and since $0 \in \mathrm{cl}\,(K)$, we obtain $\xi - \xi' \in \mathrm{cl}\,(K)$. By a symmetric argument, we find $\xi' - \xi \in \mathrm{cl}\,(K)$, hence $\xi - \xi' \in \mathrm{cl}\,(K) \cap -\mathrm{cl}\,(K)$. Since $\mathrm{cl}\,(K) \cap -\mathrm{cl}\,(K) = \{0\}$ by assumption, we conclude that $\xi = \xi'$.
Now let $\xi \in X \setminus G$ and assume that there exists $\xi' \in X$ such that $G \cap (K+\xi) = (G \cap K) + \xi'$. From what precedes, we have $\xi' = \xi$ and hence $G \cap (K+\xi) = (G+\xi) \cap (K+\xi)$. On the other hand, the assumption $\xi \in X \setminus G$ implies that the sets G and $G + \xi$ are disjoint. We deduce that $G \cap (K+\xi) = \emptyset$, a contradiction since the nonempty set K is open and the set G is dense in X. □

Let us now build an example of sets Γ, $\Delta \subset X$ satisfying $\Delta \in \mathcal{I}^{\Gamma}$ and $\Gamma \in \mathcal{I}^{\Delta}$, but with Δ and Γ not equal up to a translation. We are given an open set $K \subset X$ such that $K + K \subset K$ and $\mathrm{cl}\,(K) \cap -\mathrm{cl}\,(K) = \{0\}$, along with a dense additive subgroup $G \subset X$ such that $G \neq X$. Define the sets C, U, $V \subset X$ respectively by

$$C = G \cap K; \quad U = G \cap (K + \xi); \quad V = G \cap (K - \xi),$$

where $\xi \in X \setminus G$. In view of Lemma 5.4 (i), the set $D = C + U$ satisfies

$$D = G \cap (K + \xi) \quad \text{and} \quad D + V = G \cap K = C.$$

Lemma 5.4 (ii) shows that the set D is not translated from C. Defining the complementary sets $\Gamma = X \setminus C$ and $\Delta = X \setminus D$, we have

$$\Delta = X \setminus (C + U) = U^{X \setminus C} = U^{\Gamma} \in \mathcal{I}^{\Gamma} \tag{38}$$

and

$$\Gamma = X \setminus (D + V) = V^{X \setminus D} = V^{\Delta} \in \mathcal{I}^{\Delta}. \tag{39}$$

From what precedes, the set Δ is not translated from Γ. The above counterexample for sets obviously furnishes a counteraxample for functions. Indeed, we deduce from (38)-(39) that the indicator functions δ_Γ and δ_Δ satisfy $\delta_\Delta \in \mathcal{E}^{\delta_\Gamma}$ and $\delta_\Gamma \in \mathcal{E}^{\delta_\Delta}$, but the functions δ_Γ and δ_Δ are not equal up to a translation.

By particularizing the above sets G, $K \subset X$, one obtains various counterexamples. If $X = \mathbb{R}$, one can take $G = \mathbb{Q}$, $K =]0, +\infty[$ and $\xi \in \mathbb{R} \setminus \mathbb{Q}$. On the other hand, if X is infinite dimensional, one can assume that G is a dense subspace of X and that K is an open convex cone such that $\mathrm{cl}\,(K)$ is pointed. This furnishes a counterexample with convex sets C, $D \subset X$.

6. Cases of either superadditivity or subadditivity of φ

Let us first recall that a function $\varphi : X \to \overline{\mathbb{R}}$ is said to be superadditive (resp. subadditive) if for all $x, y \in X$,

$$\varphi(x+y) \geq \varphi(x) \underset{\cdot}{+} \varphi(y) \quad (\text{resp. } \varphi(x+y) \leq \varphi(x) \dot{+} \varphi(y)).$$

Let us start with a preliminary result.

Lemma 6.1. *Let X be a vector space. Let $h, k : X \to \overline{\mathbb{R}}$ and assume that $k(0) = 0$. Then we have*

$$\begin{aligned} h = h \bigtriangleup k \quad &\Longleftrightarrow \quad h(x) \geq h(y) \dot{+} k(x-y) \quad \textit{for all } x, y \in X \\ &\Longleftrightarrow \quad h(y) \leq h(x) \dot{+} (-k_-)(y-x) \quad \textit{for all } x, y \in X \\ &\Longleftrightarrow \quad h = h \bigtriangledown (-k_-). \end{aligned}$$

As a consequence, the function k is superadditive if and only if $k = k \bigtriangleup k$, which is in turn equivalent to $k = k \bigtriangledown (-k_-)$.

Proof. If $h = h \bigtriangleup k$, then the definition of $h \bigtriangleup k$ entails that $h(x) \geq h(y) \dot{+} k(x-y)$ for all $x, y \in X$. Conversely, if this inequality holds true for every $x, y \in X$, we have

$$h(x) \geq \sup_{y \in X} h(y) \dot{+} k(x-y) \geq h(x) + k(0) = h(x),$$

for every $x \in X$. This implies that $h(x) = (h \bigtriangleup k)(x)$ for every $x \in X$ and the first equivalence is proved.
For the second equivalence, observe that for all $x, y \in X$

$$h(x) \geq h(y) \dot{+} k(x-y) \quad \Longleftrightarrow \quad h(y) \leq h(x) \dot{+} (-k)(x-y) = h(x) \dot{+} (-k_-)(y-x).$$

The proof of the third equivalence follows the same lines as the first one. For the last assertion, observe that k is superadditive if and only if $k(x) \geq k(y) \dot{+} k(x-y)$ for all $x, y \in X$. It suffices then to use what precedes with $h = k$. $\square$

Through the above lemma, the following theorem provides, in particular, various characterizations of the class $\mathcal{E}^{\varphi}$ when φ is superadditive.

Theorem 6.1. *Let X be a vector space. Let $\varphi : X \to \overline{\mathbb{R}}$ be a superadditive function satisfying $\varphi(0) = 0$.*

(*a*) *For a function $g : X \to \overline{\mathbb{R}}$, the following assertions are equivalent*
  - (*i*) $g \in \mathcal{E}^{\varphi}$;
  - (*ii*) $g = g \bigtriangleup \varphi$;
  - (*iii*) $g(x) \geq g(y) \dot{+} \varphi(x-y)$ *for all* $x, y \in X$;
  - (*iv*) $g(y) \leq g(x) \dot{+} (-\varphi_-)(y-x)$ *for all* $x, y \in X$;
  - (*v*) $g = g \bigtriangledown (-\varphi_-)$;
  - (*vi*) $-g \in \mathcal{E}^{\varphi_-}$.

(*b*) *For every function $f : X \to \overline{\mathbb{R}}$, $f \bigtriangledown (-\varphi_-)$ is the greatest φ-envelope that is majorized by f, while $f \bigtriangleup \varphi$ is the lowest φ-envelope that is minorized by f.*

(*c*) *The following inclusion holds true $\mathcal{E}^{-\varphi} \subset \mathcal{E}^{\varphi_-}$.*

Proof. (*a*) Let us assume that $g \in \mathcal{E}^{\varphi}$. Then there exists $f : X \to \overline{\mathbb{R}}$ such that $g = f^{\varphi} = (-f) \bigtriangleup \varphi$. Using the superadditivity of φ and the last assertion of Lemma 6.1, we have

$$g \bigtriangleup \varphi = ((-f) \bigtriangleup \varphi) \bigtriangleup \varphi = (-f) \bigtriangleup (\varphi \bigtriangleup \varphi) = (-f) \bigtriangleup \varphi = g.$$

This shows that (*i*) $\Longrightarrow$ (*ii*). Conversely, if $g = g \bigtriangleup \varphi$ then $g = (-g)^{\varphi}$ and clearly $g \in \mathcal{E}^{\varphi}$. The equivalences (*ii*) $\Longleftrightarrow$ (*iii*) $\Longleftrightarrow$ (*iv*) $\Longleftrightarrow$ (*v*) follow directly from

Lemma 6.1. For the equivalence $(v) \iff (vi)$, observe that

$$g = g \bigtriangledown (-\varphi_-) \iff -g = (-g) \bigtriangleup \varphi_-,$$

and invoke the equivalence $(i) \iff (ii)$.
(b) Let $f : X \to \overline{\mathbb{R}}$. Observe that

$$\begin{aligned}(f \bigtriangledown (-\varphi_-)) \bigtriangledown (-\varphi_-) &= f \bigtriangledown ((-\varphi_-) \bigtriangledown (-\varphi_-))\\ &= f \bigtriangledown (-(\varphi_- \bigtriangleup \varphi_-))\\ &= f \bigtriangledown (-\varphi_-) \quad \text{by Lemma 6.1 since } \varphi_- \text{ is superadditive.}\end{aligned}$$

In view of the implication $(v) \Longrightarrow (ii)$ in (a), we deduce that

$$f \bigtriangledown (-\varphi_-) = (f \bigtriangledown (-\varphi_-)) \bigtriangleup \varphi = ((-f) \bigtriangleup \varphi_-)^{\varphi} = (f^{\varphi_-})^{\varphi}.$$

Hence $f \bigtriangledown (-\varphi_-)$ coincides with $(f^{\varphi_-})^{\varphi}$, which is by property (6) the greatest element of $\mathcal{E}^{\varphi}$ that is majorized by f. Replacing f (resp. φ) with $-f$ (resp. φ_-) and taking the opposite, we deduce that $f \bigtriangleup \varphi$ is the lowest element of $-\mathcal{E}^{\varphi_-}$ that is minorized by f. It suffices then to recall that $\mathcal{E}^{\varphi_-} = -\mathcal{E}^{\varphi}$, see the equivalence $(i) \iff (vi)$ in (a).
(c) Since $\varphi \in \mathcal{E}^{\varphi}$, we have $-\varphi \in -\mathcal{E}^{\varphi} = \mathcal{E}^{\varphi_-}$. In view of Proposition 3.2 (iii), we infer that $\mathcal{E}^{-\varphi} \subset \mathcal{E}^{\varphi_-}$. □

Example 6.1. Assume that $(X, \|\cdot\|)$ is a normed space. For $k \geq 0$ and $\alpha \in]0,1]$, take $\varphi = -k\|\cdot\|^{\alpha}$. Observe that for all $x, y \in X$

$$\|x+y\|^{\alpha} \leq (\|x\| + \|y\|)^{\alpha} \leq \|x\|^{\alpha} + \|y\|^{\alpha}. \tag{40}$$

It ensues that the function $\|\cdot\|^{\alpha}$ is subadditive, hence φ is superadditive. From Theorem 6.1 (a), we deduce that

$$f \in \mathcal{E}^{-k\,\|\cdot\|^{\alpha}} \iff f(x) \geq f(y) - k\,\|x-y\|^{\alpha} \quad \text{for all } x, y \in X. \tag{41}$$

By reversing the roles of x and y, we immediately obtain

$$f \in \mathcal{E}^{-k\,\|\cdot\|^{\alpha}} \iff f(x) \leq f(y) + k\,\|x-y\|^{\alpha} \quad \text{for all } x, y \in X. \tag{42}$$

If $f(y) = +\infty$ (resp. $f(y) = -\infty$) for some $y \in X$, we deduce from (41) (resp. (42)) that $f = \omega_X$ (resp. $f = -\omega_X$). On the other hand, if the function f is finite-valued, we infer from (41)-(42) that $|f(x) - f(y)| \leq k\,\|x-y\|^{\alpha}$ for all $x, y \in X$. This implies that

$$\begin{aligned}\mathcal{E}^{-k\,\|\cdot\|^{\alpha}} &= \{f : X \to \mathbb{R},\, |f(x) - f(y)| \leq k\,\|x-y\|^{\alpha} \text{ for all } x, y \in X\} \cup \{\omega_X, -\omega_X\}\\ &= \{f : X \to \mathbb{R},\, f \text{ is } \alpha\text{-Hölderian with constant } k\} \cup \{\omega_X, -\omega_X\}.\end{aligned}$$

From Theorem 6.1 (b), we deduce that $f \bigtriangledown k\|\cdot\|^{\alpha}$ (resp. $f \bigtriangleup (-k\|\cdot\|^{\alpha})$) is the greatest (resp. lowest) φ-envelope that is majorized (resp. minorized) by f. Since the map $\|\cdot\|^{\alpha}$ is even, Theorem 6.1 (c) shows that $\mathcal{E}^{k\|\cdot\|^{\alpha}} \subset \mathcal{E}^{-k\|\cdot\|^{\alpha}}$.
Now assume that $\alpha = 1$. From what precedes, we obtain that

$$\mathcal{E}^{-k\,\|\cdot\|} = \{f : X \to \mathbb{R},\, f \text{ is } k\text{-Lipschitz continuous}\} \cup \{\omega_X, -\omega_X\}.$$

The Pasch-Hausdorff regularization of f, defined by $l_k(f) = f \bigtriangledown k\|\cdot\|$, is the greatest function of $\mathcal{E}^{-k\,\|\cdot\|}$ that is majorized by f. On the other hand, $f \bigtriangleup (-k\,\|\cdot\|)$ is the lowest function of $\mathcal{E}^{-k\,\|\cdot\|}$ that is minorized by f. The inclusion $\mathcal{E}^{k\,\|\cdot\|} \subset \mathcal{E}^{-k\,\|\cdot\|}$ shows that the $k\|\cdot\|$-envelopes are either k-Lipschitz continuous or equal to $\pm\omega_X$. The convexity of $\|\cdot\|$ implies that $k\|\cdot\|$-envelopes are also convex, therefore the inclusion $\mathcal{E}^{k\,\|\cdot\|} \subset \mathcal{E}^{-k\,\|\cdot\|}$ is strict. This ensures that the inclusion in (c) of the above theorem generally fails to be an equality.

As regards the function $\varphi = -k\|\cdot\|^\alpha$ it is also worth mentioning that, for $\eta(x,y) := \|x-y\|^\alpha$ with $\alpha > 0$ (even with more general coupling functions) and taking

$$\mathbf{\Phi} := \{r - \sigma\,\eta(\cdot, y) : r \in \mathbb{R},\, \sigma > 0,\, y \in X\},$$

a lower semicontinuous function on the normed space X is shown in [7, Theorem 4.2] to be $\mathbf{\Phi}$-convex (i.e., a pointwise supremum of functions in $\mathbf{\Phi}$), whenever it is bounded from below by a function in $\mathbf{\Phi}$. The latter property with $\alpha = 2$ was previously proved in [29, Theorem 2]. The function $(x,y) \mapsto -k\|x-y\|^\alpha$ is also used as a particular important example of coupling functions arising in the framework of generalized conjugacy in many papers, see for example [23, p. 204].

Remark 6.1. Given a nonincreasing convex function $\theta : \mathbb{R}_+ \to \mathbb{R}$ such that $\theta(0) = 0$, one can easily check that the function $\theta(\|\cdot\|)$ is superadditive. Hence the previous example can be generalized by taking $\varphi = \theta(\|\cdot\|)$.

Example 6.2. Let X be a vector space. Let $\Lambda \subset X$ be a set containing the origin and such that $\Lambda + \Lambda \subset \Lambda$. The function δ_Λ is clearly subadditive. This implies that the function $\varphi = -\delta_\Lambda$ is superadditive. By Theorem 6.1 (a) it follows that

$$\begin{aligned} f \in \mathcal{E}^{-\delta_\Lambda} \quad &\Longleftrightarrow \quad f(x) \geq f(y) \dotplus (-\delta_\Lambda)(x-y) \quad \text{ for all } x, y \in X \\ &\Longleftrightarrow \quad f(x) \geq f(y) \quad \text{ if } x - y \in \Lambda \\ &\Longleftrightarrow \quad f \text{ is } \Lambda\text{-nondecreasing.} \end{aligned}$$

This and Theorem 6.1 (b) entail that $f \bigtriangledown \delta_{-\Lambda}$ (resp. $f \bigtriangleup (-\delta_\Lambda)$) is the greatest (resp. lowest) Λ-nondecreasing function that is majorized (resp. minorized) by f. Further, Theorem 6.1 (c) says that $\mathcal{E}^{\delta_\Lambda} \subset \mathcal{E}^{(-\delta_\Lambda)_-} = \mathcal{E}^{-\delta_{-\Lambda}}$, hence the functions of $\mathcal{E}^{\delta_\Lambda}$ are Λ-nonincreasing. In fact, this can be recovered directly by using the characterization of $\mathcal{E}^{\delta_\Lambda}$ given by Proposition 4.4 (ii).

7. Case $\varphi \in \Gamma(X)$

7.1. **Expressions of φ-envelopes as Legendre-Fenchel conjugates.** Let us start with the following elementary lemma.

Lemma 7.1. *Let X be a locally convex space. For every function $f : X \to \overline{\mathbb{R}}$, we have $(f^*)_- = (f_-)^*$.*

Proof. It suffices to use the definition of the Legendre-Fenchel conjugate. For every $\xi^* \in X^*$, we have

$$\begin{aligned} (f^*)_-(\xi^*) = (f^*)(-\xi^*) \quad &= \quad \sup_{x \in X}\{\langle -\xi^*, x\rangle - f(x)\} \\ &= \quad \sup_{y \in X}\{\langle \xi^*, y\rangle - f(-y)\} \\ &= \quad \sup_{y \in X}\{\langle \xi^*, y\rangle - f_-(y)\} = (f_-)^*(\xi^*). \end{aligned}$$

□

Theorem 7.1. *Let X be a locally convex space. Let us assume that $\varphi \in \Gamma(X)$ and let $\psi : X^* \to \overline{\mathbb{R}}$ be such that $\psi^* = \varphi$. Then we have for every function $f : X \to \overline{\mathbb{R}}$*

$$f^\varphi = (\psi \dot{-} (f_-)^*)^*. \tag{43}$$

Moreover the following equivalences hold

$$\begin{aligned} g \in \mathcal{E}^\varphi \quad &\iff \quad g = (\psi \dot{-} h)^* \quad \text{for some } h \in \Gamma(X^*) \\ &\iff \quad g = (\psi \dot{-} (\psi \dot{-} g^*)^{**})^*. \end{aligned}$$

Proof. For every $x \in X$,

$$\begin{aligned} f^\varphi(x) &= \sup_{y\in X}\{\varphi(x-y) \dot{-} f(y)\} \\ &= \sup_{y\in X}\left\{\sup_{\xi^*\in X^*}\{\langle \xi^*, x-y\rangle - \psi(\xi^*)\} \dot{-} f(y)\right\} \quad \text{since } \varphi = \psi^* \\ &= \sup_{y\in X}\sup_{\xi^*\in X^*}\{\langle \xi^*, x-y\rangle - \psi(\xi^*) \dot{-} f(y)\} \\ &= \sup_{\xi^*\in X^*}\sup_{y\in X}\{\langle \xi^*, x-y\rangle - \psi(\xi^*) \dot{-} f(y)\} \\ &= \sup_{\xi^*\in X^*}\left\{\sup_{y\in X}\{\langle \xi^*, -y\rangle - f(y)\} \dot{-} \psi(\xi^*) + \langle \xi^*, x\rangle\right\} \\ &= \sup_{\xi^*\in X^*}\{f^*(-\xi^*) \dot{-} \psi(\xi^*) + \langle \xi^*, x\rangle\} \\ &= \Big(\psi \dot{-} (f^*)_-\Big)^*(x) \\ &= \Big(\psi \dot{-} (f_-)^*\Big)^*(x) \quad \text{in view of Lemma 7.1.} \end{aligned}$$

For the first equivalence, recall that $g \in \mathcal{E}^\varphi$ if and only if there exists $f : X \to \overline{\mathbb{R}}$ such that $g = f^\varphi$. Then use the equality $f^\varphi = \Big(\psi \dot{-} (f_-)^*\Big)^*$ and the fact that the range of the Legendre-Fenchel transform is equal to $\Gamma(X^*)$, see, e.g., [20].
For the second equivalence, observe that

$$\begin{aligned} g \in \mathcal{E}^\varphi \quad &\iff \quad g = (g^{\varphi_-})^\varphi \\ &\iff \quad g = \Big[\big(\psi_- \dot{-} (g_-)^*\big)^*\Big]^\varphi \quad \text{from formula (43)} \\ &\iff \quad g = \Big[\big((\psi \dot{-} g^*)^*\big)_-\Big]^\varphi \quad \text{by Lemma 7.1} \\ &\iff \quad g = \Big(\psi \dot{-} (\psi \dot{-} g^*)^{**}\Big)^* \quad \text{from formula (43) again.} \end{aligned}$$

□

Remark 7.1. Since $\varphi \in \Gamma(X)$ by assumption, we have $\varphi^{**} = \varphi$, hence we can take $\psi = \varphi^*$ in the statement of Theorem 7.1.

Remark 7.2. Formula (43) can be recovered partially by using a formula on the conjugate of the difference of functions. Recall that for $\psi : X \to \mathbb{R} \cup \{+\infty\}$ and

$h \in \Gamma_0(X)$,

$$\forall x^* \in X^*, \quad (\psi \dot{-} h)^*(x^*) = \sup_{y^* \in \operatorname{dom} h^*} \{\psi^*(x^* + y^*) - h^*(y^*)\} = (\psi^* \ominus h^*)(x^*). \tag{44}$$

This formula is due to Hiriart-Urruty [11]. It was established first by Pshenichnyi [26], assuming that both ψ and h are finite-valued convex functions. Now let $\varphi \in \Gamma_0(X)$ and $f \in \Gamma_0(X)$. By reversing the roles of X and X^* and by using equality (44) with $h = (f_-)^*$ and $\psi : X^* \to \mathbb{R} \cup \{+\infty\}$ such that $\psi^* = \varphi$, we find

$$\begin{aligned}(\psi \dot{-} (f_-)^*)^* &= \varphi \ominus (f_-)^{**} \\ &= \varphi \ominus f_- = f^\varphi.\end{aligned}$$

Hence we recover formula (43) in the case where both functions φ and f are in $\Gamma_0(X)$.

The next corollary says in particular that the φ-envelope of a function coincides with the φ-envelope of its lower semicontinuous convex hull whenever $\varphi \in \Gamma(X)$.

Corollary 7.1. *Let X be a locally convex space and $\varphi \in \Gamma(X)$. Then we have for every function $f : X \to \overline{\mathbb{R}}$ and every function $g : X \to \overline{\mathbb{R}}$ satisfying $\overline{\mathrm{co}} f \leq g \leq f$,*

$$f^\varphi = (\overline{\mathrm{co}} f)^\varphi = g^\varphi.$$

Proof. For the first equality, it suffices to use Theorem 7.1 and the fact that the functions f and $\overline{\mathrm{co}} f$ have the same Legendre-Fenchel conjugate. On the other hand, since $\overline{\mathrm{co}} f \leq g \leq f$, we see that $f^\varphi \leq g^\varphi \leq (\overline{\mathrm{co}} f)^\varphi$. Recalling that $f^\varphi = (\overline{\mathrm{co}} f)^\varphi$, the second equality immediately follows. □

For every set $D \subset X^*$, we define as in section 3 the classes Σ_D and Σ_D^* by

$$\Sigma_D = \{f : X^* \to \overline{\mathbb{R}}, \operatorname{dom} f \subset D\} \quad \text{and} \quad \Sigma_D^* = \{f^*, f \in \Sigma_D\}.$$

In the same vein, let us define the classes $\widehat{\Sigma}_D$ and $\widehat{\Sigma}_D^*$ by

$$\widehat{\Sigma}_D = \{f : X^* \to \overline{\mathbb{R}}, \operatorname{dom} f = D\} \quad \text{and} \quad \widehat{\Sigma}_D^* = \{f^*, f \in \widehat{\Sigma}_D\}.$$

The following proposition allows us to characterize the classes $\widehat{\Sigma}_D^*$ and Σ_D^*.

Proposition 7.1. *Let X be a locally convex space and let $D \subset X^*$ be such that $D = \{a_i^*, i \in I\}$ for some set I. Then for every function $f : X \to \overline{\mathbb{R}}$, we have $f \in \widehat{\Sigma}_D^*$ (resp. Σ_D^*) if and only if there exists a family $(\alpha_i)_{i \in I} \subset \mathbb{R} \cup \{+\infty\}$ (resp. $\overline{\mathbb{R}}$) such that $f = \sup_{i \in I} \langle a_i^*, \cdot \rangle + \ \alpha_i$.*

Proof. Assume that $f \in \widehat{\Sigma}_D^*$ (resp. Σ_D^*). By definition, there exists $g : X^* \to \overline{\mathbb{R}}$ such that $f = g^*$ and $\operatorname{dom} g = D$ (resp. $\operatorname{dom} g \subset D$). Hence we have

$$f = \sup_{x^* \in D} \langle x^*, \cdot \rangle - g(x^*) = \sup_{i \in I} \langle a_i^*, \cdot \rangle - g(a_i^*).$$

By setting $\alpha_i = -g(a_i^*)$ for every $i \in I$, we obtain $f = \sup_{i \in I} \langle a_i^*, \cdot \rangle + \alpha_i$ with $\alpha_i \in \mathbb{R} \cup \{+\infty\}$ (resp. $\overline{\mathbb{R}}$).

Conversely, assume that there exists $(\alpha_i)_{i\in I} \subset \mathbb{R}\cup\{+\infty\}$ (resp. $\overline{\mathbb{R}}$) such that $f = \sup_{i\in I}\langle a_i^*, \cdot\rangle + \ \alpha_i$. Then we have

$$\begin{aligned} f &= \sup_{x^*\in D}\Big[\sup_{\{i\in I,\, a_i^*=x^*\}} \langle a_i^*, \cdot\rangle + \alpha_i\Big] \\ &= \sup_{x^*\in D}\Big[\langle x^*, \cdot\rangle + \sup_{\{i\in I,\, a_i^*=x^*\}} \alpha_i\Big]. \end{aligned}$$

Defining the function $h : X^* \to \overline{\mathbb{R}}$ by

$$h(x^*) = \begin{cases} \sup\limits_{\{i\in I,\, a_i^*=x^*\}} \alpha_i & \text{if} \quad x^*\in D \\ -\infty & \text{if} \quad x^* \notin D, \end{cases}$$

we obtain

$$\begin{aligned} f &= \sup_{x^*\in D}\langle x^*, \cdot\rangle + h(x^*) \\ &= \sup_{x^*\in X^*}\langle x^*, \cdot\rangle + h(x^*). \end{aligned}$$

We conclude that $f = (-h)^*$ with $\operatorname{dom}(-h) = D$ (resp. $\operatorname{dom}(-h) \subset D$), hence $f \in \widehat{\Sigma}_D^*$ (resp. $f \in \Sigma_D^*$). $\square$

Example 7.1. Take $D = \{a_1^*, \cdots, a_n^*\} \subset X^*$ for some $n \geq 1$. The previous proposition shows that, for every function $f : X \to \overline{\mathbb{R}}$,

$$f \in \Sigma_D^* \quad \Longleftrightarrow \quad f = \sup_{i=1}^{n}\langle a_i^*, \cdot\rangle + \alpha_i \quad \text{for some } \alpha_1, \cdots, \alpha_n \in \overline{\mathbb{R}}. \tag{45}$$

On the other hand, if $f \in \Gamma_0(X)$, the following equivalence holds true

$$\operatorname{dom} f^* \subset D \quad \Longleftrightarrow \quad \operatorname{dom} f^* \subset \{a_i^*\} \quad \text{for some } i \in \{1, \cdots, n\}$$

because the set $\operatorname{dom} f^*$ is convex. Since f^* is proper, this is in turn equivalent to $f^* = \delta_{\{a_i^*\}} - \alpha_i$ for some $\alpha_i \in \mathbb{R}$. Taking the conjugate, we find $f = \langle a_i^*, \cdot\rangle + \alpha_i$. It ensues that the set $\{f \in \Gamma_0(X), \operatorname{dom} f^* \subset D\}$ coincides with the set of affine continuous functions with slopes in $D = \{a_1^*, \cdots, a_n^*\}$. This yields an example for which the inclusion (14) is strict. By applying again Proposition 7.1, we obtain that

$$f \in \Sigma_{\operatorname{co}(D)}^* \quad \Longleftrightarrow \quad f = \sup_{x^*\in \operatorname{co}(D)}\langle x^*, \cdot\rangle + \alpha_{x^*}, \tag{46}$$

with $\alpha_{x^*} \in \overline{\mathbb{R}}$ for every $x^* \in \operatorname{co}(D)$. The comparison of (45) and (46) clearly shows that the inclusion $\Sigma_D^* \subset \Sigma_{\operatorname{co}(D)}^*$ is strict as soon as the set $D = \{a_1^*, \cdots, a_n^*\}$ is not a singleton. This easily implies that the inclusion (15) is strict for such a set D.

The next result gives several upper bounds (in the sense of inclusion) for the set $\mathcal{E}^\varphi$, respectively when $\varphi \in \Gamma(X)$, $\varphi \in \widehat{\Sigma}_D^*$ and $\varphi \in \Sigma_D^*$.

Corollary 7.2. *Let X be a locally convex space and let $\varphi \in \Gamma(X)$.*

(*i*) *The following inclusions hold true*

$$\mathcal{E}^\varphi \subset \bigcap_{\{\psi,\, \varphi=\psi^*\}} \left(\widehat{\Sigma}_{\operatorname{dom}\psi}^* \cup \{-\omega_X\}\right) \subset \bigcap_{\{\psi,\, \varphi=\psi^*\}} \Sigma_{\operatorname{dom}\psi}^*. \tag{47}$$

(*ii*) *For every subset $D \subset X^*$, we have*

$$\varphi \in \widehat{\Sigma}^*_D \iff \mathcal{E}^\varphi \subset \widehat{\Sigma}^*_D \cup \{-\omega_X\} \qquad \text{if } \varphi \neq -\omega_X$$
$$\varphi \in \Sigma^*_D \iff \mathcal{E}^\varphi \subset \Sigma^*_D.$$

(*iii*) *Assume that there exist families $(a_i^*)_{i\in I} \subset X^*$ and $(\alpha_i)_{i\in I} \subset \mathbb{R}\cup\{+\infty\}$ (resp. $\overline{\mathbb{R}}$) such that*

$$\varphi = \sup_{i\in I} \langle a_i^*, \cdot\rangle + \alpha_i.$$

Then for every $g \in \mathcal{E}^\varphi \setminus \{-\omega_X\}$ (resp. $g \in \mathcal{E}^\varphi$), there exists $(\beta_i)_{i\in I} \subset \mathbb{R}\cup\{+\infty\}$ (resp. $\overline{\mathbb{R}}$) such that

$$g = \sup_{i\in I} \langle a_i^*, \cdot\rangle + \beta_i.$$

In particular, if the set I is finite, every φ-envelope is polyhedral.

Proof. (*i*) Let $\psi : X^* \to \overline{\mathbb{R}}$ be such that $\varphi = \psi^*$. Assuming that $g \in \mathcal{E}^\varphi$, Theorem 7.1 shows that $g = (\psi \dot{-} h)^*$ for some $h \in \Gamma(X^*)$. If $h = -\omega_{X^*}$, we have $\psi \dot{-} h = \omega_{X^*}$ and therefore $g = -\omega_X$. If $h \neq -\omega_{X^*}$, we see that $\operatorname{dom}(\psi \dot{-} h) = \operatorname{dom}\psi$, hence $g \in \widehat{\Sigma}^*_{\operatorname{dom}\psi}$. We deduce the inclusion $\mathcal{E}^\varphi \subset \widehat{\Sigma}^*_{\operatorname{dom}\psi} \cup \{-\omega_X\}$. Since this is true for every function $\psi : X^* \to \overline{\mathbb{R}}$ such that $\varphi = \psi^*$, the first inclusion of (47) follows. For the second inclusion, it suffices to notice that $\widehat{\Sigma}^*_{\operatorname{dom}\psi} \cup \{-\omega_X\} \subset \Sigma^*_{\operatorname{dom}\psi}$.
(*ii*) Let us fix $D \subset X^*$ and assume that $\varphi \in \widehat{\Sigma}^*_D$. Then there exists $\psi : X^* \to \overline{\mathbb{R}}$ such that $\varphi = \psi^*$ and $\operatorname{dom}\psi = D$. We deduce from the first inclusion of (47) that

$$\mathcal{E}^\varphi \subset \widehat{\Sigma}^*_{\operatorname{dom}\psi} \cup \{-\omega_X\} = \widehat{\Sigma}^*_D \cup \{-\omega_X\}.$$

Conversely, if $\mathcal{E}^\varphi \subset \widehat{\Sigma}^*_D \cup \{-\omega_X\}$ and if $\varphi \neq -\omega_X$, then we obtain $\varphi \in \widehat{\Sigma}^*_D$ according to the inclusion $\varphi \in \mathcal{E}^\varphi$. The proof of the second equivalence is analogous and left to the reader.
(*iii*) Let $(a_i^*)_{i\in I} \subset X^*$ and $(\alpha_i)_{i\in I} \subset \mathbb{R}\cup\{+\infty\}$ (resp. $\overline{\mathbb{R}}$) be such that $\varphi = \sup_{i\in I}\langle a_i^*, \cdot\rangle + \alpha_i$. Let us set $D = \{a_i^*,\, i \in I\}$. Proposition 7.1 shows that $\varphi \in \widehat{\Sigma}^*_D$ (resp. Σ^*_D). If $g \in \mathcal{E}^\varphi \setminus \{-\omega_X\}$ (resp. $g \in \mathcal{E}^\varphi$), we deduce from (*ii*) that $g \in \widehat{\Sigma}^*_D$ (resp. Σ^*_D). By applying Proposition 7.1 again, we derive the existence of $(\beta_i)_{i\in I} \subset \mathbb{R}\cup\{+\infty\}$ (resp. $\overline{\mathbb{R}}$) such that $g = \sup_{i\in I}\langle a_i^*, \cdot\rangle + \beta_i$. Finally, if the set I is finite and if g is a φ-envelope, then either $g = \pm\omega_X$ or the function g is the supremum of a finite collection of continuous affine functions. We then conclude that g is polyhedral. □

By applying the second equivalence of Corollary 7.2 (*ii*) with $D = X^*$, we obtain that $\varphi \in \Gamma(X)$ if and only if $\mathcal{E}^\varphi \subset \Gamma(X)$. Corollary 7.3 below shows that in this case the set $\mathcal{E}^\varphi$ is strictly included in $\Gamma(X)$. Notice that for $\varphi \in \Gamma_0(X)$ satisfying a suitable condition (named generating condition), the functions of the class $\mathcal{E}^\varphi$ have been studied in [24] under the terminology of φ-strongly convex functions.

Following Theorem 7.1 and Remark 7.1, we have $g \in \mathcal{E}^\varphi$ if and only if $g = (\varphi^* \dot{-} h)^*$ for some $h \in \Gamma(X^*)$. Let us now have a look at the class of the functions equal to $(\varphi^* \dot{-} h)^*$ for some $h : X^* \to \mathbb{R}\cup\{+\infty\}$ not necessarily in $\Gamma(X^*)$.

Proposition 7.2. *Let X be a locally convex space. Assume that $\varphi \in \Gamma_0(X)$ and $g \in \Gamma_0(X)$.*

(*i*) *If* $g = (\varphi^* \dot{-} h)^*$ *for some* $h : X^* \to \mathbb{R} \cup \{+\infty\}$ *, then we have* $g^\infty = \varphi^\infty$*, which is equivalent to* $\mathrm{cl}^{w*}(\mathrm{dom}\, g^*) = \mathrm{cl}^{w*}(\mathrm{dom}\, \varphi^*)$*.*

(*ii*) *If* $\mathrm{dom}\, g^* = \mathrm{dom}\, \varphi^*$*, then* $g = (\varphi^* \dot{-} h)^*$ *for* $h : X^* \to \mathbb{R} \cup \{+\infty\}$ *given by* $h = \varphi^* \dot{-} g^*$*.*

Proof. (*i*) Assume that $g = (\varphi^* \dot{-} h)^*$ for some $h : X^* \to \mathbb{R} \cup \{+\infty\}$. By definition of the Legendre-Fenchel transform, we obtain

$$\begin{aligned} g &= \sup_{\xi^* \in X^*} \{\langle \xi^*, \cdot \rangle + h(\xi^*) \dot{-} \varphi^*(\xi^*)\} \\ &= \sup_{\xi^* \in \mathrm{dom}\, \varphi^*} \{\langle \xi^*, \cdot \rangle + h(\xi^*) - \varphi^*(\xi^*)\}. \end{aligned} \tag{48}$$

Observe that the function h cannot take the value $+\infty$ on $\mathrm{dom}\, \varphi^*$ (otherwise we would have $g = \omega_X$). Therefore the values $-\varphi^*(\xi^*)$ and $h(\xi^*)$ are finite for every $\xi^* \in \mathrm{dom}\, \varphi^*$. By taking the recession function of each member of (48), we obtain

$$g^\infty = \sup_{\xi^* \in \mathrm{dom}\, \varphi^*} [\langle \xi^*, \cdot \rangle + h(\xi^*) - \varphi^*(\xi^*)]^\infty.$$

The recession function of the affine map $\langle \xi^*, \cdot \rangle + h(\xi^*) - \varphi^*(\xi^*)$ is equal to $\langle \xi^*, \cdot \rangle$, thus implying that $g^\infty = \sup_{\xi^* \in \mathrm{dom}\, \varphi^*} \langle \xi^*, \cdot \rangle = \sigma_{\mathrm{dom}\, \varphi^*}$. Recalling that $\sigma_{\mathrm{dom}\, \varphi^*} = \varphi^\infty$, we deduce that $g^\infty = \varphi^\infty$, which is in turn equivalent to the equality $\mathrm{cl}^{w*}(\mathrm{dom}\, g^*) = \mathrm{cl}^{w*}(\mathrm{dom}\, \varphi^*)$, see [20].

(*ii*) Assume that $\mathrm{dom}\, g^* = \mathrm{dom}\, \varphi^*$. It is easy to check that for every $x^* \in X^*$,

$$\big(\varphi^* \dot{-} (\varphi^* \dot{-} g^*)\big)(x^*) = \begin{cases} g^*(x^*) & \text{if} \quad x^* \in \mathrm{dom}\, g^* \\ +\infty & \text{if} \quad x^* \notin \mathrm{dom}\, g^*. \end{cases}$$

It ensues that $\varphi^* \dot{-} (\varphi^* \dot{-} g^*) = g^*$. Since $g \in \Gamma_0(X)$ by assumption, we have $g = g^{**}$, hence $g = (\varphi^* \dot{-} (\varphi^* \dot{-} g^*))^*$. The function $h = \varphi^* \dot{-} g^*$ takes its values in $\mathbb{R} \cup \{+\infty\}$ because $\mathrm{dom}\, g^* = \mathrm{dom}\, \varphi^*$. □

Combining Theorem 7.1 and Proposition 7.2, we derive a necessary (resp. sufficient) condition for a function $g \in \Gamma_0(X)$ to be a φ-envelope.

Corollary 7.3. *Let* X *be a locally convex space. Assume that* $\varphi \in \Gamma_0(X)$ *and* $g \in \Gamma_0(X)$*.*

(*i*) *If* $g \in \mathcal{E}^\varphi$ *then* $g^\infty = \varphi^\infty$*.*

(*ii*) *If* $\mathrm{dom}\, g^* = \mathrm{dom}\, \varphi^*$ *and* $\varphi^* \dot{-} g^* \in \Gamma_0(X^*)$*, then* $g \in \mathcal{E}^\varphi$*.*

Proof. (*i*) If $g \in \mathcal{E}^\varphi$, we deduce from Theorem 7.1 that $g = (\varphi^* \dot{-} h)^*$ for some $h \in \Gamma(X^*)$. Since $g \in \Gamma_0(X)$ by assumption, we have $h \neq -\omega_{X^*}$, hence the function h does not take the value $-\infty$. Proposition 7.2 (*i*) then implies that $g^\infty = \varphi^\infty$.

(*ii*) If $\mathrm{dom}\, g^* = \mathrm{dom}\, \varphi^*$, Proposition 7.2 (*ii*) shows that $g = (\varphi^* \dot{-} h)^*$ with $h = \varphi^* \dot{-} g^*$. Since $h \in \Gamma_0(X^*)$ by assumption, we conclude by Theorem 7.1 that $g \in \mathcal{E}^\varphi$. □

7.2. **Klee envelopes.** Let $(X, \|\cdot\|)$ be a normed space and let $f : X \to \overline{\mathbb{R}}$ be an extended real-valued function. For any reals $\lambda > 0$ and $p \geq 1$, we define the Klee envelope of f with index λ and power p as

$$\kappa_{\lambda,p} f(x) = \sup_{y \in X} \left(\frac{1}{p\lambda} \|x - y\|^p - f(y) \right).$$

In other words, we have $\kappa_{\lambda,p} f = f^\varphi$ with the function $\varphi : X \to \mathbb{R}$ defined by $\varphi(x) = \frac{1}{p\lambda}\|x\|^p$. Applying Theorem 7.1 with $\varphi = \frac{1}{p\lambda}\|\cdot\|^p$ and denoting by $\|\cdot\|_{X^*}$ the dual norm on X^* we obtain the following result.

Corollary 7.4. *Let $(X, \|\cdot\|)$ be a normed space. For any $\lambda > 0$, $p > 1$ and for every function $f : X \to \overline{\mathbb{R}}$, we have*

$$\kappa_{\lambda,p} f = \left(\frac{\lambda^{q-1}}{q} \|\cdot\|^q_{X^*} - (f_-)^* \right)^*, \tag{49}$$

where $q > 1$ is the conjugate exponent of p. Moreover the following assertions are equivalent

(*i*) *g is a Klee envelope with index λ and power p;*
(*ii*) $g = \left(\frac{\lambda^{q-1}}{q} \|\cdot\|^q_{X^*} - h \right)^*$ *for some $h \in \Gamma(X^*)$;*
(*iii*) $g = \left(\frac{\lambda^{q-1}}{q} \|\cdot\|^q_{X^*} - \left(\frac{\lambda^{q-1}}{q} \|\cdot\|^q_{X^*} - g^* \right)^{**} \right)^*$.

These assertions are satisfied whenever the following stronger condition is fulfilled

(*iv*) $g \in \Gamma(X)$ *and* $\frac{\lambda^{q-1}}{q} \|\cdot\|^q_{X^*} - g^* \in \Gamma(X^*)$.

Proof. It suffices to apply Theorem 7.1 with $\varphi = \frac{1}{p\lambda}\|\cdot\|^p$ and $\psi = \varphi^* = \frac{\lambda^{q-1}}{q}\|\cdot\|^q_{X^*}$. Let us now establish the implication $(iv) \implies (ii)$. Assume that $g \in \Gamma(X)$ and that $\frac{\lambda^{q-1}}{q}\|\cdot\|^q_{X^*} - g^* \in \Gamma(X^*)$. The function g^* can be written as $g^* = \frac{\lambda^{q-1}}{q}\|\cdot\|^q_{X^*} - h$ for some $h \in \Gamma(X^*)$. Since $g \in \Gamma(X)$ by assumption, we have $g = g^{**}$. Hence we deduce that $g = \left(\frac{\lambda^{q-1}}{q}\|\cdot\|^q_{X^*} - h \right)^*$ and assertion (ii) is proved. $\square$

Corollary 7.5. *Let $(X, \|\cdot\|)$ be a normed space. For every $p > 1$ and every $C \subset X$, the farthest distance function $\Delta_C = \sup_{y \in C} \|\cdot - y\|$ satisfies*

$$\frac{1}{p}\Delta_C^p = \left(\frac{1}{q} \|\cdot\|^q_{X^*} - \sigma_{-C} \right)^*.$$

Proof. Observe that

$$\kappa_{1,p}\, \delta_C = \sup_{y \in X} \left\{ \frac{1}{p}\|\cdot - y\|^p - \delta_C(y) \right\} = \sup_{y \in C} \frac{1}{p}\|\cdot - y\|^p = \frac{1}{p}\Delta_C^p.$$

It suffices then to apply formula (49) of Corollary 7.4 with $f = \delta_C$ and $\lambda = 1$. $\square$

Additional properties of the Klee envelopes can be obtained in the case when $(X, \|\cdot\|)$ is a Hilbert space and $p = 2$.

Theorem 7.2. *Assume that X is a Hilbert space endowed with the scalar product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\|\cdot\|$.*

(*a*) *For every $\lambda > 0$ and every function $f:\ X \to \overline{\mathbb{R}}$, we have*

$$\begin{aligned} \kappa_{\lambda,2} f &= \left(\frac{\lambda}{2}\|\cdot\|^2 - (f_-)^*\right)^* && (50)\\ &= \left(f - \frac{1}{2\lambda}\|\cdot\|^2\right)^* \left(-\frac{\cdot}{\lambda}\right) + \frac{1}{2\lambda}\|\cdot\|^2; && (51) \end{aligned}$$

$$\kappa_{\lambda,2}\left(\kappa_{\lambda,2} f\right) = \left(f - \frac{1}{2\lambda}\|\cdot\|^2\right)^{**} + \frac{1}{2\lambda}\|\cdot\|^2. \qquad (52)$$

(*b*) *For $\lambda > 0$ and $f : X \to \overline{\mathbb{R}}$ the following assertions are equivalent*

(*i*) *f is a Klee envelope with index λ and power 2;*

(*ii*) *$f = \left(\frac{\lambda}{2}\|\cdot\|^2 - h\right)^*$ for some $h \in \Gamma(X)$;*

(*iii*) *$f = \left(\frac{\lambda}{2}\|\cdot\|^2 - \left(\frac{\lambda}{2}\|\cdot\|^2 - f^*\right)^{**}\right)^*$;*

(*iv*) *$f - \frac{1}{2\lambda}\|\cdot\|^2 \in \Gamma(X)$;*

(*v*) *$f \in \Gamma(X)$ and $\frac{\lambda}{2}\|\cdot\|^2 - f^* \in \Gamma(X)$.*

Proof. (*a*) For the equality (50), it suffices to apply Corollary 7.4 with $p = 2$. For the equality (51), observe that for every $x \in X$,

$$\begin{aligned} \kappa_{\lambda,2} f(x) &= \sup_{y \in X}\left\{\frac{1}{2\lambda}\|x-y\|^2 - f(y)\right\}\\ &= \sup_{y \in X}\left\{\frac{1}{2\lambda}\|x\|^2 + \frac{1}{2\lambda}\|y\|^2 - \frac{1}{\lambda}\langle x, y\rangle - f(y)\right\}\\ &= \left(f - \frac{1}{2\lambda}\|\cdot\|^2\right)^*(-x/\lambda) + \frac{1}{2\lambda}\|x\|^2. \end{aligned}$$

By iterating we deduce that

$$\begin{aligned} \kappa_{\lambda,2}\left(\kappa_{\lambda,2} f\right) &= \left(\kappa_{\lambda,2} f - \frac{1}{2\lambda}\|\cdot\|^2\right)^* \left(-\frac{\cdot}{\lambda}\right) + \frac{1}{2\lambda}\|\cdot\|^2\\ &= \left[\left(f - \frac{1}{2\lambda}\|\cdot\|^2\right)^* \left(-\frac{\cdot}{\lambda}\right)\right]^* \left(-\frac{\cdot}{\lambda}\right) + \frac{1}{2\lambda}\|\cdot\|^2\\ &= \left(f - \frac{1}{2\lambda}\|\cdot\|^2\right)^{**} + \frac{1}{2\lambda}\|\cdot\|^2, \end{aligned}$$

which proves the equality (52).

(*b*) We now show that assertions (*i*) to (*v*) are equivalent. The equivalences (*i*) $\Longleftrightarrow$ (*ii*) $\Longleftrightarrow$ (*iii*) are consequences of Corollary 7.4 applied with $p = 2$. Let us show the equivalence (*i*) $\Longleftrightarrow$ (*iv*). Observe that f is a Klee envelope with index λ and power 2 if and only if $f \in \mathcal{E}^{\varphi}$ with $\varphi = \frac{1}{2\lambda}\|\cdot\|^2$. From the equivalence (7) $\Leftrightarrow$ (8) and the fact that $\varphi_- = \varphi$, this is in turn equivalent to $f = (f^{\varphi})^{\varphi}$. Since $(f^{\varphi})^{\varphi} = \kappa_{\lambda,2}\left(\kappa_{\lambda,2} f\right)$ and using the equality (52), we infer that

$$f \text{ is a Klee envelope with index } \lambda \text{ and power } 2$$

$$\Updownarrow$$

$$f - \frac{1}{2\lambda}\|\cdot\|^2 = \left(f - \frac{1}{2\lambda}\|\cdot\|^2\right)^{**}$$

$$\Updownarrow$$

$$f - \frac{1}{2\lambda}\|\cdot\|^2 \in \Gamma(X).$$

Hence the equivalence $(i) \Longleftrightarrow (iv)$ is proved. Let us now show that $(iv) \Longrightarrow (v)$. If $f - \frac{1}{2\lambda}\|\cdot\|^2 = \pm\omega_X$, then assertion (v) is trivially satisfied. Hence we can assume that $f = \frac{1}{2\lambda}\|\cdot\|^2 + h$ with $h \in \Gamma_0(X)$. This clearly implies that $f \in \Gamma_0(X)$. Taking the conjugate, we obtain that $f^* = \frac{\lambda}{2}\|\cdot\|^2 \bigtriangledown h^*$ since the classical qualification condition is satisfied. It ensues that for every $x \in X$,

$$\begin{aligned} f^*(x) &= \inf_{y\in X}\left\{\frac{\lambda}{2}\|x-y\|^2 + h^*(y)\right\} \\ &= \frac{\lambda}{2}\|x\|^2 + \inf_{y\in X}\left\{-\lambda\langle x, y\rangle + \frac{\lambda}{2}\|y\|^2 + h^*(y)\right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\lambda}{2}\|x\|^2 - f^*(x) &= \sup_{y\in X}\left\{\lambda\langle x, y\rangle - \frac{\lambda}{2}\|y\|^2 - h^*(y)\right\} \\ &= \left(h^* + \frac{\lambda}{2}\|\cdot\|^2\right)^*(\lambda x). \end{aligned}$$

This clearly implies that $\frac{\lambda}{2}\|\cdot\|^2 - f^* \in \Gamma_0(X)$ and (v) is proved. Let us finally observe that the implication $(v) \Longrightarrow (ii)$ has been established in Corollary 7.4. As a conclusion, we have shown the equivalences $(i) \Longleftrightarrow (ii) \Longleftrightarrow (iii) \Longleftrightarrow (iv)$ along with the implications $(iv) \Longrightarrow (v) \Longrightarrow (ii)$, which clearly establishes that all assertions (i) to (v) are equivalent. □

The equalities (51) and (52) have been previously established by Wang [37] respectively in Proposition 4.5 and at the end of the proof of Proposition 4.13. As noticed in [37, Proposition 4.13] those equalities directly yield, for f proper and lower semicontinuous, that $\kappa_{\lambda,2}(\kappa_{\lambda,2} f) = f$ if and only if $f - \frac{1}{2\lambda}\|\cdot\|^2$ is convex.

Taking f as the indicator function of a set C gives the following corollary.

Corollary 7.6. *Assume that X is a Hilbert space. For every $C \subset X$, the farthest distance function Δ_C satisfies*

$$\frac{1}{2}\Delta_C^2 = \left(\frac{1}{2}\|\cdot\|^2 - \sigma_{-C}\right)^* = \left(\delta_{-C} - \frac{1}{2}\|\cdot\|^2\right)^* + \frac{1}{2}\|\cdot\|^2.$$

Proof. It suffices to apply formulas (50)-(51) of Theorem 7.2 with $f = \delta_C$ and $\lambda = 1$. □

7.3. **Case of a positively homogeneous function φ.** In this subsection, we assume that X is a locally convex space and that the function $\varphi \in \Gamma_0(X)$ is positively homogeneous, *i.e.* $\varphi = \sigma_D$ for a nonempty set $D \subset X^*$. By applying Theorem 7.1 with $\psi = \delta_D$, we immediately obtain the following result.

Corollary 7.7. *Let X be a locally convex space. Take $\varphi = \sigma_D$ for a nonempty set $D \subset X^*$. Then we have for every function $f : X \to \overline{\mathbb{R}}$,*

$$f^\varphi = (\delta_D \dot{-} (f_-)^*)^* = \sup_{\xi^*\in D}\{\langle \xi^*, \cdot\rangle + f^*(-\xi^*)\}.$$

Moreover,

$$g \in \mathcal{E}^{\varphi} \iff g = (\delta_D \dot{-} h)^* = \sup_{\xi^* \in D} \{\langle \xi^*, \cdot \rangle + h(\xi^*)\} \quad \textit{for some } h \in \Gamma(X^*)$$
$$\iff g = (\delta_D \dot{-} (\delta_D \dot{-} g^*)^{**})^*.$$

Let us now particularize to the case of a normed space $(X, \|\cdot\|)$ and take $\varphi = \|\cdot\|$.

Corollary 7.8. *Let $(X, \|.\|)$ be a normed space. For every function $f : X \to \overline{\mathbb{R}}$, we have*

$$\begin{aligned} \kappa_{1,1}\, f &= (\delta_{\mathbb{B}_{X^*}} \dot{-} (f_-)^*)^* = \sup_{\xi^* \in \mathbb{B}_{X^*}} \{\langle \xi^*, \cdot \rangle + f^*(-\xi^*)\} \\ &= (\delta_{\mathbb{S}_{X^*}} \dot{-} (f_-)^*)^* = \sup_{\xi^* \in \mathbb{S}_{X^*}} \{\langle \xi^*, \cdot \rangle + f^*(-\xi^*)\}. \end{aligned}$$

Moreover,

$$g \textit{ is a Klee envelope with index } 1 \textit{ and power } 1$$
$$\Updownarrow$$
$$g = (\delta_{\mathbb{B}_{X^*}} \dot{-} h)^* = \sup_{\xi^* \in \mathbb{B}_{X^*}} \{\langle \xi^*, \cdot \rangle + h(\xi^*)\} \quad \textit{for some } h \in \Gamma(X^*)$$
$$\Updownarrow$$
$$g = (\delta_{\mathbb{B}_{X^*}} \dot{-} (\delta_{\mathbb{B}_{X^*}} \dot{-} g^*)^{**})^*$$
$$\Updownarrow$$
$$g = (\delta_{\mathbb{S}_{X^*}} \dot{-} h)^* = \sup_{\xi^* \in \mathbb{S}_{X^*}} \{\langle \xi^*, \cdot \rangle + h(\xi^*)\} \quad \textit{for some } h \in \Gamma(X^*)$$
$$\Updownarrow$$
$$g = (\delta_{\mathbb{S}_{X^*}} \dot{-} (\delta_{\mathbb{S}_{X^*}} \dot{-} g^*)^{**})^*.$$

Proof. For the equalities $\kappa_{1,1}\, f = \left(\delta_{\mathbb{B}_{X^*}} \dot{-} (f_-)^*\right)^*$ and $\kappa_{1,1}\, f = \left(\delta_{\mathbb{S}_{X^*}} \dot{-} (f_-)^*\right)^*$, use Corollary 7.7 respectively with $D = \mathbb{B}_{X^*}$ and $D = \mathbb{S}_{X^*}$. The characterizations of Klee envelopes with index 1 and power 1 follow immediately. $\square$

Assuming that $f = \delta_C$, we have

$$\kappa_{1,1}\, \delta_C = \sup_{x \in X} \{\| \cdot - x\| - \delta_C(x)\} = \sup_{x \in C} \| \cdot - x\| = \Delta_C,$$

where Δ_C is the farthest distance function. Taking into account the previous corollary, we then obtain

$$\Delta_C = \left(\delta_{\mathbb{B}_{X^*}} \dot{-} \sigma_{-C}\right)^* = \left(\delta_{\mathbb{S}_{X^*}} \dot{-} \sigma_{-C}\right)^*.$$

It is interesting to compare this expression with the one of the signed distance sgd defined by $\mathrm{sgd}(\cdot, C) := d(\cdot, C) - d(\cdot, X \setminus C)$, for which it is known that $\mathrm{sgd}(\cdot, C) = (\delta_{\mathbb{S}_{X^*}} + \sigma_C)^*$, see [22].

Consider now the case of a finite set $D = \{a_1^*, \ldots, a_n^*\} \subset X^*$ for $n \geq 1$. By applying Corollary 7.7, we obtain the following result.

Corollary 7.9. *Let X be a locally convex space. Take $\varphi = \sigma_{\{a_1^*,\dots,a_n^*\}}$ with $a_1^*,\dots,a_n^* \in X^*$ and $n \geq 1$. Then we have for every function $f : X \to \overline{\mathbb{R}}$*

$$f^\varphi = \sup_{i=1}^{n} \langle a_i^*, \cdot\rangle + f^*(-a_i^*).$$

Moreover,

$$g \in \mathcal{E}^\varphi \quad \Longleftrightarrow \quad g = \sup_{i=1}^{n} \langle a_i^*, \cdot\rangle + h(a_i^*) \quad \text{for some } h \in \Gamma(X^*).$$

8. Case $\varphi \in -\Gamma(X)$

8.1. Links between φ-envelopes and Legendre-Fenchel conjugates.

Proposition 8.1. *Let X be a locally convex space and let φ, $g : X \to \overline{\mathbb{R}}$ be extended real-valued functions.*

(i) If $g \in \mathcal{E}^\varphi$, then there exists $h \in \Gamma(X^*)$ such that $(-g)^* = (-\varphi)^* \dotplus h$. If in addition $g \in -\Gamma(X)$, then $-g = ((-\varphi)^* \dotplus h)^*$.

(ii) Assume that X is normed. If $\varphi \in -\Gamma(X)$ and if there exists $h \in \Gamma(X^*)$ satisfying the equality $-g = ((-\varphi)^* \dotplus h)^*$ along with the condition $0 \in \operatorname{int}(\operatorname{dom} h - \operatorname{dom}(-\varphi)^*)$, then $g \in \mathcal{E}^\varphi$.

Proof. (i) Since $g \in \mathcal{E}^\varphi$, there exists $f : X \to \overline{\mathbb{R}}$ such that $g = f^\varphi$, hence $-g = (-\varphi) \bigtriangledown f$ by (3). Taking the conjugate of each member, we find $(-g)^* = (-\varphi)^* \dotplus f^*$. Hence the expected equality holds with $h = f^* \in \Gamma(X^*)$. If in addition $g \in -\Gamma(X)$, we have $-g = (-g)^{**}$, hence we deduce from what precedes that $-g = ((-\varphi)^* \dotplus h)^*$.
(ii) Assume that $-g = ((-\varphi)^* \dotplus h)^*$ for some $h \in \Gamma(X^*)$. If $h = -\omega_{X^*}$ or if $(-\varphi)^* = -\omega_{X^*}$, then $-g = (-\omega_{X^*})^* = \omega_X$ and the inclusion $g \in \mathcal{E}^\varphi$ trivially holds. Now assume that $h \neq -\omega_{X^*}$ and $(-\varphi)^* \neq -\omega_{X^*}$. Since $0 \in \operatorname{int}(\operatorname{dom} h - \operatorname{dom}(-\varphi)^*)$, the functions $(-\varphi)^*$ and h are proper and according to the fact that X^* is a Banach space, we have

$$\begin{aligned} -g &= (-\varphi)^{**} \bigtriangledown h^* \\ &= (-\varphi) \bigtriangledown h^* \quad \text{because } \varphi \in -\Gamma(X). \end{aligned}$$

We conclude that $g = \varphi \bigtriangleup (-h^*) = (h^*)^\varphi \in \mathcal{E}^\varphi$. □

Corollary 8.1. *Let X be a normed space and let $\varphi \in -\Gamma_0(X)$ be such that $\operatorname{dom}(-\varphi)^* = X^*$. For every $g \in -\Gamma(X)$, the following equivalences hold true*

$$\begin{aligned} g \in \mathcal{E}^\varphi &\iff (-g)^* - (-\varphi)^* \in \Gamma(X^*) \\ &\iff -g = ((-\varphi)^* + h)^* \quad \text{for some } h \in \Gamma(X^*). \end{aligned}$$

Proof. Fix $g \in -\Gamma(X)$. Since $\operatorname{dom}(-\varphi)^* = X^*$ and $-\varphi \in \Gamma_0(X)$, the function $(-\varphi)^*$ is finite-valued on X^*, so the implication

$$g \in \mathcal{E}^\varphi \Longrightarrow h := (-g)^* - (-\varphi)^* \in \Gamma(X^*)$$

follows from Proposition 8.1 (i). Recalling that $g \in -\Gamma(X)$, the right-hand inclusion implies in turn that $-g = ((-\varphi)^* + h)^*$.
Now assume that $-g = ((-\varphi)^* + h)^*$ for some $h \in \Gamma(X^*)$. If $\operatorname{dom} h \neq \emptyset$, the qualification assumption $0 \in \operatorname{int}(\operatorname{dom} h - \operatorname{dom}(-\varphi)^*)$ is automatically satisfied. We then deduce from Proposition 8.1 (ii) that $g \in \mathcal{E}^\varphi$. On the other hand, if

$\operatorname{dom} h = \emptyset$, then we have $h = \omega_{X^*}$ and hence $-g = (\omega_{X^*})^* = -\omega_X$. Then the inclusion $g \in \mathcal{E}^\varphi$ trivially holds. $\square$

8.2. **Moreau envelopes.** Let $(X, \|\cdot\|)$ be a normed space and let $f : X \to \overline{\mathbb{R}}$ be an extended real-valued function. For $\lambda > 0$ and $p \geq 1$, we define the Moreau envelope of f with index λ and power p as

$$e_{\lambda,p} f = \inf_{y \in X} \left(\frac{1}{p\lambda} \| \cdot - y \|^p + f(y) \right) = \frac{1}{p\lambda} \| \cdot \|^p \triangledown f.$$

Observe that $-e_{\lambda,p} f = \left(-\frac{1}{p\lambda} \| \cdot \|^p \right) \vartriangle (-f) = f^\varphi$, with the function $\varphi : X \to \mathbb{R}$ defined by $\varphi = -\frac{1}{p\lambda} \| \cdot \|^p$. It ensues that g is a Moreau envelope with index λ and power p if and only if $-g \in \mathcal{E}^\varphi$, for $\varphi = -\frac{1}{p\lambda} \| \cdot \|^p$. By applying the results of the previous subsection with $\varphi = -\frac{1}{p\lambda} \| \cdot \|^p$, we obtain the following statement.

Corollary 8.2. *Assume that $(X, \|\cdot\|)$ is a normed space. Let $\lambda > 0$, $p > 1$ and let q be the conjugate exponent of p.*

(*i*) *If g is a Moreau envelope with index λ and power p, then the function $g^* - \frac{\lambda^{q-1}}{q} \| \cdot \|_{X^*}^q \in \Gamma(X^*)$.*

(*ii*) *If moreover $g \in \Gamma(X)$, the following equivalences hold true*

g is a Moreau envelope with index λ and power p

$$\Updownarrow$$

$$g^* - \frac{\lambda^{q-1}}{q} \| \cdot \|_{X^*}^q \in \Gamma(X^*)$$

$$\Updownarrow$$

$$g = \left(\frac{\lambda^{q-1}}{q} \| \cdot \|_{X^*}^q + h \right)^* \quad \text{for some } h \in \Gamma(X^*).$$

Proof. (*i*) It suffices to apply Proposition 8.1 (*i*) with $\varphi = -\frac{1}{p\lambda} \| \cdot \|^p$ and to recall that $\left(\frac{1}{p\lambda} \| \cdot \|^p \right)^* = \frac{\lambda^{q-1}}{q} \| \cdot \|_{X^*}^q$.
(*ii*) The equivalences follow from Corollary 8.1 applied with $\varphi = -\frac{1}{p\lambda} \| \cdot \|^p$. $\square$

When X is a Hilbert space, we obtain a more precise characterization of Moreau envelopes with power 2, as shown by the following proposition.

Proposition 8.2. *Assume that X is a Hilbert space endowed with the scalar product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$.*

(*a*) *For every $\lambda > 0$ and every function $f : X \to \overline{\mathbb{R}}$, we have*

$$e_{\lambda,2} f = -\left(f + \frac{1}{2\lambda} \| \cdot \|^2 \right)^* \left(\frac{\cdot}{\lambda} \right) + \frac{1}{2\lambda} \| \cdot \|^2. \tag{53}$$

The λ-proximal hull of f defined by $h_\lambda f = -e_{\lambda,2}(-e_{\lambda,2} f)$ is given by

$$-e_{\lambda,2}(-e_{\lambda,2} f) = \left(f + \frac{1}{2\lambda} \| \cdot \|^2 \right)^{**} - \frac{1}{2\lambda} \| \cdot \|^2. \tag{54}$$

(*b*) *A function $f : X \to \overline{\mathbb{R}}$ is a Moreau envelope with index λ and power 2 if and only if $f - \frac{1}{2\lambda} \| \cdot \|^2 \in -\Gamma(X)$.*

Proof. (a) For every $x \in X$, we have

$$\begin{aligned} e_{\lambda,2} f(x) &= \inf_{y\in X} \left\{ \frac{1}{2\lambda}\|x-y\|^2 + f(y) \right\} \\ &= \inf_{y\in X} \left\{ \frac{1}{2\lambda}\|x\|^2 + \frac{1}{2\lambda}\|y\|^2 - \frac{1}{\lambda}\langle x, y\rangle + f(y) \right\} \\ &= -\left(f + \frac{1}{2\lambda}\|\cdot\|^2 \right)^* (x/\lambda) + \frac{1}{2\lambda}\|x\|^2, \end{aligned}$$

which proves the equality (53). By iterating we deduce that

$$\begin{aligned} -e_{\lambda,2}\left(-e_{\lambda,2} f\right) &= \left(-e_{\lambda,2} f + \frac{1}{2\lambda}\|\cdot\|^2 \right)^* \left(\frac{\cdot}{\lambda} \right) - \frac{1}{2\lambda}\|\cdot\|^2 \\ &= \left[\left(f + \frac{1}{2\lambda}\|\cdot\|^2 \right)^* \left(\frac{\cdot}{\lambda} \right) \right]^* \left(\frac{\cdot}{\lambda} \right) - \frac{1}{2\lambda}\|\cdot\|^2 \\ &= \left(f + \frac{1}{2\lambda}\|\cdot\|^2 \right)^{**} - \frac{1}{2\lambda}\|\cdot\|^2, \end{aligned}$$

which proves the equality (54).

(b) Observe that f is a Moreau envelope with index λ and power 2 if and only if $-f \in \mathcal{E}^\varphi$ with $\varphi = -\frac{1}{2\lambda}\|\cdot\|^2$. From the equivalence (7) $\Leftrightarrow$ (8) and the fact that $\varphi_- = \varphi$, this is in turn equivalent to $-f = ((-f)^\varphi)^\varphi$. Since $((-f)^\varphi)^\varphi = -e_{\lambda,2}\left(-e_{\lambda,2}\left(-f\right)\right)$ and using the equality (54), we infer that

$$f \text{ is a Moreau envelope with index } \lambda \text{ and power } 2$$

$$\Updownarrow$$

$$-f + \frac{1}{2\lambda}\|\cdot\|^2 = \left(-f + \frac{1}{2\lambda}\|\cdot\|^2 \right)^{**}$$

$$\Updownarrow$$

$$-f + \frac{1}{2\lambda}\|\cdot\|^2 \in \Gamma(X).$$

$$\Updownarrow$$

$$f - \frac{1}{2\lambda}\|\cdot\|^2 \in -\Gamma(X).$$

□

The coupling functional $(x, y) \mapsto -\frac{1}{2\lambda}\|x-y\|^2$ was considered in [7, Section 5] in the framework of generalized conjugacy. Equalities (53)-(54) were established by Penot and Volle [23, p. 206] and Martinez-Legaz [17, p. 182-184]. These equalities were also observed in [30, Example 11.26(c)] and [37, Lemma 3.3]. The characterization (b) above has been noticed in the aforementioned references, and it amounts to the previous characterization in [7, p. 288] of Q^c-convex functions with $c := 1/(2\lambda)$.

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