# Reduction of sweeping process to unconstrained differential inclusion

T. Haddad *[1], A. Jourani [2] and L. Thibault [3]

[1] Université de Jijel, Laboratoire de Mathématiques Pures et Appliquées, Algérie

haddadtr2000@yahoo.fr

[2] Université de Dijon, Institut de Mathématiques de Bourgogne

UMR 5584 CNRS, BP. 47870, 21078 Dijon cedex, France

jourani@u-bourgogne.fr

[3] Université Montpellier II, Institut de Mathématiques et de Modélisation de Montpellier

UMR 5149 CNRS, Case Courrier 051, Place Eugène Bataillon,

34095 Montpellier Cedex, France

thibault@math.univ-montp2.fr

**Abstract**

For a general class of nonconvex and non prox-regular sets, we associate with any sweeping process differential inclusion with such sets an unconstraint differential inclusion whose any solution is a solution of the sweeping process too. The case with bounded and unbounded pertubations is also investigated and an application to an existence theorem is provided.

*Keywords:* Normal cone; $\rho$-prox-regular set; $\alpha$-far property for sets; subdifferential; sweeping process; fixed point theorem; mixed semicontinuity.

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# 1 Introduction

In a series of seminal papers (see [21], [22], [23], and [24]) J.J. Moreau introduced and developped the sudy of the constrained differential inclusion

$$\begin{cases} -\dot{u}(t) \in N(C(t), u(t)) \ a.e \ t \in [T_0, T] \\ u(T_0) = u_0 \in C(T_0) \end{cases} \tag{1.1}$$

where $T, T_0$ are two real numbers with $0 \leq T_0 < T$ and $N(C(t), \cdot)$ denotes the normal cone. In those papers the sets $C(t)$ are closed convex sets of a Hilbert space $H$ moving in an absolutely continuous way, see the next section for details (or in a way with bounded variation). General motivation arising from Mechanics appeared in [24] and [19]. Any absolutely continuous solution $u(\cdot)$ of the differential inclusion must stay in $C(t)$ (since the multimapping $C$ satisfies the above continuity property), that is, $u(t) \in C(t)$ for all $t \in [T_0, T]$.... Moreau coined the name of *sweeping process* to designate such a differential inclusion.

Two years later the first paper by Moreau, planning procedures in mathematical economy led C. Henry [16] to the differential inclusion

$$\dot{u}(t) \in \mathrm{Proj}_{T(S,u(t))}(\Gamma(u(t)), \quad u(T_0) = u_0 \in S,$$

where $\Gamma$ is an upper semicontinuous multimapping with nonempty compact convex values, $S$ is a closed convex set, and $T(S, u(t))$ is the tangent cone to $S$. Henry reduced in some sense the problem to the investigation of the existence of a solution of the differential inclusion

$$\dot{u}(t) \in -N(S, u(t)) + \Gamma(u(t)) \quad \text{with } u(T_0) = u_0 \in S. \tag{1.2}$$

The latter differential inclusion appears as a perturbation of the differential inclusion (1.1) with the multimapping $\Gamma$. The extension of the study of (1.2) when $S$ is merely Clarke regular has been done by B. Cornet [9]. The investigation of the more general form

$$-\dot{u}(t) \in N(C(t), u(t)) + F(t, u(t)) \ \text{ and } \ u(T_0) = u_0 \in C(T_0) \tag{1.3}$$

seems to be started with Castaing, Duc Ha and Valadier [4] where $F(t, \cdot)$ is convex compact valued and upper semicontinuous, and all sets $C(t)$ are convex or all are complement of open convex sets. As for (1.1), because of the presence of the normal cone, for any solution $u(\cdot)$, we see that $u(t)$ is constrained to stay in $C(t)$. This means in particular that (1.3) appears as a constrained differential inclusion.

Actually, general existence results concerning (1.1) with nonconvex sets $C(t)$ in finite dimension have been established by Benabdellah [2], Colombo and Goncharov [10], Thibault [28], Valadier [30], and we also refer to the references in [28]. The existence of solution of the general perturbed differential inclusion (1.3) with merely the closedness of the sets $C(t)$ has been proved in [28] under the upper semicontinuity and the convex valuedness of $F(t,\cdot)$. When the multimapping $F(t,\cdot)$ is lower semicontinuous with nonconvex values, it is shown in Faik and Syam [12] that (1.3) still has, in the finite dimensional setting, at least a solution provided the sets $C(t)$ are uniformly $\rho$-prox-regular (see the next section for the concept). In that paper it is also assumed that $F$ satisfies a linear growth inclusion condition of the type

$$F(t,x) \subset (\beta(t) + \gamma(t)\|x\|)\mathbb{B},$$

where $\mathbb{B}$ denotes the closed unit ball centered at the origin.

Our aim in the present paper is to show for a large class of closed sets $C(t)$ of a Hilbert space, that with any sweeping process differential inclusion (1.1) or its perturbed form (1.3) (which are in fact constrained differential inclusions, as seen above) one can associate an unconstrained differential inclusion whose any absolutely continuous solution is also a solution of the original differential inclusion (1.1) or (1.3). In section 2 we introduce that class. It consists of closed subsets $S$ of a Hilbert space for which the origin is kept uniformly *positively far* from the Clarke subdifferential of the distance function, say $\partial^C d(\cdot,S)(x)$, for all $x$ in some open tube $\{x' : 0 < d(x',S) < \rho\}$ for some $\rho \in ]0,+\infty]$. The class is much larger than that of uniformly prox-regular sets. Sets of that class are not even required to be Clarke regular. Several properties of the class are examined in a great generality. Section 3 provides the reduction results. They are applied in the last section to prove an existence theorem for (1.3) in the finite dimensional framework when the perturbation $F(t,\cdot)$ enjoys a mixed semicontinuity property. The theorem extends significantly similar results in [12, 15].

## 2 Notation and preliminaries

For any $x \in H$ and $r \geq 0$ the closed ball centered at $x$ with radius $r$ will be denoted by $B_H[x, r]$. For $x = 0$ and $r = 1$ we will put $\mathbb{B}_H$ or $\mathbb{B}$ in place of $B_H[0, 1]$.

We give first some background material on variational analysis. We state only the definitions and results which will be needed in the development of the paper. For more details on the subject, we refer to [7, 20, 26]. Throughout $H$ is a real Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and $\| \, \|$ is the associated norm. Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a function and $x \in H$ be a point where it is finite. A vector $v \in H$ is a Fréchet subgradient of $f$ at $x$ (see, e.g., [20, 26]) if for each number $\varepsilon > 0$ there exists some neighborhood $U$ of $x$ such that

$$\langle v, x' - x \rangle \leq f(x') - f(x) + \varepsilon \|x' - x\| \quad \text{for all } x' \in U.$$

The set $\partial^F f(x)$ of all Fréchet subgradients of $f$ at $x$ is the Fréchet subdifferential of $f$ at $x$. The Fréchet subdifferential enjoys only (see, e.g.,[20]) fuzzy calculus rules. A limiting process is needed to obtain pointbased calculus rules. A vector $v \in H$ is a limiting (Fréchet) subgradient or (basic subgradient) of $f$ at $x$ if there exist a sequence $((x_n, f(x_n)))_n$ converging to $(x, f(x))$ and a sequence $(v_n)_n$ converging weakly to $v$ such that $v_n \in \partial^F f(x_n)$. The set $\partial^L f(x)$ of all limiting subgradients of $f$ at $x$ is the Mordukhovich limiting subdifferential of $f$ at $x$. As usual, if $f(x)$ is not finite, one puts $\partial^F f(x) = \emptyset$ and $\partial^L f(x) = \emptyset$. When $f$ is the indicator function $\psi_S$ of a subset $S$ of $H$ (defined by $\psi_S(x') = 0$ if $x' \in S$ and $\psi_S(x') = \infty$ otherwise), $\partial^F \psi_S(x)$ and $\partial^L \psi_S(x)$ are the Fréchet normal cone $N^F(S, x)$ and the Mordukhovich limiting normal cone $N^L(S, x)$ of $S$ at $x$. This means that a vector $v \in H$ is a Fréchet normal vector to $S$ at $x \in S$ if and only if for every number $\varepsilon > 0$ there is a neighborhood $U$ of $x$ such that

$$\langle v, x' - x \rangle \leq \varepsilon \|x' - x\| \quad \text{for all } x' \in S \cap U, \tag{2.1}$$

and

$$N^L(S, x) = {}^{\mathrm{seq}} \operatorname*{Lim\,sup}_{S \ni y \to x} N^F(S, y). \tag{2.2}$$

In the latter equality, the second member denotes the $\| \, \| \times w(H, H)$ sequential Painlevé-Kuratowski upper limit, that is,

$${}^{\mathrm{seq}} \operatorname*{Lim\,sup}_{S \ni y \to x} N^F(S, y) := \left\{ \begin{array}{c} v \in H : \; \exists \text{ sequences } S \ni y_n \to x, v_n \xrightarrow{w} v, \\ \text{with } v_n \in N(S, y_n) \end{array} \right\}.$$

Let $d(\cdot, S)$ or $d_S(\cdot)$ be the usual distance function associated with $S$, i.e., $d(x,S) := \inf_{y\in S} \|x-y\|$. The Fréchet normal cone (see [20]) is related to the distance function through the equality

$$N^F(S,x) = \mathbb{R}_+\partial^F d(\cdot,S)(x) \text{ and } \partial^F d(\cdot,S)(x) = N^F(S,x)\cap \mathbb{B}_H \qquad (2.3)$$

and the Mordukhovich limiting normal cone as well (see [27]) through the similar equality

$$N^L(S,x) = \mathbb{R}_+\partial^L d(\cdot,S)(x), \qquad (2.4)$$

where $\mathbb{R}_+$ denotes the set of non negative real numbers.

The proximal normal cone can be defined in a simlar way as (2.1). A vector $v \in H$ is a proximal normal vector to $S$ at $x \in S$ whenever there exist a constant $\sigma \geq 0$ and a neighborhood $U$ of $x$ such that

$$\langle v, x'-x\rangle \leq \sigma\|x'-x\|^2 \quad \text{for all } x' \in U \cap S.$$

The set of such vectors is the proximal normal cone $N^P(S,x)$ to $S$ at $x$. Obviously

$$N^P(S,x) \subset N^F(S,x) \subset N^L(S,x).$$

Nevertheless, the limiting process leads, in our Hilbert setting, to equalities (see, e.g. [17, 20, 26]), that is,

$$N^L(S,x) = {}^{\mathrm{seq}}\operatorname*{Lim\,sup}_{S\ni y\to x} N^F(S,y) = {}^{\mathrm{seq}}\operatorname*{Lim\,sup}_{S\ni y\to x} N^P(S,y). \qquad (2.5)$$

The proximal normal cone enjoys a geometrical characterization (see, e.g., [7]) given by the equality

$$N^P(S,x) = \{v \in H :\ \exists \rho > 0 \text{ s.t. } x \in \operatorname{Proj}_S(x+\rho v)\}, \qquad (2.6)$$

where

$$\operatorname{Proj}_S(u) := \{y \in S :\ d(u,S) := \|u-y\|\}.$$

The uniformity of the positive constant $\rho$ in (2.6) for the unit proximal normal vectors to $S$ leads to the concept of uniformly prox-regular sets. For a given $\rho \in ]0,+\infty]$ the closed subset $S$ is uniformly $\rho$-prox-regular (see [25]) (or equivalently $\rho$-proximally smooth (see [8]) or $\rho$-positively reached (see [13])) if and only if every unit proximal normal vector to $S$ can be realized by a $\rho$-ball, which can be translated in the fact that for all $x \in S$ and all $0 \neq v \in N^P(S,x)$ one has

$$\left\langle \frac{v}{\|v\|}, x' - x \right\rangle \leq \frac{1}{2\rho} \|x' - x\|^2 ,$$

for all $x' \in S$. We make the convenion $\frac{1}{\rho} = 0$ for $\rho = +\infty$. Recall that for $\rho = +\infty$ the uniform $\rho$-prox-regularity of the closed set $S$ is equivalent to the convexity of the set.

We will also need the concept of Clarke normal cone (see [6]). One of the best ways to define it is as the polar of a tangent cone. Recall that a vector $h \in H$ is in the Clarke tangent cone $T^C(S,x)$ to the set $S$ at $x \in S$ when for every sequence $(x_n)_n$ in $S$ converging to $x$ and every sequence of positive numbers $(t_n)_n$ converging to 0, there exists some sequence $(h_n)_n$ in $H$ converging to $h$ such that $x_n + t_n h_n \in S$ for all integers $n$. This cone is closed and convex and its negative polar $N^C(S,x)$ is the Clarke normal cone to $S$ at $x \in S$, that is,

$$N^C(S,x) = \{v \in H : \langle v, h \rangle \leq 0 \;\; \forall h \in T^C(S,x)\}.$$

As usual, one puts $N^C(S,x) = \emptyset$ if $x \notin S$. In the Hilbert space $H$, the equality (see, e.g., [20])

$$N^C(S,x) = \overline{\text{co}}\,(N^L(S,x))$$

(where $\overline{\text{co}}$ denotes the closed convex hull) provides the relationship between the Clarke normal cone and the Mordukhovich limiting normal cone. Through that normal cone, the Clarke subdifferential of the function $f : H \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\partial^C f(x) := \{v \in H : (v, -1) \in N^C(\text{epi}\, f, (x, f(x)))\},$$

where $\text{epi}\, f := \{(y,r) \in H \times \mathbb{R} : f(y) \leq r\}$ is the epigraph of $f$. When the function $f$ is finite and locally Lipschitzian around $x$, the Clarke subdifferential is characterized (see [6]) in the following simple and amenable way

$$\partial^C f(x) = \{v \in H : \langle v, h \rangle \leq f^0(x;h) \;\; \forall h \in H\}, \tag{2.7}$$

where

$$f^0(x;h) := \limsup_{(t,x') \to (0^+,x)} t^{-1}[f(x'+th) - f(x')] \tag{2.8}$$

is the generalized directional derivative of the locally Lipschitzian function $f$ at $x$ in the direction $h$. The funtion $f^0(x;\cdot)$ is in fact the support function of $\partial^C f(x)$. That characterization easily yields that the Clarke subdifferential of any locally Lipschitzian function enjoys the important property of upper (or outer) semicontinuity. The equality (see [6])

$$N^C(S,x) = \mathrm{cl}\,(\mathbb{R}_+ \partial^C d(\cdot,S)(x)) \quad \text{for } x \in S, \tag{2.9}$$

giving the expression of the Clarke normal cone in terms of the distance function, will allow us to take advantage, in the next section, of the upper semicontinuity of the Clarke subdifferential of the particular Lipschitzian function $d(\cdot,S)$. As usual, it will be convenient to write below $\partial^C d(x,S)$ in place of $\partial^C d(\cdot,S)(x)$. When $f$ is locally Lipschitzian one also has $\partial^C f(x) = \overline{\mathrm{co}}\,(\partial^L f(x))$ (see, e.g., [20]), and hence in particular for the distance function

$$\partial^C d(x,S) = \overline{\mathrm{co}}\,(\partial^L d(x,S)) \quad \text{for all } x \in H. \tag{2.10}$$

The following proposition provides some properties of the Fréchet and Clarke subdifferentials of the distance function $d(\cdot,S)$ when the set $S$ is prox-regular. It also summarizes some important consequences of the uniform prox-regularity which will be needed in the paper. For the proofs of these results we refer the reader to [25].

**Proposition 1** *Let $S$ be a nonemty closed subset in $H$ and let $\rho \in ]0,\infty]$. Then the following are equivalent.*
***(a)*** *$S$ is uniformly $\rho$-prox-regular;*
***(b)*** *For any point $x'$ in the open enlargement $\{x \in H :\ d(x,S) < \rho\}$ the set $\mathrm{Proj}_S(x')$ is a singleton and the mapping $\mathrm{Proj}_S(\cdot)$ is continuous on this open enlargement;*
***(c)*** *The Fréchet subdifferential of $d(\cdot,S)$ coincides with its Clarke subdifferential at all points $x \in H$ satisfying $d(x,S) < \rho$;*
***(d)*** *The distance function $d(\cdot,S)$ is continuously Fréchet differentiable on the open tube $\{x \in H :\ 0 < d(x,S) < \rho\}$;*
***(e)*** *For all $x_i \in S$ and all $v_i \in N^P(S,x_i)$ with $\|v_i\| \le \rho$ $(i = 1,2)$ one has*

$$\langle v_1 - v_2, x_1 - x_2\rangle \ge -\|x_1 - x_2\|^2 .$$

As a consequence of **(c)**, **(e)**, (2.5), and the above equalities relating the Fréchet and Clarke normal cones to the distance function, we obtain that, for uniformly $\rho$-prox-regular sets, the proximal normal cone to $S$ coincides with all the normal cones contained in the Clarke normal cone at all points $x \in S$, i.e., $N^P(S,x) = N^F(S,x) = N^C(S,x)$.

The properties **(b)** and **(e)** have been among the key points in the proofs of existence of solution of (perturbation of) sweeping process associated with $\rho$-prox-regular moving sets in [3] and [11]. As we will see in the next section, some new results can be reached with the larger class of sets in Definition 1. It is a localization of the class of *subdifferentially behaved* sets in [14]. The localization will be crucial for our development.

Let us denote, for $\rho \in ]0,+\infty]$, by $U_\rho(S)$, the open $\rho$-tube around the set $S$ involving in the statement of (d) in Proposition 1, that is,

$$U_\rho(S) := \{x \in H : \ 0 < d(x,S) < \rho\}.$$

**Definition 1** *Let $\alpha$ be a real number in the interval $]0,1]$ and $\rho \in ]0,+\infty]$, and let $\partial$ be the Mordukhovich or Clarke subdifferential. Let $S$ be a nonempty closed subset of $H$ with $S \neq H$. We say that the $\partial$-subdifferential of the distance function $d(\cdot,S)$ keeps the origin $\alpha$-far-off on the open $\rho$-tube around $S$ provided*

$$\alpha \leq \inf_{x \in U_\rho(S)} d(0, \partial d(x,S)). \tag{2.11}$$

Observe that for any number $\alpha$ satisfying (2.11) we have $\alpha \leq 1$. Indeed, it suffices to take some point $x_0$ in the open set $U_\rho(S)$ such that $\partial^F d(x_0,S) \neq \emptyset$ (such a point exists, see, e.g., [20]) and to see that for any $v_0 \in \partial^F d(x_0,S)$ we have $\alpha \leq \inf\limits_{x \in U_\rho(S)} d(0, \partial^F d(x,S)) \leq \|v_0\| = 1$. That justifies the requirement of the interval $]0,1]$ in the property of the definition.

We note that the above property could be of course localized around a point $\bar{x} \in S$ in the form

$$\alpha \leq \inf_{x \in W \setminus S} d(0, \partial d(x,S))$$

for a certain open neighborhood $W$ of $\bar{x}$ in $H$. This will be developed elsewhere.

As we will see below, that property ensures a good behavior of sets in the study of some evolution differential inclusions of sweeping process type. When $\rho = \infty$, that is, the origin is kept $\alpha$-far from $\partial d(\cdot,S)$ at the exterior of $S$, we recover the concept of $\partial$-*behaved* sets of [14].

We know (see [18]) that for any $x$ outside of any closed set $S$ and any $v \in \partial^F d(x,S)$ we have $\|v\| = 1$. Thus the property in Definition 1 with $\partial = \partial^F$ has no influence in the behavior of the set $S$. It is the same for $\partial = \partial^L$ whenever $H$ is finite dimensional. At the opposite, the property with Clarke subdifferential may avoid in some sense some critical behavior.

The following proposition examines the case of sets whose distance functions are Clarke subdifferentially regular. Recall that a locally Lipschitzian function $f$ is Clarke subdifferentially regular on an open set $W$ if and only if $\partial^C f(x) = \partial^F f(x)$ for all $x \in W$. By **(c)** of Proposition 1, the Clarke subdifferential regularity of the distance function $d(\cdot, S)$ on the open $\rho$-tube $U_\rho(S)$ is equivalent to the uniform $\rho$-prox regularity of the closed set $S$.

**Proposition 2** *Let $S$ be a nonempty closed subset of $H$ with $S \neq H$. If $S$ is uniformly $\rho$-prox-regular, then*

$$\inf_{x \in U_\rho(S)} d(0, \partial^C d(x,S)) = \inf_{x \in U_\rho(S)} d(0, \partial^L d(x,S)) = 1,$$

*which means that the origin is kept 1-far from $\partial^C d(\cdot, S)$ on $U_\rho(S)$.*

**Proof.** We have already recalled above that for any $x \notin S$, any vector $v \in \partial^F d(x,S)$ is a unit vector. The equalities of the proposition follow then from the above equivalence between the Clarke subdifferential regularity of $d(\cdot, S)$ on $U_\rho(S)$ and the uniform $\rho$-prox-regularity of $S$. $\square$

The proposition says that any uniformly prox-regular set $S$ has the property that the origin is kept positively far from $\partial^C d(\cdot, S)$ on an open tube around $S$. The converse does not hold in the sense that there are sets with such a property which are not uniformly prox-regular.

**Example 1** *The set $S := \{(r,s) \in \mathbb{R}^2 : \ s \geq -|r|\}$ in $\mathbb{R}^2$ is not uniformly $\rho$-prox-regular for any $\rho \in ]0, +\infty]$ but the Clarke subdifferential of its distance function keeps the origin positively far-off at the exterior of $S$.*

For any $(r,s) \notin S$ we have

$$d((r,s), S) = \frac{\sqrt{2}}{2} ||r| + s|.$$

This equality confirms first, according to **(d)** of Proposition 1 , that $S$ is not $\rho$-prox-regular for any $\rho \in ]0, +\infty]$. Further, observe that for $(r, s) \notin S$ we have $s < 0$ and that the above equality may be reformulated as

$$d((r,s),S) = \frac{\sqrt{2}}{2}(-r-s) \text{ if } r \geq 0 \ \text{ and } \ d((r,s),S) = \frac{\sqrt{2}}{2}(r-s) \text{ if } r \leq 0.$$

So at any point $(r, s) \notin S$ where the gradient $\nabla d(\cdot, S)(r, s)$ exists, it is equal to $\frac{\sqrt{2}}{2}(-1, -1)$ if $r > 0$ and to $\frac{\sqrt{2}}{2}(1, -1)$ if $r < 0$. Taking into account (see [6]) the representation $\partial^C f(x) = \mathrm{co}\{\lim \nabla f(x_n) : x_n \to x\}$ for any locally Lipschitzian function $f$ on a finite dimensional space, we see that at any point $(r, s) \notin S$, with $r = 0$, we have

$$\partial^C d((r,s),S) = \{\frac{\sqrt{2}}{2}(1-2\lambda, -1) \in \mathbb{R}^2 : \lambda \in [0,1]\}$$

hence for any non-empty open set $U \subset \mathbb{R}^2 \setminus S$, with $U \cap \{0\} \times \mathbb{R} \neq \emptyset$

$$\inf_{(r,s)\in U} d(0, \partial^C d((r,s),S) = \frac{\sqrt{2}}{2}.$$

In particular the Clarke subdifferential of $d(., S)$ keeps the origin $\frac{\sqrt{2}}{2}$-far-off at the exterior of $S$. $\square$

Recall that a closed set $S$ is Clarke tangentially regular at $x \in S$ when the Clarke tangent cone $T^C(S, x)$ and the Bouligand contingent cone $K(S, x)$ coincide. When the coincidence holds at every $x \in S$, one says that the set $S$ is Clarke regular. A vector $v$ is in the contingent cone $K(S, x)$ if there exist a sequence of positive numbers $(t_n)_n$ converging to zero and a sequence of vectors $(v_n)_n$ converging to $v$ with respect to the norm $\| \ \|$ such that $x + t_n v_n \in S$ for all integers $n$. When the convergence of $(v_n)_n$ to $v$ is required with respect to the weak topology in $H$, the vector $v$ is in the weak contingent cone $WK(S, x)$. The above example even makes clear that the property in Definition 1 is stricly weaker than Clarke tangential regularity in the finite dimensional setting.. Indeed, for the set $S$ in that example, it is easily seen that $T^C(S, (0, 0)) \neq K(S, (0, 0))$.

# 3 Reduction theorem

Let $T_0$ and $T$ be two non negative real numbers with $T_0 < T$. Throughout $C : [T_0, T] \rightrightarrows H$ will be a multimapping with *nonempty closed* values. The two following assumptions concerning the multimapping $C(\cdot)$ will be involved in the paper:

$(\mathbf{H}_1)$: there exist two constants $\alpha \in ]0,1]$ and $\rho \in ]0,+\infty]$ such that, for each $t \in [T_0,T]$, the origin is kept $\alpha$-far from the Clarke subdifferential of the distance function $d(\cdot, C(t))$ on the open $\rho$-tube $U_\rho(C(t))$;

$(\mathbf{H}_2)$: the set $C(t)$ varies in an absolutely continuous way, that is, there exists an absolutely continuous non negative function $\zeta : [T_0,T] \to \mathbb{R}_+$ such that

$$|d(x,C(t)) - d(y,C(s))| \leq \|x-y\| + |\zeta(t) - \zeta(s)|$$

for all $x, y \in H$ and all $s, t \in [T_0,T]$

We note that the assumption $(\mathbf{H}_2)$ is entailed by (and, in the finite dimensional setting, is in fact equivalent to) the geometric inclusion

$$C(s) \subset C(t) + |\zeta(t) - \zeta(s)|\, \mathbb{B}_H \tag{3.1}$$

for all $s, t \in [T_0,T]$.

In the remaining of this section, unless otherwise stated, $N(\cdot,\cdot)$ will denote either the *Clarke normal cone* or the *Mordukhovich limting normal cone*, and $\partial$ will be the *subdifferential* associated with this normal cone.

Consider the constrained differential inclusion

$$\begin{cases} -\dot{u}(t) \in N(C(t), u(t)) \;\; \text{a.e. } t \in [T_0,T] \\ u(T_0) = u_0 \in C(T_0). \end{cases} \tag{3.2}$$

We recall that a solution of (3.2) is an absolutely continuous mapping $u : [T_0,T] \to H$ such that $\dot{u}(t) \in -N(C(t),u(t))$ for almost every $t \in [T_0,T]$ and $u(t) \in C(t)$ for all $t \in [T_0,T]$. The differential inclusion is then implicitly subject to the constraints $u(t) \in C(t)$. Under the assumption $(H_2)$, the constraints are completely implicit in the inclusion $\dot{u}(t) \in -N(C(t),u(t))$ since it is not hard to see that under $(H_2)$ when $u(t) \in C(t)$ for a.e. $t \in [T_0,T]$, then $u(t) \in C(t)$ for all $t \in [T_0,T]$ because of the continuity of $u(\cdot)$.

Our aim in this section is to study the relationship between the above constrained differential inclusion and the following unconstrained one

$$\begin{cases} -\dot{u}(t) \in \dfrac{1}{\alpha^2}|\dot{\zeta}(t)|\, \partial d(u(t), C(t)) \;\; \text{a.e. } t \in [T_0,T] \\ u(T_0) = u_0 \in C(T_0). \end{cases} \tag{3.3}$$

Let us start with the following lemma which is in the line of Lemma 2 in [14]. To keep the paper self-contained, we provide a proof.

**Lemma 1** *Let $\partial$ be any subdifferential included in the Clarke one for locally Lipschitzian functions. Let $S$ be a nonempty closed subset of $H$ for which the $\partial$-subdifferential of the distance function $d(\cdot, S)$ keeps the origin $\alpha$-far-off in the open $\rho$-tube $U_\rho(S)$. Then for any $x \in U_\rho(S)$ and any $v \in \partial d(x, S)$ one has*

$$\lim_{s \downarrow 0} s^{-1}[d(x - sv, S) - d(x, S)] \leq -\alpha^2.$$

**Proof.** Fix any $x \in U_\rho(S)$. By the inclusion of $\partial$ in the Clarke subdifferential, it is easily seen that $2d(x, S)v \in \partial^C d^2(x, S)$ and hence, by a well-known property of Clarke subdifferential $-2d(x, S)v \in \partial^C(-d(\cdot, S))^2(x)$. Further the equality

$$-(d(x, S))^2 = -\|x\|^2 + \sup_{y \in S}(2\langle x, y\rangle - \|y\|^2)$$

makes clear that the function $-d^2(\cdot, S)$ is Clarke directionally regular (as the sum of two locally Lipschitzian Clarke directionally regular functions at $x$, the second one being convex continuous). So, writing

$$s^{-1}[d(x-sv, S)-d(x, S)] = s^{-1}[-d^2(x-sv, S)+d^2(x, S)][-d(x-sv, S)-d(x, S)]^{-1},$$

we see first that the limit in the statement of the lemma that we denote by $L$ exists and

$$-2L\, d(x, S) = \lim_{s \downarrow 0} s^{-1}[-d^2(x - sv, S) + d^2(x, S)] = (-d^2(\cdot, S))^0(x; -v),$$

the second equality being due to the Clarke directional regularity of the function $-d^2(\cdot, S)$ at $x$. Therefore, by (2.7)

$$\langle -2d(x, S)v, -v\rangle \leq -2L\, d(x, S), \text{ i.e., } L \leq -\|v\|^2.$$

Since $x \in U_\rho(S)$, according to Definition 1 , the latter inequality ensures us that $L \leq -\alpha^2$. $\square$

Under the uniform prox-regularity of the sets $C(t)$ instead of the assumption $(H_1)$ above, the following result has been initiated in [28] and slightly extended in [15]. We show that it still holds under the weaker assumption $(H_1)$.

**Theorem 1** *Assume that $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ are satisfied. Then any absolutely continuous solution $u : [T_0, T] \to H$ of (3.3) is a solution of (3.2).*

**Proof.** Fix any solution $u(\cdot)$ of the unconstrained differential inclusion (3.3). Since $\partial d(x,S) \subset N(S,x)$ for any $x \in S$ (see (2.4) and 2.10)), it is enough to prove that $u(t) \in C(t)$ for all $t \in [T_0, T]$.

Let us recall that $\dot{\zeta}(\cdot) \in L^1([T_0,T],\mathbb{R})$ and hence

$$\lim_{\lambda(A)\to 0} \int_A |\dot{\zeta}(t)|\, dt = 0,$$

where $\lambda$ denotes here the Lebesgue measure. Let us fix $\delta > 0$ such that

$$\int_A |\dot{\zeta}(t)|\, dt < (1 + \frac{1}{\alpha^2})^{-1}\rho \tag{3.4}$$

for all Lebesgue measurable sets $A \subset [T_0, T]$ satisfying $\lambda(A) < \delta$.

**Step 1.** Suppose for a moment that $T - T_0 < \delta$ and let us show that $u(t) \in C(t)$ for every $t \in [T_0, T]$.

We first observe that, for any $t \in [T_0, T]$, according to the assumption $(H_2)$ and to the initial condition $u(T_0) \in C(T_0)$ we have

$$\begin{array}{rl} d(u(t), C(t)) & \leq \|u(t) - u(T_0)\| + |\zeta(t) - \zeta(T_0)| + d(u(T_0), C(T_0)) \\ & = \| \int_{T_0}^t \dot{u}(s)\, ds\| + | \int_{T_0}^t \dot{\zeta}(s)\, ds| \end{array}$$

and hence, since $\|\dot{u}(s)\| \leq \frac{|\dot{\zeta}(s)|}{\alpha^2}$ for a.e. $s \in [T_0, T]$ because $u(\cdot)$ is (by assumption) a solution of the differential inclusion (3.3), we get

$$\begin{array}{rl} d(u(t), C(t)) & \leq \int_{T_0}^t \|\dot{u}(s)\|\, ds + \int_{T_0}^t |\dot{\zeta}(s)|\, ds \\ & \leq (1 + \frac{1}{\alpha^2}) \int_{T_0}^t |\dot{\zeta}(s)|\, ds, \end{array}$$

which entails by (3.4) and by the inequality $T - T_0 < \delta$

$$d(u(t), C(t)) < \rho, \ \text{ i.e., } u(t) \in U_\rho(S) \text{ for all } t \in [T_0, T]. \tag{3.5}$$

We follow now the proof of Theorem 2.1 in [28]. By $(H_2)$ the function $h$ given by $h(t) := d(u(t), C(t))$ is absolutely continuous on the interval $[T_0, T]$. Let $t$ be any element of $]T_0, T[$ where $\dot{h}(t)$, $\dot{\zeta}(t)$, and $\dot{u}(t)$ exist and where the inclusion in (3.3) holds. Then writing

$$\begin{array}{rcl} h(t+s) - h(t) & = & d(u(t+s), C(t+s)) - d(u(t), C(t)) \\ & = & d(u(t+s), C(t+s)) - d(u(t+s), C(t)) \\ & & +d(u(t+s), C(t)) - d(u(t), C(t)) \end{array}$$

and having in mind the assumption $(H_2)$ we obtain for $s > 0$ sufficiently small

$$h(t+s) - h(t) \le |\zeta(t+s) - \zeta(t)| + d(u(t+s), C(t)) - d(u(t), C(t)). \quad (3.6)$$

Further, for $s > 0$ small enough

$$\begin{aligned}
&s^{-1}[d(u(t+s), C(t)) - d(u(t), C(t))] \\
&= s^{-1}[d(u(t) + s\dot{u}(t) + s\varepsilon(s,t), C(t)) - d(u(t), C(t))] \\
&= s^{-1}[d(u(t) + s\dot{u}(t), C(t)) - d(u(t), C(t))] + \eta(s,t),
\end{aligned}$$

for some mappings $\varepsilon(\cdot, t)$ and $\eta(\cdot, t)$ with $\lim_{s\downarrow 0} \varepsilon(s,t) = 0$ and $\lim_{s\downarrow 0} \eta(s,t) = 0$ in $H$. Since $u(t) \in U_\rho(C(t))$ by (3.5), taking the inclusion in (3.3) into account, we then obtain thanks to Lemma 1

$$\lim_{s\downarrow 0} s^{-1}[d(u(t+s), C(t)) - d(u(t), C(t))] \le -|\dot{\zeta}(t)|.$$

Using the latter inequality and (3.6) we see that

$$\dot{h}(t) \le |\dot{\zeta}(t)| - |\dot{\zeta}(t)| = 0.$$

Thus $\dot{h}(s) \le 0$ for a.e. $s \in ]T_0, T[$. Since $h(T_0) = 0$, this yields for every $t \in [T_0, T]$

$$0 \le h(t) = \int_{T_0}^{t} \dot{h}(s)\, ds \le 0$$

and hence $h(t) = 0$. This means that $u(t) \in C(t)$ for all $t \in [T_0, T]$.

**Step 2.** Let go back to the general case without any restriction on the lenght of the interval $[T_0, T]$.

On the basis of Step 1, fix like in [15] an integer $N$ such that $\frac{T-T_0}{N} < \delta$ and fix also the subdivision of $[T_0, T]$ with $T_0, T_1, \cdots, T_N = T$ given, for each $k \in \{0, 1, \cdots, N\}$ by

$$T_k = T_0 + k\frac{(T - T_0)}{N}.$$

For each $k = 1, \cdots, N$ denote by $u^k(\cdot)$ the restriction of $u(\cdot)$ to $[T_{k-1}, T_k]$, that is, $u^k := u|_{[T_{k-1}, T_k]}$. We observe first that $u^1 : [T_0, T_1] \to H$ is a solution of the unconstrained differential inclusion

$$\begin{cases} -\dot{u}^1(t) \in \dfrac{1}{\alpha^2}|\dot{\zeta}(t)|\, \partial d(u^1(t), C(t)) \ \text{ a.e } t \in [T_0, T_1] \\ \qquad\qquad u^1(T_0) = u_0 \in C(T_0). \end{cases}$$

Since $T_1 - T_0 < \delta$, it follows from Step 1 that the mapping $u^1(\cdot)$ also satisfies

$$u^1(t) \in C(t) \text{ for all } t \in [T_0, T_1].$$

Suppose that, for each $i = 1, \cdots, k-1$, we have $u_i(t) \in C(t)$ for all $t \in [T_{i-1}, T_i]$. Taking into account the equalities $u^{k-1}(T_{k-1}) = u(T_{k-1}) = u^k(T_{k-1})$, we see that the mapping $u^k : [T_{k-1}, T_k] \to H$ is a solution of the differential inclusion

$$\begin{cases} -\dot{u}^k(t) \in \dfrac{1}{\alpha^2}|\dot{\zeta}(t)|\,\partial d(u^k(t), C(t)) \text{ a.e. } t \in [T_{k-1}, T_k] \\ u^k(T_{k-1}) = u^{k-1}(T_{k-1}) \in C(T_{k-1}). \end{cases}$$

Because of the inequality $T_k - T_{k-1} < \delta$, we may apply again the result of Step 1 to get

$$u^k(t) \in C(t) \text{ for all } t \in [T_{k-1}, T_k].$$

Inductively we then obtain for each $k = 1, \cdots, N$ that

$$u^k(t) \in C(t) \text{ for all } t \in [T_{k-1}, T_k].$$

These inclusions can be reformulated as $u(t) \in C(t)$ for all $t \in [T_0, T]$, which implies that $u(\cdot)$ is a solution of the differential inclusion (3.2). □

The following example shows the necessity of the assumption $(\mathbf{H}_1)$.

**Example 2** *Let $C(t) = \{x \in \mathbb{R}^2 : \|x\|_2 = t\}$. Take $T_0 = 0$, and $N(\cdot, \cdot)$ and $\partial$ for the Clarke normal cone and Clarke subdifferential, respectively. Then the sets $(C(t))$ fulfill assumption $(\mathbf{H}_2)$, but not $(\mathbf{H}_1)$. Moreover the arc $u(t) = 0$ for all $t$, is a solution of the differential inclusion (3.3), while $0 \notin C(t)$, for all $t > 0$. Hence $u$ is not a solution of (3.2).*

As a direct consequence of the above theorem and of Proposition 2 we recover the reduction theorem of [28, Theorem 2.1] and [15, Theorem 2] in (b) of the next corollary. The assertion (a) of the corollary follows from the fact that, in the finite dimensional setting, $\|v\| = 1$ for all $v \in \partial^L d(x, S)$ when $x \notin S$ (because as recorded in section 2 any vector in $\partial^F d(x', S)$ is a unit vector if $x' \notin S$).

**Corollary 1** *Assume that* $(H_2)$ *is satisfied. Then the following hold.*
*(a) Each absolutely solution of (3.3) with* $\alpha = 1$ *is also a solution of (3.2) provided that* $\partial = \partial^L$ *and* $N = N^L$.
*(b) If all the sets* $C(t)$ *are uniformly* $\rho$*-prox-regular, then each solution of (3.3) with* $\alpha = 1$ *is a solution of (3.2).*

The following theorem deals with a multivalued perturbation $F : [T_0, T] \times H \rightrightarrows H$. The concrete cases of (b) and (c) require an inclusion-type growth condition. The more general case of an intersection-type growth condition will be seen later. The theorem uses the concept of measurable multimappings for which we refer to [5]. Instead of assuming here that the multivalued perturbation $F$ is $\mathcal{L}([T_0, T]) \otimes \mathcal{B}(H)$-measurable, where $\mathcal{L}([T_0, T])$ and $\mathcal{B}(H)$ denote the Lebesgue and Borel $\sigma$-fields of $[T_0, T]$ and $H$, we merely require the measurability of the multimapping $F(\cdot, u(\cdot))$ for each $\mathcal{L}([T_0, T])$-measurable mapping $u(\cdot)$.

**Theorem 2** *Assume that* $(H_1)$ *and* $(H_2)$ *hold. Let* $F : [T_0, T] \times H \rightrightarrows H$ *be a multimapping with nonempty closed values such that the multimapping* $t \mapsto F(t, v(t))$ *is Lebesgue measurable for each Lebesgue measurable mapping* $v(\cdot)$ *from* $[T_0, T]$ *into* $H$. *Let the differential inclusions*

$$\begin{cases} -\dot{u}(t) \in N(C(t), u(t)) + F(t, u(t)) \\ \qquad u(T_0) = u_0 \in C(T_0) \end{cases} \tag{3.7}$$

*and*

$$\begin{cases} -\dot{u}(t) \in \dfrac{1}{\alpha^2} m(t)\, \partial d(C(t), u(t)) + F(t, u(t)) \\ \qquad u(T_0) = u_0 \in C(T_0), \end{cases} \tag{3.8}$$

*where* $m(\cdot)$ *is a non negative real valued Lebesgue integrable function.*
*(a) Suppose that* $u(\cdot)$ *is a solution of (3.8) and let* $z(\cdot)$ *be a Lebesgue measurable selection of the multimapping* $F(\cdot, u(\cdot))$ *such that*

$$-\dot{u}(t) \in \frac{1}{\alpha^2} m(t)\, \partial d(u(t), C(t)) + z(t),$$

*which exists since* $u(\cdot)$ *is a solution of (3.8).*

*If* $m(t) \geq |\dot{\zeta}(t)| + \|z(t)\|$ *for a.e.* $t \in [T_0, T]$, *then* $u(\cdot)$ *is also a solution of (3.7).*
*(b) If the multimapping* $F$ *satisfies, for some non negative Lebesgue integrable function* $\beta(\cdot)$ *the boundedness condition* $F(t, x) \subset \beta(t)\mathbb{B}_H$ *for all* $t \in [T_0, T]$

*and all $x \in H$, then any solution of (3.8) with $m(t) \geq |\dot{\zeta}(t)| + \beta(t)$ is a solution of (3.7).*
*(c) Let $\beta(\cdot)$ and $\gamma(\cdot)$ be non negative Lebesgue integrable functions for which the multimapping $F$ fulfills the inclusion-type growth condition*

$$F(t,x) \subset (\beta(t) + \gamma(t)\|x\|)\mathbb{B}_H$$

*for all $(t,x) \in [T_0, T] \times H$. Then any solution of (3.8) with*

$$m(t) \geq |\dot{\zeta}(t)| + \beta(t) + \gamma(t)(\|u_0\| + \exp(\int_{T_0}^{t} \gamma(s)\,ds) \int_{T_0}^{t} (\frac{1}{\alpha^2}|\dot{\zeta}(s)| + \beta(s))\,ds$$

*is also a solution of (3.7).*

**Proof.** (a) Let $z(\cdot)$ and $m(\cdot)$ as above. As in [4] (see also [28]) we put

$$Z(t) = \int_{T_0}^{t} z(s)\,ds, \; D(t) = C(t) - Z(t), \text{ and } v(t) = u(t) - Z(t)$$

for all $t \in [T_0, T]$. It is easily verified that the $\partial$-subdifferential of the distance function $d(\cdot, D(t))$ of each set $D(t)$ keeps the origin $\alpha$-far-off on the tube $U_\rho(D(t))$, that is, $(H_1)$; and it is not hard to see that for all $x, y \in H$ and $s, t \in [T_0, T]$

$$|d(x, D(t)) - d(y, D(s))| \leq \|x - y\| + |\int_{s}^{t} m(\tau)\,d\tau|,$$

that is, $(H_2)$ holds with $M(\cdot)$ in place of $\zeta(\cdot)$, where $M(t) := \zeta(T_0) + \int_{T_0}^{t} m(s)\,ds$. Further, from (3.8) and subdifferential calculus we have

$$-\dot{v}(t) \in \frac{1}{\alpha^2}|\dot{M}(t)|\,\partial d(v(t), D(t)) \quad \text{for a.e. } t \in [T_0, T]$$

and $v(T_0) = u(T_0) - Z(T_0) \in D(T_0)$. Then, applying Theorem 1 yields for a.e. $t \in [T_0, T]$ that $-\dot{v}(t) \in N(D(t), v(t))$, which is equivalent to $-\dot{u}(t) \in N(C(t), u(t)) + z(t)$. This entails that $u(\cdot)$ is a solution of (3.7).
(b) The assertion (b) follows directly from (a).
(c) Let $u(\cdot)$ be a solution of (3.8) with $m(\cdot)$ as given in ($c$). By measurability arguments of multimappings, there is a measurable selection $z(\cdot)$ of the multimapping $F(\cdot, u(\cdot))$ such that for a.e. $t \in [T_0, T]$ we have

$$-\dot{u}(t) \in \frac{1}{\alpha^2}|\dot{\zeta}(t)|\,\partial d(u(t), C(t)) + z(t),$$

which says in particular according to the inclusion-type growth condition that

$$\|\dot{u}(t)\| \leq \frac{1}{\alpha^2}|\dot{\zeta}(t)| + \beta(t) + \gamma(t)\|u(t)\|.$$

Putting $r(t) := \|u_0\| + \int_{T_0}^t \|\dot{u}(s)\|\, ds$ and $\delta(t) := \dfrac{1}{\alpha^2}|\dot{\zeta}(t)| + \beta(t)$, the latter inequality ensures us that $\dot{r}(t) \leq \delta(t) + \gamma(t)r(t)$ for a.e. $t \in [T_0, T]$. The Gronwall lemma yields for every $t \in [T_0, T]$ that

$$r(t) \leq r(T_0) + \exp(\int_{T_0}^t \gamma(s)\, ds) \int_{T_0}^t \delta(s)\, ds,$$

which implies

$$\|u(t)\| \leq \|u_0\| + \exp(\int_{T_0}^t \gamma(s)\, ds) \int_{T_0}^t \delta(s)\, ds.$$

Taking into account this inequality and the inclusion-type growth condition we obtain

$$\|z(t)\| \leq \beta(t) + \gamma(t)(\|u_0\| + \exp(\int_{T_0}^t \gamma(s)\, ds) \int_{T_0}^t (\frac{1}{\alpha^2}|\dot{\zeta}(s)| + \beta(s))\, ds$$

and hence $m(t) \geq |\dot{\zeta(t)}| + \|z(t)\|$. We then apply (a) to obtain that $u(\cdot)$ is a solution of (3.7), which completes the proof. □

The next proposition can be seen as a part of the path to the converse of the implication established in Theorem 1.

**Proposition 3** *Assume that (3.1) holds. For each absolutely continuous mapping $u : [T_0, T] \to H$ satisfying $u(t) \in C(t)$ for all $t \in [T_0, T]$ one has*

$$-\dot{u}(t) \in (WK(C(t), u(t)) + |\dot{\zeta}(t)|\, \mathbb{B}_H) \cap (-WK(C(t), u(t)) + |\dot{\zeta}(t)|\, \mathbb{B}_H)$$

*for a.e. $t \in [T_0, T]$ (in fact for every $t$ where $u(\cdot)$ and $\zeta(\cdot)$ are derivable).*

**Proof.** Fix any $t \in ]T_0, T[$ where $\dot{u}(t)$ and $\dot{\zeta}(t)$ exist. For $s > 0$ sufficiently small, from the inclusion $u(t-s) \in C(t-s)$ and from (3.1) we derive

$$u(t) - s(\dot{u}(t) + \varepsilon(s, t)) \in C(t) + |\zeta(t-s) - \zeta(t)|\, \mathbb{B}_H,$$

where $\lim_{s\downarrow 0} \varepsilon(s,t) = 0$ in $H$, and hence for some $b(s,t) \in \mathbb{B}_H$ and some $\eta(s,t) \in \mathbb{R}$ with $\lim_{s\downarrow 0} \eta(s,t) = 0$ we have

$$u(t) - s(\dot{u}(t) + \varepsilon(s,t) + (|\dot{\zeta}(t)| + \eta(s,t))\, b(s,t)) \in C(t).$$

Taking some sequence $s_n \downarrow 0$ such that $(b(s_n,t))_n$ converges weakly to a certain $b' \in \mathbb{B}_H$, the latter inclusion ensures (see the definition of the weak contingent cone $WK(\cdot,\cdot)$ recorded in section 2) that

$$-\dot{u}(t) - |\dot{\zeta}(t)|\, b' \in WK(C(t), u(t) \quad \text{hence} \quad -\dot{u}(t) \in WK(C(t), u(t)) + |\dot{\zeta}(t)|\, \mathbb{B}_H.$$

In a similar way starting instead with $u(t+s) \in C(t+s)$ we obtain that

$$\dot{u}(t) \in WK(C(t), u(t)) + |\dot{\zeta}(t)|\, \mathbb{B}_H.$$

The proof of the proposition is then complete. $\square$

From the above proposition we can derive the following result of [28, Proposition 2.1].

**Corollary 2** *Assume that for each $t \in [T_0, T]$ the set $C(t)$ is Fréchet regular in the sense that for each $x \in C(t)$ the equality $N^F(C(t),x) = N^C(C(t),x)$ holds. Then under $(H_2)$, each solution of (3.2) is a solution of (3.3).*

**Proof.** It is easily seen that $(H_2)$ ensures that for each positive real number $\varepsilon$ the inclusion (3.1) holds with the function $(1+\varepsilon)\zeta(\cdot)$ in place of $\zeta(\cdot)$. Fix any $t \in ]T_0, T[$ where $\dot{u}(t)$ and $\dot{\zeta}(t)$ exist and such that $-\dot{u}(t) \in N(C(t), u(t))$. By the above proposition, for a certain $b(t) \in \mathbb{B}_H$ (depending on $\varepsilon$) we have $-\dot{u}(t) + (1+\varepsilon)|\dot{\zeta}(t)|\, b(t) \in WK(C(t), u(t))$. Further, our regularity assumption implies that $N(C(t), u(t)) = N^F(C(t), u(t))$ and hence $-\dot{u}(t) \in N^F(C(t), u(t))$. Therefore taking into account the definitions of $N^F(\cdot,\cdot)$ and $WK(\cdot,\cdot)$ it is not hard to see that

$$\langle -\dot{u}(t), -\dot{u}(t) + (1+\varepsilon)|\dot{\zeta}(t)|\, b(t)\rangle \leq 0.$$

This being true for any $\varepsilon > 0$, we deduce that $\|\dot{u}(t)\| \leq |\zeta(t)| \leq \dfrac{1}{\alpha^2}|\zeta(t)|$ (we recall that $0 < \alpha \leq 1$). Then

$$-\dot{u}(t) \in N^F(C(t), u(t)) \cap (\frac{1}{\alpha^2}|\zeta(t)|\mathbb{B}_H) = \frac{1}{\alpha^2}|\zeta(t)|\partial^F d(u(t), C(t)),$$

the equality being due to (2.3). So the inclusion in (3.3) is fulfilled. $\square$

The second corollary is a direct consequence of the above one and of Theorem 1.

**Corollary 3** *Assume that $(H_2)$ is satisfied and that all the sets $C(t)$ are Fréchet regular. Then an absolutely continuous mapping $u : [T_0, T] \to H$ is a solution of (3.2) if and only if it is a solution of (3.3).*

# 4 An existence result

In the previous section, differential inclusion of the type

$$-\dot{u}(t) \in N(C(t), u(t)) + F(t, u(t)) \quad \text{with } u(T_0) = u_0 \in C(t_0),$$

implicitly subject to the constraints $u(t) \in C(t)$, because of the presence of the normal cone $N(C(t), u(t))$, has been reduced to the unconstrained differential one

$$-\dot{u}(t) \in \frac{1}{\alpha^2} m(t)\, \partial d(u(t), C(t)) + F(t, u(t)) \quad \text{with } u(T_0) = u_0,$$

in the sense that it has been proved that any absolutely continuous solution of the second one is a solution of the first. The reduction has been reached under the assumption that the origin be $\alpha$-far from $\partial d(\cdot, C(t))$ on a uniform open tube around $C(t)$. The principle of reduction can be seen as a penalization with an appropriate multiple of the subdifferential of the distance function $d(\cdot, C(t))$. The function $m(\cdot)$ determines the coefficient of the penalization.

In this section we turn to the illustration of how that reduction can lead to a general existence theorem in the finite dimensional space. The inclusion growth condition on the multimapping $F$ in Theorem 3.2 is even relaxed in the following intersection linear growth condition:
$(\mathbf{H}_3)$:

$$F(t, x) \cap (\beta(t) + \gamma(t)\, \|x\|)\mathbb{B}_H \neq \emptyset$$

for all $(t, x) \in [T_0, T] \times H$ for some functions $\beta(\cdot), \gamma(\cdot) \in L^1([T_0, T], \mathbb{R}^+)$.

Let $\varepsilon > 0$ and $u_0 \in C(T_0)$ be fixed and let $\eta(\cdot) : [T_0, T] \to \mathbb{R}$ be the absolutely continuous solution of the ordinary differential equation

$$\begin{cases} \dot{\eta}(t) = \delta(t) + 2\gamma(t)\eta(t) \\ \eta(T_0) = 0, \end{cases} \tag{4.1}$$

where $\delta(t) := \varepsilon + \beta(t) + \gamma(t)\|u_0\| + \gamma(t) \int_{T_0}^t |\dot{\zeta}(s)|\, ds$. We observe that

$$\eta(t) = \int_{T_0}^t \delta(s) \exp(2 \int_s^t \beta(r)\, dr)\, ds \quad \text{for all } t \in [T_0, T] \tag{4.2}$$

and $\dot{\eta}(t) \geq 0$ a.e. $t \in [T_0, T]$.
We denote by $\xi$ the absolutely continuous function defined for each $t \in [T_0, T]$ by

$$\xi(t) = \int_{T_0}^t (|\dot{\zeta}(s)| + \dot{\eta}(s))\, ds. \tag{4.3}$$

Before stating the existence theorem, recall that a multimapping $\Delta : H \rightrightarrows H$ is *graphically closed* (or closed) at a point $x$ (where $\Delta(x) \neq \emptyset$) when for every sequence $(x_n, y_n)_n$ in the graph $\operatorname{gph} \Delta := \{(x', y') : y' \in \Delta(x')\}$ converging to $(x, y)$, the limit point $(x, y)$ stays in $\operatorname{gph} \Delta$, i.e., $y \in \Delta(x)$. Obviously, upper (or outer) semicontinuity at a point of a multimapping with closed values entails its graphical closedness at the point. We will also need the concept of *mixed semicontinuity property* (see [29]) for the multimapping $\Delta$. By this, one understands that at each point $x \in H$ where $\Delta(x)$ is convex the multimapping $\Delta$ is graphically closed, and whenever $\Delta(x)$ is not convex $\Delta$ is lower semicontinuous on a certain neighborhood of $x$.

**Theorem 3** *Assume that $H$ is finite dimensional and that $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ are fulfilled. Let $F : [T_0, T] \times H \rightrightarrows H$ be a nonempty closed valued $\mathcal{L}([T_0, T]) \otimes \mathcal{B}(H)$-measurable multimapping such that $(\mathbf{H}_3)$ holds and such that $F(t, \cdot)$ satisfies the mixed semicontinuity property above for each $t \in [T_0, T]$.*

*Then, for each $u_0 \in C(T_0)$, there is an absolutely continuous solution of the constrained differential inclusion*

$$\begin{cases} -\dot{u}(t) \in N^C(C(t), u(t)) + F(t, u(t)) \text{ a.e. } t \in [T_0, T] \\ u(T_0) = u_0 \in C(T_0), \end{cases} \tag{4.4}$$

*and this solution $u(\cdot)$ satisfies the inequality*

$$\|\dot{u}(t)\| \leq \frac{1}{\alpha^2} |\dot{\zeta}(t)| + (1 + \frac{1}{\alpha^2}) \dot{\eta}(t)), \tag{4.5}$$

*for almost all $t \in [T_0, T]$, where $\dot{\zeta}(t)$ is the derivative of the function involved in $(H_2)$ and where $\dot{\eta}(t)$ is given by (4.1) and (4.2).*

The proof below will need the following theorem by Tostonogov [29]. In it and in the remaining of the paper, $C([T_0,T],H)$ denotes the Banach space of all continuous mappings from $[T_0,T]$ into $H$ endowed with the supremum norm. A Carathéodory function $g:[T_0,T]\times H\to\mathbb{R}$ will be said to be integrably bounded on bounded subsets if for each real number $r>0$ the function $t\mapsto \sup\limits_{\|x\|\le r}|g(t,x)|$ is integrable on $[T_0,T]$.

**Theorem 4** *( see [29]) Let $G:[T_0,T]\times H\rightrightarrows H$ be a nonempty closed valued $\mathcal{L}([T_0,T])\otimes\mathcal{B}(H)$-mesurable multimapping such that for each $t\in[T_0,T]$ the multimapping $G(t,\cdot)$ satisfies the mixed semicontinuity property above. Assume that there exists a function $g:[T_0,T]\times H\to\mathbb{R}_+$ of Carathéodory type which is integrably bounded on bounded subsets of $H$ and which is such that*

$$G(t,x)\cap B[0,(g(t,x)]\neq\varnothing \text{ for all } (t,x)\in[T_0,T]\times H.$$

*Then for any $\varepsilon>0$ and any compact set $K\subset C([T_0,T],H)$ there is a nonempty closed convex valued multimapping $\Phi:K\rightrightarrows L^1([T_0,T],H)$ which has a sequentially closed graph with respect to the norm (of uniform convergence) in $K$ and the weak topology $w(L^1_H,L^\infty_H)$ in $L^1([T_0,T],H)$ and which is such that for any $u\in K$ and $\varphi\in\Phi(u)$ one has for a.e. $t\in[T_0,T]$*

$$\varphi(t)\in G(t,u(t)), \tag{4.6}$$

$$\|\varphi(t)\|\le g(t,u(t))+\varepsilon. \tag{4.7}$$

**Proof of Theorem 3**.

**Step 1**. Let $u_0\in C(T_0)$. Fix $\varepsilon>0$ and consider the absolutely continuous solution $\eta(\cdot)$ of Equation (4.1) on $[T_0,T]$ with $\eta(T_0)=0$.

**Step 2** is the following proposition.

**Proposition 4** *Let the multimapping $G$ be as in Theorem 4 with $g(t,x)=\beta(t)+\gamma(t)\|x\|$, the functions $\beta(\cdot)$ and $\gamma(\cdot)$ being non negative and integrable on $[T_0,T]$. Let $\Gamma:[T_0,T]\rightrightarrows H$ be an $\mathcal{L}([T_0,T])\otimes\mathcal{B}(H)$-measurable multimapping with nonempty convex compact values and such that for each $t\in[T_0,T]$ the multimapping $\Gamma(t,\cdot)$ is upper (or outer) semicontinuous. Assume that the multimapping $\Gamma$ is integrably bounded, that is, there exists a non negative Lebesgue integrable function $\sigma$ such that*

$$\Gamma(t,x)\subset\sigma(t)\mathbb{B}_H \quad \text{for all } (t,x)\in[T_0,T]\times H.$$

*Then the differential inclusion*

$$\dot{u}(t) \in \Gamma(t, u(t)) + G(t, u(t)) \quad \textit{with } u(T_0) = u_0 \in H$$

*has a solution $u(\cdot)$ for which there is a Lebesgue measurable selection $\varphi$ of $G(\cdot, u(\cdot))$ such that $\|\varphi(t)\| \le \dot{\eta}(t)$*

$$\dot{u}(t) \in \Gamma(t, u(t)) + \varphi(t) \quad \textit{for a.e. } t \in [T_0, T]$$

*and hence $\|\dot{u}(t)\| \le \sigma(t) + \dot{\eta}(t)$ for a.e. $t \in [T_0, T]$, where $\eta(\cdot)$ is given by (4.1) and (4.2).*

**Proof.** Put $m_0(t) := \sigma(t) + \dot{\eta}(t)$ and

$$K := \left\{ \begin{array}{c} u \in C([T_0, T], H) : \; u(t) = u_0 + \int_{T_0}^t \dot{u}(s)\, ds \; \forall t \in [T_0, T]; \\ \dot{u} \in L^1([T_0, T]) \text{ and } \|\dot{u}(t)\| \le m_0(t) \text{ a.e } t \in [T_0, T] \end{array} \right\}.$$

The set $K$ is obviously convex. Further the function $m_0(\cdot)$ being integrable on $[T_0, T]$, the convex set $K$ is also compact in $C([T_0, T], H)$.

Applying Theorem 4 with $g(t, x) := \beta(t) + \gamma(t) \|x\|$ yields a nonempty closed convex valued multimapping $\Phi : K \rightrightarrows L^1([T_0, T], H)$ whose graph is sequentially closed with respect to the topology of uniform convergence in $K$ and the weak topology in $L^1([T_0, T], H)$ and such that, for any $u \in K$ and $\varphi \in \Phi(u)$, for a.e. $t \in [T_0, T]$ we have

$$\varphi(t) \in G(t, u(t)) \text{ and } \|\varphi(t)\| \le \beta(t) + \gamma(t) \|u(t)\| + \varepsilon. \tag{4.8}$$

Consider the multimapping $\Psi : K \rightrightarrows C([T_0, T], H)$ defined by

$$\Psi(u) = \left\{ \begin{array}{c} v \in C([T_0, T], H) : \; v(t) = u_0 + \int_{T_0}^t (w(s) + \varphi(s))\, ds, \forall t \in [T_0, T]; \\ w \in L^1([T_0, T]), w(t) \in \Gamma(t, u(t)) \text{ a.e. and } \varphi \in \Phi(u) \end{array} \right\}.$$

Fix any $u \in K$ and $v \in \Psi(u)$. Take by definition of $\Psi$ any $\varphi \in \Phi(u)$ and any Lebesgue measurable mapping $w$ satisfying

$$w(t) \in \Gamma(t, u(t)) \text{ and } \dot{v}(t) = w(t) + \varphi(t) \text{ for a.e. } t \in [T_0, T]. \tag{4.9}$$

We claim that

$$\|\varphi(t)\| \le \dot{\eta}(t) \text{ for a.e. } t \in [T_0, T]. \tag{4.10}$$

Indeed, by (4.8) we have for a.e. $t \in [T_0, T]$

$$\begin{aligned} \|\varphi(t)\| &\leq \varepsilon + \beta(t) + \gamma(t)\,\|u(t)\| \\ &\leq \varepsilon + \beta(t) + \gamma(t)(\|u_0\| + \int_{T_0}^{t} \|\dot{u}(s)\|\,ds), \end{aligned}$$

hence by definition of $K$

$$\|\varphi(t)\| \;\leq\; \varepsilon + \beta(t) + \gamma(t)(\|u_0\| + \int_{T_0}^{t} (\sigma(s) + \dot{\eta}(s))\,ds).$$

Using (4.1) we obtain

$$\|\varphi(t)\| \leq \dot{\eta}(t) \;\text{ for a.e. } t \in [T_0, T],$$

that is, (4.10) of the claim is proved.

Further $v(\cdot)$ is absolutely continuous and by (4.10)

$$\|\dot{v}(t)\| \leq \|\sigma(t)\| + \|\varphi(t)\| \leq \dot{\xi}(t) + \dot{\eta}(t) = m_0(t).$$

This implies the inclusion $\Psi(u) \subset K$ for each $u \in K$, i.e. , $\Psi : K \rightrightarrows K$ is a multimapping from $K$ into $K$.

Now let us prove that the graph of the multimapping $\Psi$ is upper semicontinuous. Obviously $\Psi(u)$ is convex according to the convexity of $\Phi(u)$ and of $\Gamma(t, u(t))$. Let $((u_n, v_n))_n$ be a sequence in $\mathrm{gph}\,(\Psi)$ converging to $(u, v) \in K \times K$ with respect to the topology of uniform convergence. We may write $v_n(t) = u_0 + \int_{T_0}^{t} (w_n(s) + \varphi_n(s))\,ds$ with $w_n(t) \in \Gamma(t, u_n(t))$ and $\varphi_n \in \Phi(u_n)$. Since $\|w_n(t)\| \leq \sigma(t)$, a (non relabeled) subsequence of $(w_n)$ converges $w(L^1, L^\infty)$ to a certain mapping $w(\cdot)$. Taking into account the uniform convergence of $(u_n(\cdot))_n$ to $u(\cdot)$ and the upper semicontinuity with respect to $x$ of the convex compact valued measurable multimapping $\Gamma$, by the closure theorem [5, Theorem VI.4] we obtain $w(t) \in \Gamma(t, u(t))$ for a.e. $t \in [T_0, T]$. Futhermore by (4.10) we have $\|\varphi_n(t)\| \leq \dot{\eta}(t)$ a.e. Thus a (non relabeled) subsequence of $(\varphi_n)_n$ converges $w(L^1, L^\infty)$ to some mapping $\varphi$. The graph of $\Phi$ being sequentially $\|\ \| \times w(L^1, L^\infty)$ closed in $K \times K$, we get $\varphi \in \Phi(u)$. On the other hand, it is not difficult to see that for all $t \in [T_0, T]$ we have $v(t) = u_0 + \int_{T_0}^{t} (w(s) + \varphi(s))\,ds$. This shows that $(u, v) \in \mathrm{gph}\,(\Psi)$, that is, the graph of $\Psi$ is closed and hence it is upper semicontinuous because $K$ is compact for the topology of uniform convergence.

The upper semicontinuity of the closed convex valued multimapping $\Psi$ of the convex compact set $K$ into itself allows us to apply the Kakutani fixed point theorem, which yiels a certain $u \in K$ satisfying $u \in \Psi(u)$. This inclusion along with (4.10) translates the conclusions of the proposition. $\square$

We continue the proof of Theorem 3 with Step 3.

<u>**Step 3**</u>. Put $m(t) := |\dot{\zeta}(t)| + \dot{\eta}(t)$ and note that for each $t$ the multimapping $-\dfrac{1}{\alpha^2} m(t)\, \partial^C d(\cdot, C(t))$ is upper semicontinuous (as recorded in section 2 for the Clarke subdifferential of locally Lipschitzian functions) with nonempty convex compact values. Further, it not hard to verify, through the expression of the support function (see (2.5)), that the multimapping associating with $(t, x)$ the set $-\dfrac{1}{\alpha^2} m(t)\, \partial^C d(x, C(t))$ is $\mathcal{L}([T_0, T]) \otimes \mathcal{B}(H)$-measurable. We may then apply Proposition 4 with $\Gamma(t, x) = -\dfrac{1}{\alpha^2} m(t)\, \partial^C d(x, C(t))$, $\sigma(t) = \dfrac{1}{\alpha^2} m(t)$, and $-F(t, x)$ in place of $G(t, x)$. Thus there exist an absolutely continuous mapping $u(\cdot)$ from $[T_0, T]$ into $H$ and a measurable selection $\varphi(\cdot)$ of $F(\cdot, u(\cdot))$ with $\|\varphi(t)\| \le \dot{\eta}(t)$ for a.e. $t \in [T_0, T]$ and such that

$$-\dot{u}(t) \in \frac{1}{\alpha^2} m(t)\, \partial^C d(u(t), C(t)) + \varphi(t) \ \text{ a.e. and } u(T_0) = u_0 \in C(T_0).$$

Since $m(t) \ge |\dot{\zeta}(t)| + \|\varphi(t)\|$ a.e., by Theorem 2 the mapping $u(\cdot)$ is a solution of the differential inclusion (4.4) and for a.e. $t \in [T_0, T]$ one has

$$\|\dot{u}(t)\| \le \frac{1}{\alpha^2}(|\dot{\zeta}(t)| + \dot{\eta}(t)) + \dot{\eta}(t).$$

This completes the proof of Theorem 3. $\square$

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