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NECESSARY OPTIMALITY CONDITIONS IN MULTIOBJECTIVE DYNAMIC OPTIMIZATION*

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Abstract. We consider a nonsmooth multiobjective optimal control problem related to a general preference. Both differential inclusion and endpoint constraints are involved. Necessary conditions and Hamiltonian necessary conditions expressed in terms of the limiting Fréchet subdifferential are developed. Examples of useful preferences are given.

Key words. multiobjective optimal control, necessary conditions, Hamiltonian necessary conditions, preference, utility function, differential inclusions

AMS subject classifications. 49K24, 90C29

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1. Introduction.

This paper is mainly concerned with the following multiobjective dynamic optimization problem with the dynamic governed by a differential inclusion:

$$\text{(P)} \qquad \begin{aligned} &\min f(x(a), x(b)),\\ &(x(a), x(b)) \in S,\\ &\dot{x}(t) \in F(t, x(t)) \quad \text{a.e.} \quad t \in [a, b], \end{aligned}$$

where $f\colon \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^m$ is a mapping, $S \subset \mathbb{R}^n \times \mathbb{R}^n$ is a closed nonempty set, and $F\colon [a, b] \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is a closed-valued multivalued mapping which is measurable in $t \in [a, b]$.

These problems naturally arise, for example, in economics (economic growth models) (see [16] and references therein), in chemical engineering (polymerization processes) (see [3], [4], and references therein), and in multiobjective control design (see [45], [9], and references therein). Problems considered in this paper use preferences determined by cones (Pareto and weak Pareto optimum), use preferences determined by utility function, or use the concept of Nash equilibrium.

Our aim in this paper is to use a general preference including the previous ones in order to state necessary and Hamiltonian necessary conditions for multiobjective optimal control problems (P).

The concept of preference appeared in the value theory in economics. Many authors in the early studies often defined the preference by a utility function, i.e., given a preference whether it is always possible to find a utility function that can determine the preference.

In [17] the author proved that a preference $\prec$ can be determined by a continuous utility function if and only if for any x the sets

$$\{y : x \prec y\} \quad \text{and} \quad \{y : y \prec x\} \quad \text{are closed.} \tag{1}$$

This theorem is not general and besides this it is an existence theorem (i.e., does provide methods for determing a utility function), and there are some useful preferences

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that do not satisfy (1) (like the preference determined by lexicographical order). There are different approaches and various results on necessary conditions for (P). Several researches have been devoted to the weak Pareto solution and its generalization (see [5], [11], [15], [32], [42], [46], [47], and references therein). Other research gets refinements of necessary optimality conditions for real-valued objective optimal control problems (see [23], [29], [30], [31], [44], [43], and [26]) or Hamiltonian necessary conditions (see [37], [19], [20], [13], [35], [36], [39], [48], and [49]).

These results are expressed in terms of various generalized derivatives including Clarke's generalized subgradient [11], a limiting subgradient which is also known under other names: limiting subgradient set in [12], approximate subdifferential in Ioffe [22], subdifferential in Mordukhovich [36], and subgradient set in the general sense in Rockafellar [40]. Most of these results are obtained for Lipschitz, integrably sub-Lipschitz, bounded, or unbounded differential inclusions.

In [23], Ioffe used results of [40] and [24] to obtain general necessary optimality conditions and Hamiltonian optimality conditions for single-objective optimal control problems.

In [49], Zhu used recent progress in nonsmooth analysis, in particular calculus for smooth subdifferentials of lower semicontinuous (l.s.c.) functions (see [6], [7], [14], [24]), the methods for proving the extremal principle (see [27], [28], [33], [38]), and techniques in handling the Hamiltonian for a differential inclusion, to prove Hamiltonian necessary conditions that extend the classical Hamiltonian necessary conditions for optimal control problems that had previously been derived for uniformly Lipschitz, bounded, and convex-valued differential inclusions related to a general preference. The obtained conditions are expressed in terms of Clarke's generalized gradient which is larger than the limiting Fréchet subdifferential. The regularity conditions ($A3$) imposed in [49], which use the usual limiting normal cone, are too strong to include the preference defined by a utility function (see Example 3).

In this paper we propose a different approach. We introduce a definition of regularity modified from that introduced in [49]. To solve the problem of regularity of preference determined by a utility function, we define a larger limiting normal cone to replace the usual one in [49]. Under our regularity condition of the general preference and a sub-Lipschitz property of multivalued mappings, introduced by Loewen and Rockafellar in [29], we obtain Euler–Lagrange necessary optimality conditions for multiobjective optimal control problems with nonconvex differential inclusion constraints in terms of the limiting Fréchet subdifferential. Necessary optimality conditions for the weak Pareto solution and its generalization can be derived and refined by using our necessary conditions.

Our main result extends the necessary optimality condition of Ioffe (see Theorem 1 in [23]) from a single objective optimal control of differential inclusion problem to a multiobjective one. This is also an extension of the Hamiltonian necessary optimality conditions for convex differential inclusions obtained in [49].

The paper is organized as follows. Section 2 contains the key definitions, normals, subgradients, and coderivatives used in what follows. In section 3 we state our main result and establish necessary optimality conditions for multiobjective control problems with some examples and discussions. Then we derive necessary conditions for these examples of preferences. In section 4 we give a technical proof of the main result.

2. Background. Now we state basic tools of generalized differentiation that are more appropriate for our main purpose. Details may be found in [33].

Let C be a closed subset of $\mathbb{R}^n$ containing some point c. The ε-normal cone to C at c is the set

$$\hat{N}_\varepsilon(C,c) := \left\{\zeta \in \mathbb{R}^n \ : \ \liminf_{x\in C\to c} \frac{\langle -\zeta, x-c\rangle}{\|x-c\|} \geq -\varepsilon\right\}.$$

The normal cone to C at c is the set

$$N(C;c) := \limsup_{\substack{x\in C\to c\\ \varepsilon\to 0^+}} \hat{N}_\varepsilon(C,c).$$

Now let $f : \mathbb{R}^n \longrightarrow \mathbb{R}\cup\{\infty\}$ be an l.s.c. function, and let $c \in \mathbb{R}^n$ such that $f(c) < \infty$. The limiting Fréchet subdifferential of f at c is the set

$$\partial f(c) = \{\zeta \in \mathbb{R}^n \ : \ (\zeta, -1) \in N(epi\, f; (c, f(c)))\},$$

where $epi\, f$ denotes the epigraph of f. We have the following analytic characterization of $\partial f(c)$:

$$\partial f(c) = \limsup_{\substack{x\to c\\ f(x)\to f(c)\\ \varepsilon\to 0^+}} \partial_\varepsilon f(x),$$

where

$$\partial_\varepsilon f(x) = \left\{x^* \in X^* \ : \ \liminf_{h\to 0} \frac{f(x+h) - f(x) - \langle x^*, h\rangle}{\|h\|} \geq -\varepsilon\right\}.$$

The singular subdifferential of f at c is the set

$$\partial^\infty f(c) = \{\zeta \in \mathbb{R}^n \ : \ (\zeta, 0) \in N(epi\, f; (c, f(c)))\}.$$

Next we consider a multivalued mapping F from $\mathbb{R}^n$ to $\mathbb{R}^m$ of the closed graph

$$\text{Gr}F := \{(x,y) : \ y \in F(x)\}.$$

The multivalued mapping $D^*F(x,y) : \mathbb{R}^m \mapsto \mathbb{R}^n$ defined by

$$D^*F(x,y)(y^*) := \{x^* \in \mathbb{R}^n : \ (x^*, -y^*) \in N(\text{Gr}F; (x,y))\}$$

is called the coderivative of F at the point $(x,y) \in \text{Gr}F$.

The domain over which our study occurs is typically one of the functions $W^{1,1}([a,b],\mathbb{R}^n)$ (abbreviated $W^{1,1}$) consisting of all absolutely continuous functions $x\colon [a,b] \mapsto \mathbb{R}^n$ for which $|\dot{x}|$ is integrable on $[a,b]$ ($\dot{x}$ denotes the derivative (a.e.) of x). An *arc* is a function in $W^{1,1}$. The space $W^{1,1}$ is endowed with the norm

$$\|x\| = |x(a)| + \int_a^b |\dot{x}(t)|dt,$$

where $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^n$. Here $\mathbb{B}$ stands for the closed unit ball in $\mathbb{R}^n$ and

$$B(z,r) = \{x \in W^{1,1} : \ \|x - z\| \leq r\}.$$

The distance function on $W^{1,1}$, $\mathbb{R}^n$ or $\mathbb{R}^n \times \mathbb{R}^n$ will be denoted by $d(\cdot\,,\cdot)$. The convex hull and the closed convex hull are denoted by co and $\bar{\text{co}}$, respectively.

The following lemma is needed.

LEMMA 2.1. *Let G be pseudo-Lipschitzian* [1], [41] *around $(x_0, y_0) \in \mathrm{Gr}G$ with modulus K; i.e., there exists $r > 0$ such that for all $x, u \in x_0 + r\mathbb{B}$*

$$G(x) \cap (y_0 + r\mathbb{B}) \subset G(u) + K|x - u|\mathbb{B}.$$

Then for all $y^* \in \mathbb{R}^n$, with $D^*G(x_0, y_0)(y^*) \neq \emptyset$, one has

$$\sup\{|x^*| : \quad x^* \in D^*G(x_0, y_0)(y^*)\} \leq K|y^*|.$$

If in addition G is closed-valued, then for all $(x, y) \in (x_0 + \frac{r}{12}\mathbb{B}) \times (y_0 + \frac{r}{12}\mathbb{B})$, with $(x, y) \notin \mathrm{Gr}G$, and all $(x^*, y^*) \in \partial d(\cdot\,; G(\cdot))(x, y)$ we have

$$|y^*| = 1 \text{ and } |x^*| \leq K|y^*|.$$

Proof. It suffices to establish the second part; the first one follows from the definition of limiting Fréchet subdifferential. Let $(x, y) \in (x_0 + \frac{r}{12}\mathbb{B}) \times (y_0 + \frac{r}{12}\mathbb{B})$, with $(x, y) \notin \mathrm{Gr}G$, and let $(x^*, y^*) \in \partial d(\cdot; G(\cdot))(x, y)$. Then there are sequences $x_k \to x$, $y_k \to y$, $x_k^* \to x^*$, $y_k^* \to y^*$, $\varepsilon_k \to 0^+$, and $r_k \to 0^+$ such that

$$d(v; G(u)) - d(y_k; G(x_k)) - \langle x_k^*, u - x_k\rangle - \langle y_k^*, v - y_k\rangle + \varepsilon_k[|u - x_k| + |v - y_k|] \geq 0$$

for all $u \in x_k + r_k\mathbb{B}$ and $v \in y_k + r_k\mathbb{B}$. For each integer k, there exists $v_k \in G(x_k)$ such that

$$d(y_k; G(x_k)) = |y_k - v_k|.$$

So

$$|y' - v| - |y_k - v_k| - \langle x_k^*, u - x_k\rangle - \langle y_k^*, v - y_k\rangle + \varepsilon_k[|u - x_k| + |v - y_k|] \geq 0$$

for all $u \in x_k + r_k\mathbb{B}$, $v \in y_k + r_k\mathbb{B}$, and $y' \in G(u)$.

Consider the function g defined by

$$g(u, y', v) = |y' - v| - \langle x_k^*, u - x_k\rangle - \langle y_k^*, v - y_k\rangle + \varepsilon_k[|u - x_k| + |v - y_k|].$$

Then [34]

$$(0, 0, 0) \in \partial g(x_k, v_k, y_k) + N(\mathrm{Gr}G; (x_k, v_k)) \times \{0\}.$$

As for k large enough $y_k \neq v_k$, then

$$\partial g(x_k, v_k, y_k) \subset \{(0, v^*, -v^*) \colon |v^*| = 1\} + (-x_k^*, 0, -y_k^*) + \varepsilon_k\mathbb{B} \times \{0\} \times \varepsilon_k\mathbb{B},$$

and hence we obtain $(u_k^*, v_k^*) \in N(\mathrm{Gr}G; (x_k, v_k))$, with $|v_k^*| = 1$, such that

$$|x_k^* - u_k^*| \leq \varepsilon_k \text{ and } |y_k^* - v_k^*| \leq \varepsilon_k.$$

Now since $d(y_k; G(x_k))) = |y_k - v_k|$, we get for k sufficiently large

$$|y_k - v_k| \leq \frac{r}{2},$$

and hence

$$|x_0 - x_k| + |y_0 - v_k| \leq \frac{5r}{6}.$$

Thus for all $u, u' \in x_k + \frac{r}{6}\mathbb{B}$

$$G(u) \cap \Big(v_k + \frac{r}{6}\mathbb{B}) \subset G(u'\Big) + K|u - u'|\mathbb{B}.$$

So the first part of the lemma ensures that

$$|u_k^*| \leq K|v_k^*|,$$

and since $u_k^* \to x^*$ and $v_k^* \to y^*$ we get $|x^*| \leq K|y^*|$, and the proof is complete.

LEMMA 2.2. *Let $G\colon V \mapsto \mathbb{R}^m$ be a multivalued mapping, where V is a nonempty set in $\mathbb{R}^n$. Suppose that*

(i) GrG *is closed and*

(ii) *there exists a compact set K in $\mathbb{R}^m$ such that*

$$G(x) \subset K \quad \forall x \in V.$$

Then G is upper semicontinuous (u.s.c.) on V; that is, for all $u \in V$ and all $\varepsilon > 0$ there exists a neighborhood U of u in V such that

$$G(x) \subset G(u) + \varepsilon\mathbb{B} \quad \forall x \in U.$$

With the help of the last lemma, we can prove the following one.

LEMMA 2.3. *Suppose that the mapping $f\colon (x_0, y_0) + r\mathbb{B} \mapsto \mathbb{R}$ is Lipschitzian with constant K. Define the multivalued mapping $\Gamma\colon (x_0, y_0) + r\mathbb{B} \times \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$ by*

$$\Gamma(x, y, p, s) = co\{q\colon (q, p) \in \partial f(x, y) + s\mathbb{B}\}.$$

Then for all $\lambda \in]0, 1[$, all $(x, y, s) \in (x_0, y_0, 0) + \lambda r\mathbb{B}$, and all $p \in \mathbb{R}^n$, with $\Gamma(x, y, p, s) \neq \emptyset$, Γ is u.s.c. at (x, y, p, s) in the sense of Lemma 2.2.

Proof. Note that (ii) of Lemma 2.2 is satisfied. It is not difficult to show that Γ is of closed graph and to apply Lemma 2.2.

LEMMA 2.4 (see [11]). *Let $\varepsilon > 0$ and $\Gamma : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$ be a multivalued mapping such that for almost all $t \in [a, b]$, $\Gamma(t, \cdot)$ has nonempty, compact, and convex values around $(z(t), \dot{z}(t), p, s)$, with $s \in [0, \varepsilon]$ and $\Gamma(t, z(t), \dot{z}(t), p, s) \neq \emptyset$. For sequences (z_k) and (p_k) in $W^{1,1}$, (ϕ_k) in $L^1([a, b],]0, +\infty[)$, (α_k) and (s_k) in $\mathbb{R}_+$ with $z_k \to z$ in $W^{1,1}$, $\phi_k \to \phi$ in $L^1([a, b],]0, +\infty[)$ for some integrable function ϕ, $\alpha_k \to 0$, and $s_k \to 0$ we suppose the following:*

(i) *For every (x, y, p, s) in the interior of the set*

$$\{(x', y', p', s') : t \in [a, b],\, x' \in z(t) + \varepsilon\mathbb{B},\, y' \in \dot{z}(t) + \varepsilon\mathbb{B},$$
$$s' \in [0, \varepsilon],\, \Gamma(t, x', y', p', s') \neq \emptyset\}$$

the multivalued mapping $t' \mapsto \Gamma(t', x, y, p, s)$ is measurable.

(ii) *For all k, $|\dot{p}_k(t)| \leq \phi_k(t)$ for almost all $t \in [a, b]$.*

(iii) *For all k, $\dot{p}_k(t) \in \Gamma(t, z_k(t), \dot{z}_k(t), p_k(t), s_k) + \alpha_k\mathbb{B}$ a.e. $t \in [a, b]$.*

(iv) *For almost all $t \in [a, b]$ for every $p \in \mathbb{R}^n$ with $\Gamma(t, z(t), \dot{z}(t), p, 0) \neq \emptyset$, the multivalued mapping $(x', y', p', s') \mapsto \Gamma(t, x', y', p', s')$ is u.s.c. at $(z(t), \dot{z}(t), p, 0)$.*

(v) *The sequence $(p_k(a))$ is bounded.*

(vi) *There exists an integrable function ψ such that*

$$\sup_{\{(p', s') : s' \in [0, \varepsilon],\ \Gamma(t, z(t), \dot{z}(t), p', s') \neq \emptyset\}} \ \max_{y \in \Gamma(t, z(t), \dot{z}(t), p', s')} |y| \leq \psi(t) \ a.e.$$

Then there is a subsequence of (p_k) *which converges uniformly to an arc* p *satisfying*

$$\dot{p}(t) \in \Gamma(t, z(t), \dot{z}(t), p(t), 0) \quad a.e.\ t \in [a, b].$$

We conclude this section by recalling necessary optimality conditions for the following generalized problem of Bolza:

$$(\mathrm{P_B}) \qquad \min \left\{ \ell(x(a), x(b)) + \int_a^b L(t, x(t), \dot{x}(t))\, dt \right\},$$

where the functions $L : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R} \cup \{+\infty\}$ and $\ell : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R} \cup \{+\infty\}$ are such that for each $t \in [a, b]$, the functions $L(t, \cdot, \cdot)$ and ℓ are l.s.c. on $\mathbb{R}^n \times \mathbb{R}^n$.

The function L is *epi-Lipschitz* [10] at an arc z if there exist an integrable function $k : [a, b] \mapsto \mathbb{R}$ and a positive ε satisfying the following conditions: for almost all $t \in [a, b]$, given two points z_1 and z_2 within ε of $z(t)$ and $u_1 \in \mathbb{R}^n$ such that $L(t, z_1, u_1)$ is finite, there exist a point $u_2 \in \mathbb{R}^n$ and $\delta \geq 0$ such that $L(t, z_2, u_2)$ is finite and

$$|u_1 - u_2| + |L(t, z_1, u_1) - L(t, z_2, u_2) - \delta| \leq k(t)|z_1 - z_2|.$$

This is equivalent to saying that the multivalued mapping

$$E(t, s) = \{(u, r) \in \mathbb{R}^n \times \mathbb{R} : L(t, s, u) \leq r\}$$

is Lipschitzian in s on $z(t) + \varepsilon\mathbb{B}$ with constant $k(t)$ (i.e., for all $s, s' \in z(t) + \varepsilon\mathbb{B}$ we have $E(t, s') \subset E(t, s) + k(t) \mid s' - s \mid \mathbb{B}$).

L is said to be epimeasurable (in t) [10] if for each $s \in \mathbb{R}^n$ the multivalued mapping $E(t, s)$ is Lebesgue measurable in t.

The notation ∂L will denote the limiting Fréchet subdifferential of the function $L(t, \cdot, \cdot)$.

Now we may state a variant of the necessary conditions for the generalized Bolza problem established in Jourani [26].

THEOREM 2.1. *Let* z *solve locally the generalized problem of Bolza* $(\mathrm{P_B})$ *(in* $W^{1,1}$*). Suppose that* $L(t, z, u)$ *is epimeasurable in* t*, and* $L(t, \cdot, \cdot)$ *is epi-Lipschitzian at* z*, and* ℓ *is locally Lipschitzian around* $(z(a), z(b))$*. Then there exists an arc* p *such that one has*

$$\dot{p}(t) \in co\{q : \quad (q, p(t)) \in \partial L(t, z(t), \dot{z}(t))\} \quad a.e.\ t \in [a, b],$$

$$(p(a), -p(b)) \in \partial \ell(z(a), z(b)),$$

$$\langle p(t), \dot{z}(t) \rangle - L(t, z(t), \dot{z}(t)) = \max\{\langle p(t), v \rangle - L(t, z(t), v) :\ v \in \mathbb{R}^n\}.$$

3. The main result.

DEFINITION 3.1. F *is said to be sub-Lipschitzian in the sense of Loewen and Rockafellar* [29] *at* z *if there exist* $\beta > 0$*,* $\varepsilon > 0$*, and a summable function* $k : [a, b] \mapsto \mathbb{R}$ *such that for almost all* $t \in [a, b]$*, for all* $N > 0$*, for all* $x, x' \in z(t) + \varepsilon\mathbb{B}$*, and* $y \in \dot{z}(t) + N\mathbb{B}$ *one has*

$$d(y, F(t, x)) - d(y, F(t, x')) \leq (k(t) + \beta N)|x - x'|.$$

Let $\prec$ be a (nonreflexive) preference for vectors in $\mathbb{R}^m$. We consider the following multiobjective optimization problem:

$$\text{(P)} \qquad \begin{array}{c} \min f(x(a), x(b)), \\ (x(a), x(b)) \in S, \\ \dot{x}(t) \in F(t, x(t)) \quad \text{a.e.} \quad t \in [a, b], \end{array}$$

where $f\colon \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^m$ is a mapping, $S \subset \mathbb{R}^n \times \mathbb{R}^n$ is a closed nonempty set, and $F\colon [a, b] \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is a closed-valued multivalued mapping which is measurable in $t \in [a, b]$.

We say that an arc $x \in W^{1,1}$ is a feasible trajectory for problem (P) if x satisfies $(x(a), x(b)) \in S$ and $\dot{x}(t) \in F(t, x(t))$ a.e. $t \in [a, b]$.

z is a solution to (P), provided that it is feasible and there does not exist any feasible trajectory x of (P) such that $f(x(a), x(b)) \prec f(z(a), z(b))$. For all $r \in \mathbb{R}^m$, we denote

$$\mathcal{L}(r) := \{s \in \mathbb{R}^m \,:\, s \prec r\}.$$

We will need the following regularity assumptions on the preference modified from [49].

DEFINITION 3.2. *We say that a preference $\prec$ is regular at $r \in \mathbb{R}^m$, provided that*

(A_1) for any $s \in \mathbb{R}^m$, $s \in \mathrm{cl}\mathcal{L}(s)$;

(A_2) for any $r \prec s$, $t \in \mathrm{cl}\mathcal{L}(r)$ implies that $t \prec s$.

Remark 3.1. The preference determined by the lexicographical order $\prec$ is defined by $r \prec s$ if there exists an integer $q \in \{0, 1, \dots, m-1\}$ such that $r_i = s_i,\ i = 1, \dots, q$, and $r_{q+1} < s_{q+1}$. This preference is not regular. Indeed we consider in $\mathbb{R}^3$ the vectors $r = (1, 1, 3)$, $s = (1, 1, 5)$, and $t = (1, 1, 6)$. We have $r \prec s$ and $t \in \mathrm{cl}\mathcal{L}(r)$, but $s \prec t$; then (A_2) does not hold so that $\prec$ is not regular at r.

Note that a preference determined by the lexicographical order does not correspond to any real utility function [16].

Remark 3.2. Our definition of regularity is different from that given by Zhu in [49], where the following third condition is in force: for any sequences $r_k, \theta_k \mapsto r$ in $\mathbb{R}^m$

$$\limsup_{k \to +\infty} N(\mathrm{cl}\mathcal{L}(r_k); \theta_k) \subset N(\mathrm{cl}\mathcal{L}(r); r).$$

But with this condition, preferences defined by a utility function (e.g., u) are not regular at any $r \in \mathbb{R}^m$ even if

$$\lim_{s \to r} d(0, \partial u(s)) > 0.$$

For more details, see Example 3.

We consider the following enlargement cone of the limiting Fréchet normal cone:

$$\tilde{N}(\mathrm{cl}\mathcal{L}(x), x) = \limsup_{y, x' \to x} N(\mathrm{cl}\mathcal{L}(y); x').$$

Before stating our main result we recall that the Hamiltonian associated with F is defined by

$$H(t, x, y) = \sup_{v \in F(t,x)} \langle y, v \rangle.$$

THEOREM 3.1. *Let z be a local solution to the multiobjective optimal control problem* (P). *Suppose that F is sub-Lipschitzian at z and that the preference $\prec$ is regular at $f(z(a), z(b))$. Then there exist $p \in W^{1,1}$, $\lambda \geq 0$, and $w \in \tilde{N}(\mathrm{cl}\mathcal{L}(f(z(a), z(b))), f(z(a), z(b)))$, with $|\omega| = 1$ such that $(\lambda, p) \neq 0$ and*

$$\dot{p}(t) \in coD^*F(t, z(t), \dot{z}(t))(-p(t)) \ \ a.e. \ \ t \in [a, b]; \tag{2}$$

$$(p(a), -p(b)) \in \lambda\partial(\langle\omega, f(\cdot, \cdot)\rangle)(z(a), z(b)) + N(S; (z(a), z(b))); \tag{3}$$

$$\langle p(t), \dot{z}(t)\rangle = H(t, z(t), p(t)) \ \ a.e. \ \ t \in [a, b]. \tag{4}$$

If in addition F is convex-valued, then (2) *may be replaced by the following one:*

$$\dot{p}(t) \in co \ \{q : \ (-q, \dot{z}(t)) \in \partial H(t, (z(t), p(t)))\} \quad a.e. \ t \in [a, b]. \tag{5}$$

The aim of Theorem 3.1 is to extend the necessary optimality conditions of Ioffe (Theorem 1 in [23]) from a single objective optimal control of differential inclusion problem to a multiobjective one. By using the large class of sub-Lipschitz differential inclusion, Theorem 3.1 also extends the Hamiltonian necessary optimality conditions for convex-valued differential inclusions obtained in [49].

In the remainder of this section we now examine a few examples. The proof of Theorem 3.1 is postponed to the next section.

Example 1 (a generalized Pareto optimal). Let K be a pointed convex cone ($K \cap (-K) = \{0\}$). We define the preference $\prec$ by $r \prec s$ if and only if $r - s \in K$ and $r \neq s$. A multiobjective optimal control problem with this preference is called a generalized Pareto optimal control problem. Notice that if $K = \mathbb{R}^m_-$ (resp., $K = \mathrm{int}\,\mathbb{R}^m_-$, where $\mathbb{R}^m_- = \{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : x_i \leq 0 \text{ for all } i = 1, \ldots, m\}$) we get Pareto (resp., weak Pareto) optimal control problems. This preference is regular at any $r \in \mathbb{R}^m$. Moreover, for any $r \in \mathbb{R}^m$ we have $\tilde{N}(\mathrm{cl}\mathcal{L}(r), r) = K^0$ with $K^0 = \{s \in \mathbb{R}^m : \langle s, q\rangle \leq 0 \ \text{ for all } \ q \in K\}$.

COROLLARY 3.1. *Let z be a local solution to the generalized Pareto multiobjective optimal control problem* (P). *Then there exist $p \in W^{1,1}$, $\lambda \geq 0$, and $\omega \in K^0$ with $|\ \omega\ |= 1$ such that $(\lambda, p) \neq 0$ and*

$$\dot{p}(t) \in coD^*F(t, z(t), \dot{z}(t))(-p(t)) \ \ a.e. \ \ t \in [a, b]; \tag{6}$$

$$(p(a), -p(b)) \in \lambda\partial(\langle\omega, f(\cdot, \cdot)\rangle)(z(a), z(b)) + N(S; (z(a), z(b))); \tag{7}$$

$$\langle p(t), \dot{z}(t)\rangle = H(t, z(t), p(t)) \ \ a.e. \ \ t \in [a, b]. \tag{8}$$

Example 2 (a preference determined by a utility function). Let u be a continuous function; we define the preference $\prec$ determined by utility function u by $r \prec s$ if and only if $u(r) < u(s)$.

LEMMA 3.1. *Let u be a continuous utility function determining the preference $\prec$. Suppose that*

$$\liminf_{s \to r} d(0, \partial u(s)) > 0. \tag{9}$$

Then the preference $\prec$ is regular at r and

$$\tilde{N}(\mathrm{cl}\mathcal{L}(r), r) = \limsup_{r' \to r} N(\mathrm{cl}\mathcal{L}(r'); r') = \partial^\infty u(r) \bigcup \Big(\bigcup_{a>0} a\partial u(r) \Big).$$

Proof. The proof of Lemma 3.1 is similar to that given in [49]. From (9), $\mathcal{L}(r)$ is nonempty, and from the continuity of u it follows that $\prec$ satisfies (A_1) and (A_2) in Definition 3.2, and thus $\prec$ is regular. Now for $r^{'}$ sufficiently close to r, $\mathrm{cl}\mathcal{L}(r^{'}) = \{s \in \mathbb{R}^m \,:\, u(s) - u(r^{'}) \leq 0\}$. Then

$$\partial_\varepsilon u(r^{'}) \subset \hat{N}_\varepsilon(\mathrm{cl}\mathcal{L}(r^{'}), r^{'}).$$

By passing to the limits we have

$$\partial^\infty u(r) \bigcup \Big(\bigcup_{a>0} a\partial u(r) \Big) \subset \limsup_{r' \to r} N(\mathrm{cl}\mathcal{L}(r'); r') \subset \tilde{N}(\mathrm{cl}\mathcal{L}(r), r).$$

Conversely, let $\zeta \in \tilde{N}(\mathrm{cl}\mathcal{L}(r), r)$ such that $\zeta \neq 0$. Then there are sequences $\zeta_k \to \zeta$, $r_k, r_k^{'} \to r$ such that $\zeta_k \in N(\mathrm{cl}\mathcal{L}(r_k); r_k^{'})$. By the definition of limiting Fréchet normal cone, we may assume that $\zeta_k \in \hat{N}_{\varepsilon_k}(\mathrm{cl}\mathcal{L}(r_k), r_k^{'})$. We must have $u(r_k) = u(r_k^{'})$. Indeed, $\hat{N}_{\varepsilon_k}(\mathrm{cl}\mathcal{L}(r_k), r_k^{'}) = \{0\}$ when $u(r_k^{'}) < u(r_k)$ and is empty when $u(r_k^{'}) > u(r_k)$. Then $\hat{N}_{\varepsilon_k}(\mathrm{cl}\mathcal{L}(r_k), r_k^{'}) = \hat{N}_{\varepsilon_k}(\mathrm{cl}\mathcal{L}(r_k), r_k)$. From $\hat{N}_{\varepsilon_k}(\mathrm{cl}\mathcal{L}(r_k), r_k) = \hat{N}_{\varepsilon_k}(\{s \,:\, u(s) - u(r_k) \leq 0\}, r_k)$ and [8], there exist $a_k > 0$ and $\theta_k \in \partial_{\varepsilon_k} u(r)$ such that $\mid a_k\theta_k - \zeta_k \mid < \frac{1}{k}$ so that

$$\lim_{k\to\infty} a_k\theta_k = \zeta.$$

We claim that (a_k) is bounded. Indeed, suppose the contrary. Then (a_k) has a subsequence going to infinity. But in this case (θ_k) must have a subsequence converging to zero, and this contradicts (9). So (a_k) is bounded, and we can assume that $a_k \to a$. If $a \neq 0$, then $\zeta \in a\partial u(r)$. If $a = 0$, then $\zeta \in \partial^\infty u(r)$, and the proof is complete.

From Lemma 3.1 and Theorem 3.1 we have the following corollary.

COROLLARY 3.2. *Let $\prec$ be a preference determined by a utility function u and z be a local solution to the multiobjective optimal control problem* (P)*. Suppose that*

$$\liminf_{s \to f(z(a), z(b))} d(0, \partial u(s)) > 0.$$

Then there exist $p \in W^{1,1}$, $\lambda \geq 0$, and

$$\omega \in \partial^\infty u(f(z(a), z(b))) \bigcup \Big(\bigcup_{a>0} a\partial u(f(z(a), z(b))) \Big)$$

with $\mid \omega \mid = 1$ such that $(\lambda, p) \neq 0$, and

$$\dot{p}(t) \in coD^*F(t, z(t), \dot{z}(t))(-p(t)) \;\; a.e. \;\; t \in [a, b]; \tag{10}$$

$$(p(a), -p(b)) \in \lambda\partial(\langle \omega, f(\cdot, \cdot)\rangle)(z(a), z(b)) + N(S; (z(a), z(b))); \tag{11}$$

$$\langle p(t), \dot{z}(t)\rangle = H(t, z(t), p(t)) \;\; a.e. \;\; t \in [a, b]. \tag{12}$$

In [49], the author showed that, for a preference $\prec$ defined by a continuous utility function u, $N(\mathrm{cl}\mathcal{L}(r);r) = \partial^\infty u(r) \bigcup \big(\bigcup_{a>0} a\partial u(r)\big)$, provided that $\lim_{s\to r} d(0, \partial u(s)) > 0$. This could give him the regularity and the explicit shape of $N(\mathrm{cl}\mathcal{L}(r);r)$. But there is a gap in the proof. The following example shows that Zhu's regularity does not hold.

Example 3. Consider the function $u : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$u(x,y) = \mid x \mid - \mid y \mid .$$

Then u is Lipschitz continuous and satisfies $\partial u(0,0) = [-1,1] \times \{-1,1\}$, so that $(0,0) \notin \partial u(0,0)$, $\partial^\infty u(0,0) = \{(0,0)\}$, and

$$N(\mathrm{cl}\mathcal{L}(0,0);(0,0)) = \{(x,y) \in \mathbb{R}^2 \, : \, |y| = |x|\}.$$

Then it is clear that

$$N(\mathrm{cl}\mathcal{L}(0,0);(0,0)) \neq \partial^\infty u(0,0) \bigcup \Big(\bigcup_{a>0} a\partial u(0,0) \Big).$$

4. Proof of Theorem 3.1. Since F is sub-Lipschitzian at z there exist $\beta > 0$, $\varepsilon > 0$, and a summable function $k : [a,b] \mapsto \mathbb{R}$ such that for almost all $t \in [a,b]$, for all $N > 0$, for all $x, x' \in z(t) + \varepsilon\mathbb{B}$, and $y \in \dot{z}(t) + N\mathbb{B}$ one has

$$d(y, F(t,x)) - d(y, F(t,x')) \leq (k(t) + \beta N)|x - x'|.$$

Let G be the solution set of the system

$$\dot{x}(t) \in F(t, x(t)) \, a.e., \;\; (x(a), x(b)) \in S. \tag{13}$$

Let ε be as above. We say that the system (13) is seminormal [25] at z if there exist $\alpha > 0$ and $r > 0$ such that for all $x \in B(z,r)$

$$d(x, G \cap B(z,\varepsilon)) \leq \alpha \left\{ d((x(a), x(b)); S) + \int_a^b d(\dot{x}(t); F(t, x(t)))dt \right\}. \tag{14}$$

Set $G_\varepsilon = G \cap B(z,\varepsilon)$.

We divide the proof into two parts and each part is divided into two steps.

Part 1 (when system (13) is not seminormal at z). The proof of this part is similar to that given in [23].

Step 1 (*application of Ekeland's variational principle* [18] *and Theorem* 2.1). Consider the function h defined by

$$h(x) = d((x(a), x(b)); S) + \int_a^b d(\dot{x}(t); F(t, x(t)))dt.$$

Since F is sub-Lipschitzian at z, then h is l.s.c. on the set $B(z,\varepsilon)$ and G_ε is closed (see the appendix). If system (13) is not seminormal at z, then there is a sequence $x_k \to z$ in $W^{1,1}$ such that for k large enough

$$d(x_k, G_\varepsilon) > kh(x_k).$$

Set $\varepsilon_k = \sqrt{h(x_k)} > 0$, $\lambda_k = \min(\varepsilon_k, k\varepsilon_k^2)$, and $s_k = \frac{\varepsilon_k^2}{\lambda_k}$. Then $\varepsilon_k \to 0^+$ and $s_k \to 0^+$. Therefore one has

$$h(x_k) \leq \inf_{x \in B(z,\varepsilon)} h(x) + \varepsilon_k^2.$$

By Ekeland variational principle we get $z_k \in B(z, \varepsilon)$ satisfying

$$\|z_k - x_k\| < \lambda_k, \tag{15}$$

$$h(z_k) \le h(x) + s_k \|x - z_k\| \quad \forall x \in B(z, \varepsilon). \tag{16}$$

Observe that for k sufficiently large $\| z_k - z \| \le \frac{\varepsilon}{2}$. By the closedness of G_ε and relation (15) $z_k \notin G$, and by (16) z_k is a local solution to the following Bolza problem:

$$\min \left\{ \ell_k(x(a), x(b)) + \int_a^b L_k(t, x(t), \dot{x}(t)) dt \right\},$$

where

$$\ell_k(u, v) = d((u, v); S) + s_k |u - z_k(a)|$$

and

$$L_k(t, x, y) = \begin{cases} d(y; F(t, x)) + s_k |y - \dot{z}_k(t)| & \text{if } (x, y) \in A(t), \\ +\infty & \text{otherwise,} \end{cases}$$

where $A(t) = (z(t) + \varepsilon \mathbb{B}) \times (\dot{z}(t) + (N + |\dot{z}(t) - \dot{z}_k(t)|)\mathbb{B})$ and $N > 0$ is an arbitrary integer.

Since $L_k(t, \cdot, \cdot)$ is l.s.c, epi-Lipschitzian at z_k (see the appendix) and epimeasurable in t and since ℓ_k is locally Lipschitzian around $(z_k(a), z_k(b))$, then Theorem 2.1 yields the existence of an arc p_k in $W^{1,1}$ satisfying

$$\dot{p}_k(t) \in \text{co}\{q : \quad (q, p_k(t)) \in \partial L_k(t, z_k(t), \dot{z}_k(t))\} \quad a.e. \quad t \in [a, b] \tag{17}$$

$$(p_k(a), -p_k(b)) \in \partial \ell_k(z_k(a), z_k(b)), \tag{18}$$

$$\langle p_k(t), \dot{z}_k(t) \rangle - L_k(t, z_k(t), \dot{z}_k(t)) = \max_{v \in \mathbb{R}^n} \{\langle p_k(t), v \rangle - L_k(t, z_k(t), v)\}. \tag{19}$$

From (17), (18), and (19) we have

$$(p_k(a), -p_k(b)) \in \partial d((z_k(a), z_k(b)); S) + s_k \mathbb{B} \times \{0\}, \tag{20}$$

$$\dot{p}_k(t) \in \text{ co } \{q : (q, p_k(t)) \in \partial d(\cdot; F(t, \cdot))(z_k(t), \dot{z}_k(t)) + \{0\} \times s_k \mathbb{B}\} \text{ a.e.,} \tag{21}$$

$$\begin{aligned} &\langle p_k(t), \dot{z}_k(t) \rangle - d(\dot{z}_k(t); F(t, z_k(t))) \\ &\quad = \max_{v \in \dot{z}(t) + (N + |\dot{z}(t) - \dot{z}_k(t)|)\mathbb{B}} \{\langle p_k(t), v \rangle - d(v; F(t, z_k(t))) - s_k |v - \dot{z}_k(t)|\} \text{ a.e.} \end{aligned}$$

Step 2 (*application of Lemmas* 2.1–2.4). By (20) there exists $\zeta_k \in \partial d((z_k(a), z_k(b)); S)$ such that

$$(p_k(a), -p_k(b)) - \zeta_k \in s_k \mathbb{B} \times \{0\}. \tag{22}$$

Since $z_k \notin G$, we have either

$$|\zeta_k| = 1 \text{ if } (z_k(a), z_k(b)) \notin S \tag{23}$$

or (because of Lemma 2.1 and (21)) on a set of positive measure on which $\dot{z}_k(t) \notin F(t, z_k(t))$ we have

$$1 - s_k \leq |p_k(t)| \leq 1 + s_k. \tag{24}$$

It follows from (22)–(24) that

$$\frac{1}{\sqrt{2}} - s_k \leq \max_{t \in [a,b]} |p_k(t)| \leq 1 + s_k. \tag{25}$$

Now let $\Gamma : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \mapsto \mathbb{R}^n$ be the multivalued mapping defined by

$$\Gamma(t, x, y, w, s) = \text{ co } \{q : (q, w) \in \partial d(\cdot; F(t, \cdot))(x, y) + \{0\} \times s\mathbb{B}\}.$$

Then

$$(p_k(a), -p_k(b)) - \zeta_k \in s_k\mathbb{B} \times \{0\}, \tag{26}$$

$$\dot{p}_k(t) \in \Gamma(t, z_k(t), \dot{z}_k(t), p_k(t), s_k) \quad \text{a.e.}, \tag{27}$$

$$\begin{aligned} &\langle p_k(t), \dot{z}_k(t)\rangle - d(\dot{z}_k(t); F(t, z_k(t))) \\ &\quad = \max_{v \in \dot{z}(t) + (N + |\dot{z}(t) - \dot{z}_k(t)|)\mathbb{B}} \{\langle p_k(t), v\rangle - d(v; F(t, z_k(t))) - s_k|v - \dot{z}_k(t)|\} \quad \text{a.e.} \end{aligned} \tag{28}$$

Extracting a subsequence if necessary we may suppose that $\zeta_k \to \zeta$ for some ζ in $\partial d((z(a), z(b)); S)$ with

$$|\zeta| = 1 \text{ if } (z_k(a), z_k(b)) \notin S \text{ for infinite number of } k.$$

On the other hand, by Lemma 2.2, the multivalued mapping $\Gamma(t, \cdot)$ is u.s.c. with compact convex values and by the definition of the limiting Fréchet subdifferential and the sub-Lipschitz condition we have (via Lemma 2.1 and (21)) for all k

$$|\dot{p}_k(t)| \leq 1 + k(t) + \beta(1 + |\dot{z}(t) - \dot{z}_k(t)|) \quad \text{a.e.}$$

Note that $\Gamma(t, x, y, w, s)$ is measurable in t (see the appendix). By Lemma 2.4 there exists a subsequence of (p_k) converging uniformly to an arc p satisfying

$$\dot{p}(t) \in \Gamma(t, z(t), \dot{z}(t), p(t), 0) \quad \text{a.e.}, \tag{29}$$

and hence we obtain, by passing to the limit in (26) and (28),

$$(p(a), -p(b)) \in \partial d((z(a), z(b)); S), \tag{30}$$

$$\langle p(t), \dot{z}(t)\rangle = \max_{v \in F(t, z(t)) \cap (\dot{z}(t) + N\mathbb{B})} \langle p(t), v\rangle \text{ a.e.} \tag{31}$$

Now because of (25) the pair (ζ, p) must be nonzero. In fact we have

$$\frac{1}{\sqrt{2}} \leq \max_{t \in [a,b]} |p(t)| \leq 1. \tag{32}$$

As p depends on N, we obtain a sequence (p_N) satisfying (29)–(32) and

$$|\dot{p}_N(t)| \leq 1 + k(t) + \beta \quad \text{a.e.}$$

Again Lemma 2.4 produces a subsequence of (p_N) converging uniformly to some p which satisfies the following:

$$\dot{p}(t) \in \Gamma(t, z(t), \dot{z}(t), p(t), 0) \quad \text{a.e.},$$

$$(p(a), -p(b)) \in \partial d((z(a), z(b)); S),$$

$$\langle p(t), \dot{z}(t)\rangle = \max_{v \in F(t,z(t))} \langle p(t), v\rangle,$$

$$\frac{1}{\sqrt{2}} \leq \max_{t \in [a,b]} |p(t)| \leq 1.$$

Finally we have

$$\dot{p}(t) \in \text{co} D^* F(t, z(t), \dot{z}(t))(-p(t)) \;\; a.e. \;\; t \in [a, b],$$

$$(p(a), -p(b)) \in N(S; (z(a), z(b))),$$

$$\langle p(t), \dot{z}(t)\rangle = H(t, z(t), p(t)) \;\; a.e. \;\; t \in [a, b].$$

Part 2 (when system (13) is seminormal).

Step 1 (*application of Ekeland's variational principle*). Let k be a positive integer, choose $\theta_k \prec f(z(a), z(b))$ such that $|\theta_k - f(z(a), z(b))| < \frac{1}{k^2}$, and define $\Theta := \text{cl}\mathcal{L}(\theta_k)$. Define the function

$$h(x, \theta) = \begin{cases} |f(x(a), x(b)) - \theta| & \text{if } x \in B(z, s_1), \\ +\infty & \text{otherwise,} \end{cases}$$

where s_1 is such that f is Lipschitzian on $(z(a), z(b)) + s_1\mathbb{B}$ with constant k_f. From (A_1) we have $(z, \theta_k) \in G_\varepsilon \times \Theta$, and hence

$$h(z, \theta_k) \leq \inf_{(x,\theta) \in G_\varepsilon \times \Theta} h(x, \theta) + \frac{1}{k^2}.$$

Note that G_ε and Θ are closed in $W^{1,1}$ and $\mathbb{R}^m$, respectively, and that h is l.s.c. on $G_\varepsilon \times \Theta$. Then by Ekeland variational principle there exists $(z_k, \gamma_k) \in G_\varepsilon \times \Theta$ such that

$$\|z_k - z\| + |\gamma_k - \theta_k| \leq \frac{1}{k} \tag{33}$$

and

$$h(z_k, \gamma_k) \leq h(x, \theta) + \frac{1}{k}\left[\|z_k - x\| + |\gamma_k - \theta|\right] \; \forall (x, \theta) \in G_\varepsilon \times \Theta. \tag{34}$$

From (34) one gets

$$h(z_k, \gamma_k) \leq h(x, \gamma_k) + \frac{1}{k}\|z_k - x\| \, \forall x \in G_\varepsilon \tag{35}$$

and

$$h(z_k, \gamma_k) \leq h(z_k, \theta) + \frac{1}{k}|\gamma_k - \theta| \, \forall \theta \in \Theta. \tag{36}$$

Since z is an optimal local solution to problem (P), then, by (A_2) and the choice of θ_k, one has $\gamma_k \neq f(z_k(a), z_k(b))$. Set $w_k = \frac{f(z_k(a),z_k(b))-\gamma_k}{|f(z_k(a),z_k(b))-\gamma_k|}$. Extracting a subsequence we may assume that (w_k) converges to some w, with $|w| = 1$ so that by (36) one has

$$w \in \limsup_{k\to+\infty} N(\mathrm{cl}\mathcal{L}(\theta_k); \gamma_k)$$

and then

$$\omega \in \tilde{N}(\mathrm{cl}\mathcal{L}(f(z(a), z(b))), f(z(a), z(b))).$$

Now from (35) and the seminormality of (13) there exist $\alpha > 0$ and $\min(s_1, r, \varepsilon) > s > 0$ (both not depending on k) such that

$$\begin{aligned} h(z_k, \gamma_k) &\leq h(x, \gamma_k) + \frac{1}{k}\|z_k - x\| \\ &+ \alpha(k_f + 1)\left[d((x(a), x(b)); S) + \int_a^b d(\dot{x}(t), F(t, x(t))dt\right] \end{aligned}$$

for all $x \in B(z, s)$, where r and α are as in (14).

Define the functions

$$\ell_k(u, v) = |f(u, v) - \gamma_k| + \frac{1}{k}|u - z_k(a)| + \alpha(k_f + 1)d((u, v); S)$$

and

$$L_k(t, x, y) = \begin{cases} \alpha(k_f + 1)d(y; F(t, x)) + \frac{1}{k}|y - \dot{z}_k(t)| & \text{if } (x, y) \in A(t), \\ +\infty & \text{otherwise,} \end{cases}$$

where $A(t) = (z(t) + s\mathbb{B}) \times (\dot{z}(t) + (N + |\dot{z}(t) - \dot{z}_k(t)|)\mathbb{B})$ so that z_k is a local solution to the Bolza problem

$$\min\left\{\ell_k(x(a), x(b)) + \int_a^b L_k(t, x(t), \dot{x}(t))dt\right\}.$$

Step 2 (*application of Theorem* 2.1 *and Lemmas* 2.1–2.4). It is easy to check that ℓ_k is l.s.c and locally Lipschitzian around $(z_k(a), z_k(b))$, $L_k(t, \cdot, \cdot)$ is l.s.c, and L_k is epimeasurable in t and epi-Lipschitzian at z_k (see the appendix). Then by Theorem 2.1 there exists an arc p_k in $W^{1,1}$ satisfying

$$\dot{p}_k(t) \in \mathrm{co}\{q : \quad (q, p_k(t)) \in \partial L_k(t, z_k(t), \dot{z}_k(t))\} \quad a.e.\ t \in [a, b], \tag{37}$$

$$(p_k(a), -p_k(b)) \in \partial \ell_k(z_k(a), z_k(b)), \tag{38}$$

$$\langle p_k(t), \dot{z}_k(t)\rangle - L_k(t, z_k(t), \dot{z}_k(t)) = \max_{v\in\mathbb{R}^n}\{\langle p_k(t), v\rangle - L_k(t, z_k(t), v)\}. \tag{39}$$

Consider the multivalued mapping defined by

$$\Gamma(t,x,y,w,s) = \text{ co } \{q : (q,w) \in \alpha(k_f+1)\partial d(\cdot;F(t,\cdot))(x,y) + \{0\} \times s\mathbb{B}\}.$$

From (37)–(39) we have

$$(p_k(a),-p_k(b)) \in \partial(|f(\cdot)-\gamma_k|)(z_k(a),z_k(b)) + N(S;(z_k(a),z_k(b))) + \frac{1}{k}\mathbb{B} \times \{0\}, \tag{40}$$

$$\dot{p}_k(t) \in \Gamma\left(t, z_k(t), \dot{z}_k(t), p_k(t), \frac{1}{k}\right) \quad \text{a.e.,} \tag{41}$$

$$\langle p_k(t), \dot{z}_k(t)\rangle - \alpha(k_f+1)d(\dot{z}_k(t);F(t,z_k(t))) = \max_{v \in \dot{z_k}(t)+(N+|\dot{z}(t)-\dot{z_k}(t)|)\mathbb{B}} \{\langle p_k(t), v\rangle - \alpha(k_f+1)d(v;F(t,z_k(t))) - s_k|v-\dot{z}_k(t)|\} \text{ a.e.} \tag{42}$$

By Lemma 2.2, the multivalued mapping $\Gamma(t,\cdot)$ is u.s.c. with compact convex values, and by the definition of the limiting Fréchet subdifferential and the sub-Lipschitz condition we have (via Lemma 2.1 and (41)) for all k

$$|\dot{p}_k(t)| \le \alpha(k_f+1)(1+k(t)+\beta(1+|\dot{z}(t)-\dot{z}_k(t)|)) \quad \text{a.e.}$$

By Lemma 2.4 there exists a subsequence of (p_k) converging uniformly to an arc p satisfying

$$\dot{p}(t) \in \Gamma(t, z(t), \dot{z}(t), p(t), 0) \quad \text{a.e.} \tag{43}$$

Note that

$$\partial(|f(\cdot,\cdot)-\gamma_k|)(z_k(a),z_k(b)) \subset \partial(\langle w_k, f(\cdot,\cdot)\rangle)(z_k(a),z_k(b)),$$

and hence, by passing to the limit in (40) and (42) and using the same argument as in Part 1, Step 2, we have

$$(p(a),-p(b)) \in \partial(\langle \omega, f(\cdot,\cdot)\rangle)(z(a),z(b)) + N(S;(z(a),z(b))),$$

$$\langle p(t), \dot{z}(t)\rangle = H(t,z(t),p(t)) \text{ a.e.}$$

Now if we assume that F is convex-valued, then, by (29) and/or (43) and Rockafeller result [40], we obtain

$$\dot{p}(t) \in \text{co}\,\{q : (-q,\dot{z}(t)) \in \partial H(t,z(t),p(t))\} \quad \text{a.e.} \quad t \in [a,b],$$

which completes the proof.

5. Appendix.

- $h(x)=d((x(a),x(b));S)+\int_a^b d(\dot{x}(t);F(t,x(t)))dt$ is l.s.c. on $B(z,\varepsilon)$.
  Since F is sub-Lipschitzian at z, then there exist $\beta>0$, $\varepsilon>0$, and a summable function $k:[a,b]\mapsto\mathbb{R}$ such that for almost all $t\in[a,b]$, for all $N>0$, for all $x,x'\in z(t)+\varepsilon\mathbb{B}$, and $y\in\dot{z}(t)+N\mathbb{B}$ one has

$$d(y,F(t,x))-d(y,F(t,x'))\le(k(t)+\beta N)|x-x'|.$$

  Let $x\in B(z,\varepsilon)$ and $\varepsilon'>0$, and set $\delta<\frac{\varepsilon'}{1+\int_a^b k(t)\,dt+\beta(\varepsilon+b-a)}$.
  Let $x'\in B(z,\varepsilon)$ such that $\| x-x' \|<\delta$, and set $N=|\dot{x'}(t)-\dot{z}(t)|+1$. We have

$$\begin{aligned}\int_a^b d(\dot{x}(t),F(t,x(t)))\,dt-\int_a^b d(\dot{x'}(t),F(t,x'(t)))\,dt &\le| \dot{x}(t)-\dot{x'}(t) |\\ &+\int_a^b d(\dot{x'}(t),F(t,x(t)))\,dt-\int_a^b d(\dot{x'}(t),F(t,x'(t)))\,dt\\ &\le\delta+\int_a^b(k(t)+\beta N)\mid x(t)-x'(t)\mid\,dt\\ &\le\delta+\delta\left(\int_a^b k(t)\,dt+\beta(\varepsilon+b-a)\right)\le\varepsilon'.\end{aligned}$$

  Thus h is l.s.c on $B(z,\varepsilon)$.
- G_ε is closed.
  Let (x_n) be a subsequence in G_ε such that $x_n\longrightarrow x$ in $W^{1,1}$. Since S is closed $(x(a),x(b))\in S$. Set $N'=|\dot{x}(t)-\dot{z}(t)|+1$; since F is sub-Lipschitzian at z we have

$$d(\dot{x}(t),F(t,x(t)))\le(k(t)+\beta N)\mid x(t)-x_n(t)\mid+d(\dot{x}(t),F(t,x_n(t)))$$

  so that

$$\begin{aligned}\int_a^b d(\dot{x}(t),F(t,x(t)))\,dt &\le\| x-x_n \|\int_a^b(k(t)+\beta N)\,dt\\ &+\int_a^b d(\dot{x}(t),F(t,x_n(t)))\,dt\\ &\le\| x-x_n \|\left(\beta\| x-z \|+\beta(b-a)+\int_a^b k(t)\,dt\right)\\ &+\| x-x_n \|.\end{aligned}$$

  Then $d(\dot{x}(t),F(t,x(t)))=0$ a.e., and since F is closed-valued $x\in G_\varepsilon$.
- $L_k(t,\cdot,\cdot)$ is epi-Lipschitzian at z_k.
  We have

$$L_k(t,x,y)=\begin{cases}\alpha(k_f+1)d(y;F(t,x))+\frac{1}{k}|y-\dot{z}_k(t)| & \text{if } (x,y)\in A(t),\\ +\infty & \text{otherwise,}\end{cases}$$

  where $A(t)=(z(t)+s\mathbb{B})\times(\dot{z}(t)+(N+|\dot{z}(t)-\dot{z}_k(t)|)\mathbb{B})$.
  For k large enough we can suppose that $|z_k(t)-z(t)|<\frac{s}{2}$. Let $x_1,x_2\in B(z_k(t),\frac{s}{2})$ and $y\in\mathbb{R}^n$ such that $L_k(t,x_1,y)$ is finite. Then

$$| x_1-z(t) |\le s \;\text{ and }\; | y-\dot{z}(t) |\le N+|\dot{z}(t)-\dot{z}_k(t)|.$$

Since $\mid x_2 - z(t) \mid < s$, $L_k(t, x_2, y)$ is finite, and using the fact that F is sub-Lipschitzian at z we get

$$L_k(t, x_2, y) - L_k(t, x_1, y) = \alpha(k_f + 1)[d(y; F(t, x_2)) - d(y; F(t, x_1))]$$

$$\leq \alpha(k_f + 1)(k(t) + \beta(N + |\dot{z}(t) - \dot{z}_k(t)|)) \mid x_1 - x_2 \mid .$$

Then $L_k(t, \cdot, \cdot)$ is epi-Lipschitzian at z_k.

- $\Gamma(t, x, y, w, s)$ is measurable in t.

  The measurability of the multivalued mapping $\Gamma(t, x, y, w, s)$ in t follows from the two following lemmas.

  LEMMA 5.1. *Let $G : [a, b] \to \mathbb{R}^n$ be a measurable multivalued mapping, and let K be a compact set in $\mathbb{R}^n$. Then the multivalued mapping $G(\cdot) + K$ is also measurable.*

  Proof. It suffices to see that for any set A in $\mathbb{R}^n$ we have

$$(G(\cdot) + K)^{-1}(A) = G^{-1}(A - K),$$

  where $G^{-1}(A) = \{t : G(t) \cap A \neq \emptyset\}$.

  LEMMA 5.2. *Let $f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a l.s.c. function in (x, y) and measurable in (t, x, y). Consider the multivalued mapping*

$$R(t, x, y, p) = \{q : (q, -p) \in \partial f(t, x, y) + \{0\} \times s\mathbb{B}\}.$$

  Then R and $\bar{co}\,R$ are measurable in t.

  Proof. It follows from Lemma 2 in [21] that the graph of the multivalued mapping $t \to \partial f(t, x, y)$ is measurable. As this multivalued mapping is closed-valued, Theorem 8.1.4 in [2] implies that it is measurable in t. Now Lemma 5.1 asserts that the multivalued mapping

$$t \longrightarrow \partial f(t, x, y) + \{0\} \times s\mathbb{B}$$

  is measurable in t. The measurability of $t \to R(t, x, y, p)$ follows from the formula

$$(\partial f(\cdot, x, y) + \{0\} \times s\mathbb{B})^{-1}(A \times \{-p\}) = R^{-1}(\cdot, x, y, p)(A).$$

  The measurability of $\bar{co}R$ follows from Theorem 8.2.2 in [2].

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